A unified convergence analysis for a local projection stabilisation applied to the Oseen problem

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Weak formulation of Oseen equations

- domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$
- spaces: $V := H_0^1(\Omega)^d$, $Q := L_0^2(\Omega)$
- parameters: $\nu > 0$, $\sigma \geq 0$
- given divergence-free velocity field $b \in W^{1,\infty}(\Omega)$
- bilinear form

$$
\begin{aligned}
A\big((u,p);(v,q)\big) :=& \nu(\nabla u, \nabla v) + \big((b \cdot \nabla)u, v\big) + \sigma(u,v) \\
& - (p, \nabla \cdot v) + (q, \nabla \cdot u)
\end{aligned}
$$

- weak formulation

  Find $(u,p) \in V \times Q$ such that

$$
A\big((u,p);(v,q)\big) = (f,v) \qquad \forall (v,q) \in V \times Q
$$

- uniquely solvable

Finite element spaces

- shape-regular triangulation $\mathcal{T}_h$
- scalar conforming finite element space Y_h

- equal-order discrete spaces: $V_h := Y_h^d \cap V$, $Q_h := Y_h \cap Q$

Assumption A1

there exists an interpolation operator $i_h : H^1(\Omega) \to Y_h$ with

- $i_h : H_0^1(\Omega) \to \big(Y_h \cap H_0^1(\Omega)\big)$
- $\|w - i_h w\|_{0,K} + h_K |w - i_h w|_{1,K} \leq C h_K^\ell \|w\|_{\ell,\omega(K)}$,
  $\forall w \in H^\ell\big(\omega(K)\big)$, $\forall K \in \mathcal{T}_h$, $1 \leq \ell \leq r + 1$

Finite element discretisation

- discrete problem

  Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$A\big((u_h, p_h); (v_h, q_h)\big) = (f, v_h) \qquad \forall (v_h, q_h) \in V_h \times Q_h$$

- two drawbacks
  - discrete inf-sup condition is violated
  - dominating convection

- one remedy

local projection schemes

Abstract setting

- macro element M: union of neighbouring cells $K \in \mathcal{T}_h$
- macro triangulation $\mathcal{M}_h$: shape-regular ($\mathcal{M}_h = \mathcal{T}_h$ allowed)

- projection space D_h: discontinuous f.e. space w.r.t $\mathcal{M}_h$
- $D_h(M) := \{w|_M : w \in D_h\}$

- local projection $\pi_M : L^2(M) \to D_h(M)$
- $\pi_h : L^2(\Omega) \to D_h$ such that $(\pi_h w)|_M := \pi_M(w|_M)$

Assumption A2 fluctuation operator $\kappa_h := id - \pi_h$ with

$$\|\kappa_h q\|_{0,M} \leq C h_M^\ell |q|_{\ell,M}$$

$\forall q \in H^\ell(M)$, $\forall M \in \mathcal{M}_h$, $0 \leq \ell \leq r$

Stabilised problem

- stabilisation term

$$S_h\big((u_h,p_h);(v_h,q_h)\big) := \sum_{M\in\mathcal{M}_h} \Big(\tau_M\big(\kappa_h(b\cdot\nabla)u_h,\kappa_h(b\cdot\nabla)v_h\big)_M + \mu_M\big(\kappa_h(\nabla\cdot u_h),\kappa_h(\nabla\cdot v_h)\big)_M + \alpha_M\big(\kappa_h\nabla p_h,\kappa_h\nabla q_h\big)_M\Big)$$

- user-chosen parameters τ_M, μ_M, and α_M
- stabilised problem

  Find $(u_h,p_h)\in V_h\times Q_h$ such that

$$(A+S_h)\big((u_h,p_h);(v_h,q_h)\big) = (f,v_h) \qquad \forall (v_h,q_h)\in V_h\times Q_h$$

- norm

$$|\!|\!|(v,q)|\!|\!| := \Big(\nu|v|_1^2+\sigma\|v\|_0^2+(\nu+\sigma)\|q\|_0^2+S_h\big((v,q);(v,q)\big)\Big)^{1/2}$$

Special interpolant j_h

- $Y_h(M) := \{w_h|_M : w_h \in Y_h,\ w_h = 0$ on $\Omega \setminus M\}$

Assumption A3 inf-sup condition

$$\inf_{q_h \in D_h(M)} \sup_{v_h \in Y_h(M)} \frac{(v_h, q_h)_M}{\|v_h\|_{0,M} \ \|q_h\|_{0,M}} \geq \beta_1 > 0$$

$\forall M \in \mathcal{M}_h$

Lemma Let A1 and A3 be satisfied. Then, there exists an interpolation operator $j_h : H^1(\Omega) \to Y_h$ such that

- $\|w - j_h w\|_0 + h|w - j_h w|_1 \leq Ch^\ell \, \|w\|_\ell$, $\forall w \in H^\ell(\Omega)$, $1 \leq \ell \leq r + 1$
- $(w - j_h w, q_h) = 0$, $\forall q_h \in D_h$, $\forall w \in H^1(\Omega)$

Proof I

- for $w \in H^1(\Omega)$ exists $z_h(w) \in Y_h(M)$ such that
  $$\big(z_h(w), q_h\big)_M = (w - i_h w, q_h)_M \qquad \forall q_h \in D_h(M)$$
  and
  $$\|z_h(w)\|_{0,M} \leq \frac{1}{\beta_1} \|w - i_h w\|_{0,M}$$
  due to the inf-sup condition A3
- set $j_h w|_M := i_h w|_M + z_h(w)$ on each $M \in \mathcal{M}_h$
- from
  $$\bigoplus_{M \in \mathcal{M}_h} Y_h(M) \subset Y_h$$
  we get $j_h : H^1(\Omega) \to Y_h$

Proof II

- estimate

$$\|w - j_h w\|_0 \leq \left(1 + \frac{1}{\beta_1}\right) \|w - i_h w\|_0 \leq C\, h^l \|w\|_l$$

  for all $w \in H^l(\Omega)$, $1 \leq l \leq r+1$

- H^1-estimate by inverse inequality
- orthogonality follows from definition of z_h and j_h, indeed:

$$\begin{aligned}(w - j_h w, q_h) &= (w - (i_h w + z_h(w)), q_h) \\ &= (w - i_h w, q_h) - (z_h(w), q_h) = 0\end{aligned}$$

Error Analysis I

for all $(w_h, r_h) \in V_h \times Q_h$

- $A\big((u - u_h, p - p_h); (w_h, r_h)\big) = S_h\big((u_h, p_h); (w_h, r_h)\big)$
- for $(u, p) \in \big(H^{r+1}(\Omega)\big)^d \times H^{r+1}(\Omega)$, we have

$$\left|S_h\big((u,p);(v_h,q_h)\big)\right| \leq C h^{r+1/2} \big(\|u\|_{r+1} + \|p\|_{r+1}\big) \left|\!\left|\!\left|(v_h, q_h)\right|\!\right|\!\right|$$

provided $\alpha_M, \mu_M, \tau_m \sim h_M$ and b piecewise smooth

Error Analysis II

Lemma There exists a positve constant β_2 such that

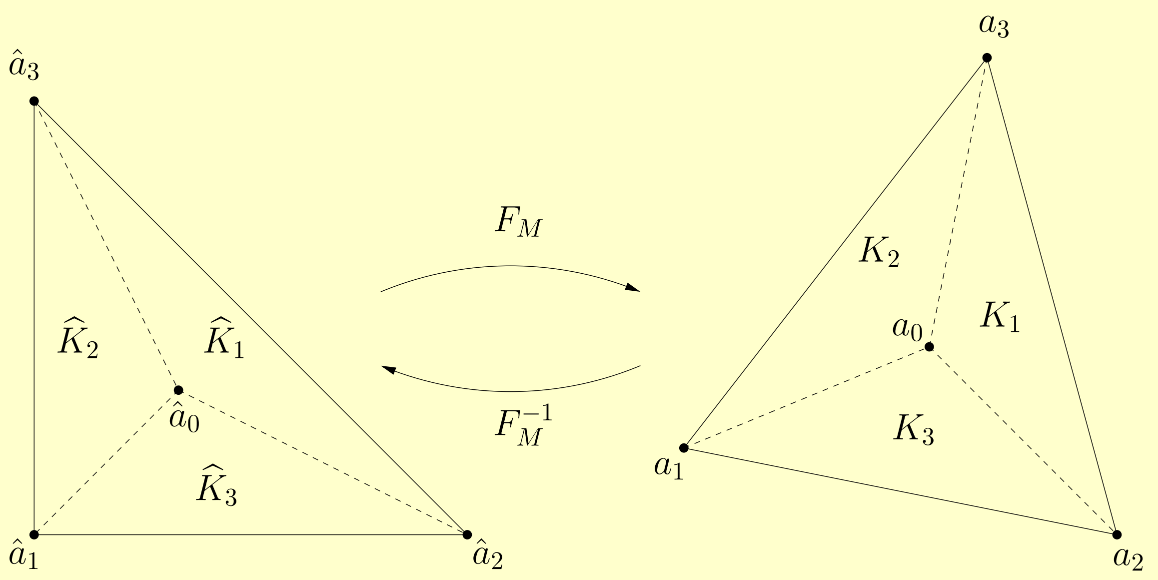
$$\inf_{(v_h,q_h)} \sup_{(w_h,r_h)} \frac{(A+S_h)\big((v_h,q_h);(w_h,r_h)\big)}{\big|\!\big|\!\big|(v_h,q_h)\big|\!\big|\!\big| \; \big|\!\big|\!\big|(w_h,r_h)\big|\!\big|\!\big|} \geq \beta_2.$$

Theorem error estimate

$$\big|\!\big|\!\big|(u-u_h,p-p_h)\big|\!\big|\!\big| \leq C(\nu^{1/2}+h^{1/2})h^r\big(\|u\|_{r+1}+\|p\|_{r+1}\big)$$

Two-level approach I

- macro simplex



- spaces

$$Y_h = P_{r,h} = \{v \in H^1(\Omega) : v|_K \in P_r(K),\ K \in \mathcal{T}_h\}$$

$$D_h = P^{\text{disc}}_{r-1,2h} = \{v \in L^2(\Omega) : v|_M \in P_{r-1}(M),\ M \in \mathcal{M}_h\}$$

Two-level approach II

A1 and A2 are satisfied

Proof of A3 (inf-sup condition)

- affine transformation to reference macro
- $\hat{b}$ piecewise linear base function of barycentre
- let $\hat{q} \in D(\widehat{M}) = P_{r-1}(\widehat{M})$ be arbitrary, set $\hat{v} := \hat{q}\hat{b}$
- $\hat{v} \in Y(\widehat{M})$ since
  - $\hat{v}|_{\partial\widehat{M}} = 0$, $\hat{v}|_{\widehat{K}_i} \in P_r(\widehat{K}_i)$, $\hat{v} \in C^0(\widehat{M})$
- norm equivalence in finite dimensional spaces gives

$$\frac{(\hat{q},\hat{v})}{\|\hat{q}\|_0\,\|\hat{v}\|_0} = \frac{(\hat{q},\hat{q}\hat{b})}{\|\hat{q}\|_0\,\|\hat{v}\|_0} \geq \beta\frac{(\hat{q},\hat{q})}{\|\hat{q}\|_0\,\|\hat{v}\|_0} \geq \beta\frac{(\hat{q},\hat{q})}{\|\hat{q}\|_0\,\|\hat{q}\|_0} = \beta$$

Enrichment method I

- quadrilaterals or hexahedra
- local space

$$Q_r^{\text{bubble}}(\widehat{K}) := Q_r(\widehat{K}) \oplus \text{span}\big(\hat{b}\,\hat{x}_i^{r-1}, i = 1, \ldots, d\big)$$

with bubble function $\hat{b} \in Q_2(\widehat{K}) \cap H_0^1(\widehat{K})$

- mapped finite element spaces

$$\begin{aligned} Y_h = Q_{r,h}^{\text{bubble}} &= \{v \in H^1(\Omega) : \\ &\qquad v|_K \circ F_K^{-1} \in Q_r^{\text{bubble}}(\widehat{K}),\ K \in \mathcal{T}_h\} \\ D_h = P_{r-1,h}^{\text{disc}} &= \{v \in L^2(\Omega) : \\ &\qquad v|_K \circ F_K^{-1} \in P_{r-1}(K),\ K \in \mathcal{T}_h\} \end{aligned}$$

Enrichment method II

A1 and A2 are satisfied

proof of A3 (inf-sup condition)

- Q_1-transformation to reference cells
- estimate determinant of Jacobian
- let $\hat{q} \in D(\widehat{K}) = P_{r-1}(\widehat{K})$ be arbitrary, set $\hat{v} := \hat{q}\hat{b}$
- $\hat{v} \in Y(\widehat{K})$ since
  - $\hat{v} \in H_0^1(\widehat{K})$
  - $\hat{v} \in Q_r^{\text{bubble}}(\widehat{K})$ due to

$$\hat{q} = \hat{q}_0 + \hat{q}_1 \quad \text{with} \quad \begin{aligned} &\hat{q}_0 \in \text{span}\big(\hat{x}_i^{r-1}, i = 1, \ldots, d\big), \\ &\hat{q}_1 \in Q_{r-2}(\widehat{K}) \end{aligned}$$

Relation to subgrid modeling

- local projection: fluctuations of gradient
- subgrid modeling: gradient of fluctuations

- linear element on simplices: both approaches coincide for
  two-level methods and enrichment method

- higher order element on simplices: spectral equivalence of both approaches for
  two-level methods and enrichment method

- on quadrilaterals and hexahedra: no spectral equivalence

Summary and outlook

- local projection methods works for large classes of finite element spaces

- works on simplices, quadrilaterals, and hexahedra

- two-level approach
- enrichment method

- spectral equivalence to subgrid modeling on simplices

- comparison to other stabilisation techniques like SUPG and edge/face stabilisation