Hardy Space Infinite Elements for Scattering and Resonance Problems

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19th Chemnitz FEM Symposium

collaborators

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outline

radiation conditions and TBC

pole condition

Hardy space method

resonances

scattering vs resonances problems

Let $K \subset \mathbb{R}^d$ be smooth, compact and $\mathbb{R}^d \subset K$ connected.

scattering problem: For given $k > 0$ and a given incident field u_i find a scattered field u_s such that

$$-\Delta u_s - k^2 u_s = 0 \qquad \text{in } \mathbb{R}^d \setminus K$$
$$-u_s = u_i \qquad \text{on } \partial K$$
$$u_s \text{ satisfies radiation condition}$$

resonance problem: Find an eigenpair (u, k^2) such that

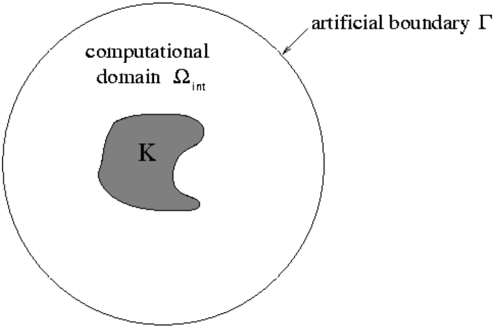
$$-\Delta u = k^2 u \qquad \text{in } \mathbb{R}^d \setminus K$$
$$u = 0 \qquad \text{on } \partial K$$
$$u \text{ satisfies radiation condition}$$

k is called a resonance.

We have $\operatorname{Im}(k) < 0$, and u grows exponentially at infinity.

transparent boundary conditions (TBC)

For finite element computations the infinite domain $\Omega := \mathbb{R}^d \setminus K$ has to be truncated to a finite computational domain Ω_{int}. At the artificial boundary Γ of Ω_{int} we have to imposed a so-called transparent boundary condition which reflects the radiation condition at infinity.



radiation conditions for real wave number k

Let $K \subset \mathbb{R}^2$ compact, $k > 0$, and $\Delta u + k^2 u = 0$ in $\Omega := \mathbb{R}^2 \setminus K$.
Then the following conditions are equivalent:

- Sommerfeld's radiation condition:

  $$\sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) \to 0 \qquad \text{as } r = |x| \to \infty$$

  uniformly for all directions $\hat{x} = \frac{x}{|x|}$.

- In polar coordinates (r, ϕ) ($r > a$, $0 \le \phi < 2\pi$) u has a series representation

  $$u(r, \phi) = \sum_{n=-\infty}^{\infty} c_n e^{in\phi} H_{|n|}^{(1)}(kr).$$

  Here $H_{|n|}^{(1)}$ is the Hankel function of the first kind of order $|n|$.

- u has an integral representation

  $$u(x) = \int_{|y|=a} \left\{ \frac{\partial \Phi(x,y,k)}{\partial n(y)} u(y) - \Phi(x,y,k) \frac{\partial u}{\partial n}(y) \right\} ds(y)$$

  in terms of the fundamental solution
  $\Phi(x, y, k) := (i/4) H_0^{(1)}(k|x-y|)$ for a sufficiently large.

radiation conditions for $\operatorname{Im} k < 0$

If $\operatorname{Im} k < 0$ and $\operatorname{Re} k > 0$, Sommerfeld's radiation condition is not a valid characterization of outgoing waves. (In particular, it does not guarantee uniqueness for exterior boundary value problems.)

The series representation and the integral representation, however, are still equivalent and lead to well-posed exterior boundary value problems in appropriate norms. (Recall that the solutions grow exponentially at infinity!)

classical TBCs

- approximation by local boundary conditions, e.g. $\frac{\partial u}{\partial n} = ik\,u$ on Γ_a
  Bayliss-Gunzburger-Turkel, Enquist-Majda, Feng, Goldberg, Grote, Keller, ...
  based on **Sommerfeld's radiation condition** or analogous higher order conditions
- boundary integral equation method (FEM/BEM coupling)
  Chandler-Wilde, Costabel, Greengard, Hackbusch, Hsiao, Kress, Nédélec, Rokhlin, Wendland, ...
  based on **integral representation** of solution
- infinite elements
  Bettess, Burnett, Demkowicz, Gerdes, Zienkiewicz, ...
  based on **series representation** of solution

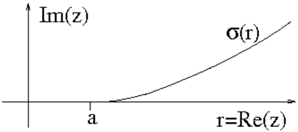
All these TBCs destroy the eigenvalue structure of the problem!

complex coordinate stretching/PML

We consider the holomorphic extension of the solution $u(r,\phi)$ in polar coordinates with respect to the radial variable r and define

$$u_\sigma(r,\phi) := u(r + i\sigma(r), \phi)$$

assumptions: $\sigma \in C^1[0,\infty)$
$\sigma \geq 0$, $\sigma' \geq 0$, $\sigma(x) = 0$ for $x \leq a$,
$\lim_{x\to\infty} \sigma(x) = \infty$.



Since for $d = 1$ the holomorphic extension is

$$u(z) = \begin{cases} c\exp(+ikz) & \text{for an outgoing solution,} \\ c\exp(-ikz) & \text{for an incoming solution,} \end{cases}$$

we get

$$u_\sigma(r) = \begin{cases} c\exp(+ikr - k\sigma(r)): & \text{exponentially decaying,} \\ c\exp(-ikr + k\sigma(r)): & \text{exponentially increasing.} \end{cases}$$

Perfectly Matched Layer Method (PML)

d = 1: Let $\gamma(r) := r + i\sigma(r)$. The chain rule applied to $u_\sigma = u \circ \gamma$ yields

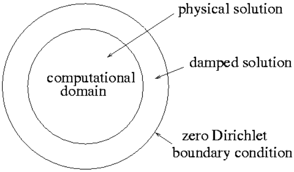
$$\gamma'(r)^{-1} u_\sigma'(r) = u'(\gamma(r)).$$

Differentiating again and using $-u''(z) = k^2 u(z)$ we obtain the differential equation

$$\frac{-1}{\gamma'(r)} \frac{d}{dr} \left(\frac{1}{\gamma'(r)} u_\sigma'(r) \right) = k^2 u_\sigma(r).$$

d > 1: A similar computation yields a pde for u_σ involving an unisotropic damping tensor in radial direction.

Imposing a zero Dirichlet condition at some finite distance, we obtain a transparent boundary condition which can be implemented by standard fem software.



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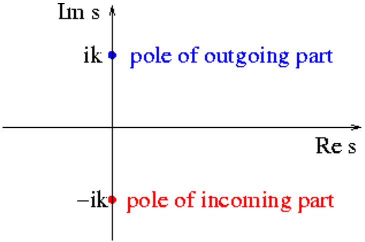
pole condition for $d = 1$

The general solutuion to the 1d Helmholtz eq. $u''(r)+k^2u(r)=0$ is

$$u(r) = u_\infty^+ e^{ikr} + u_\infty^- e^{-ikr}.$$

Its **Laplace transform** $\hat{u}(s) := \int_0^\infty e^{-sr} u(r)\,dr$ is given by

$$\hat{u}(s) = \frac{u_\infty^+}{s-ik} + \frac{u_\infty^-}{s+ik}.$$



Note: u is outgoing if and only if the Laplace transform $\hat{u}$ of u has no poles in the lower complex half–plane.

Laplace transform of the Helmholtz equation

$\Delta u + k^2 u = 0$ in $\mathbb{R}^2 \setminus K, K \subset \{x : |x| < a\}$ compact, $k > 0$

↓

polar coordinates:
$U(\rho, \hat{x}) := \sqrt{\rho} u(\rho \hat{x}), \qquad \rho > 0, \hat{x} \in S^1$

$$\left(\frac{\partial^2}{\partial r^2} + k^2 + \frac{1}{(r+a)^2} (\Delta_{\hat{x}} + \tfrac{1}{4} I) \right) U(r+a, \hat{x}) = 0$$

↓

Laplace transform:
$\hat{U}(s, \hat{x}) := \int_0^\infty e^{-sr} U(r+a, \hat{x})\, dr, \qquad \mathrm{Re}\, s > 0$

$$(s^2 + k^2) \hat{U}(s, \hat{x}) + \int_s^\infty e^{-a(s_1 - s)} (s_1 - s)(\Delta_{\hat{x}} + \tfrac{1}{4} I) \hat{U}(s_1, \hat{x})\, ds_1$$
$$= sU(a, \hat{x}) + \frac{\partial}{\partial \rho} U(a, \hat{x}), \qquad \mathrm{Re}\, s > 0$$

pole condition and Sommerfeld radiation condition

Definition

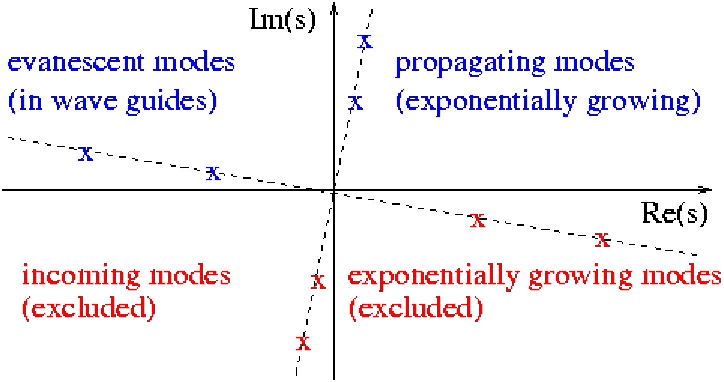
u satisfies the pole condition if the mapping $s \to \hat{U}(s,\cdot)$ defined on $\{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}$ with values in $L^2(S^{d-1})$ has a holomorphic extension to $D := \{s \in \mathbb{C} : \mathrm{Re}\, s > 0 \text{ or } \mathrm{Im}\, s < 0\}$.

Theorem

A bounded solution to the Helmholtz equation for $k > 0$ satisfies the pole condition if and only if it satisfies the Sommerfeld radiation condition.

T. Hohage, F. Schmidt, L. Zschiedrich: *Solving time-harmonic scattering problems based on the pole condition. I: Theory* SIAM J. Math. Anal., **35**:183-210 (2003)

pole condition for $\operatorname{Im} k < 0$



discussion

- The pole condition is a unifying radiation condition in particular for
  - scattering by bounded obstacles
  - rough surface scattering problems (equivalent to Upward Propagating Radiation Condition proposed by S. Chandler-Wilde as shown by T. Arens & T. Hohage)
  - scattering problems in wave guides
- independent of the differential equation and in particular the wave number
- stable representation formula of exterior solution, which is cheap to evaluate (talk by Roland Klose)
- leads to several new transparent boundary conditions

outline

Hardy space $\mathcal{H}^2_-(\mathbb{R})$

Definition

A function u, which is holomorphic in the lower complex half-plane $\mathbb{C}^- := \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$ has L^2 boundary values $v = u|_{\mathbb{R}} \in L^2(\mathbb{R})$ if

$$\int_{-\infty}^{\infty} |u(x - i\epsilon) - v(x)|^2 \, dx \xrightarrow{\epsilon \searrow 0} 0.$$

$$\mathcal{H}^2_-(\mathbb{R}) := \left\{ v \in L^2(\mathbb{R}) : \exists u : \mathbb{C}^- \to \mathbb{C} \text{ holomorphic with } v = u|_{\mathbb{R}} \right\}.$$

- $\mathcal{H}^2_-(\mathbb{R})$ equipped with the L^2 inner product is a Hilbert space.
- pole condition: $\hat{U}(\cdot, \hat{x})|_{\mathbb{R}} \in \mathcal{H}^2_-(\mathbb{R})$ for all $\hat{x} \in S^{d-1}$
- idea: Galerkin method in $\mathcal{H}^2_-(\mathbb{R})$
- problem: appropriate basis of $\mathcal{H}^2_-(\mathbb{R})$

Hardy space $\mathcal{H}^2_-(S^1)$

Definition

Let $B^1 := \{z \in \mathbb{C} : |z| < 1\}$ and $S^1 := \partial B$. A holomorphic function $u : B^1 \to \mathbb{C}$ has L^2 boundary values $v = u|_{S^1} \in L^2(S^1)$ if

$$\int_{S^1} |u(rz) - v(z)|^2 \, |dz| \xrightarrow{r \nearrow 1} 0.$$

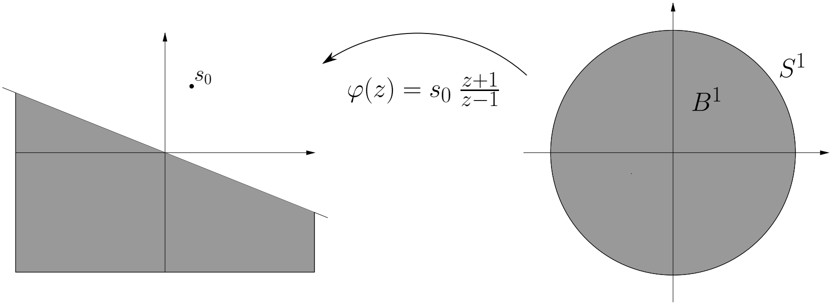
$$\mathcal{H}^2_-(S^1) := \left\{ v \in L^2(S^1) : \exists u : B^1 \to \mathbb{C} \text{ holomorphic with } v = u|_{S^1} \right\}.$$

Lemma

$\mathcal{H}^2_-(S^1)$ equipped with the L^2 inner product is a Hilbert space with orthonormal basis

$$z \mapsto \frac{1}{\sqrt{2\pi}} z^j, \qquad j = 0, 1, 2, \ldots$$

Möbius transform



Lemma

The mapping $\mathcal{M}u := (u \circ \varphi) \cdot \sqrt{\varphi'}$ is a unitary operator from $\mathcal{H}^2_-(\mathbb{R})$ to $\mathcal{H}^2_-(S^1)$.

variational formulation

With $X := H^1(\Omega_{\text{int}}) \times (\mathcal{H}^2_-(S^1) \otimes H^{1/2}(\Gamma))$ the complete problem is given by:

Find nontrivial eigenpairs $(\,(u, \widetilde{U})\,,\, k^2) \in X \times \mathbb{C}$ satisfying

$$a(\,(v,\widetilde{V})\,,\,(u,\widetilde{U})\,) = k^2 b(\,(v,\widetilde{V})\,,\,(u,\widetilde{U})\,) \quad \forall\ (v,\widetilde{V}) \in X.$$

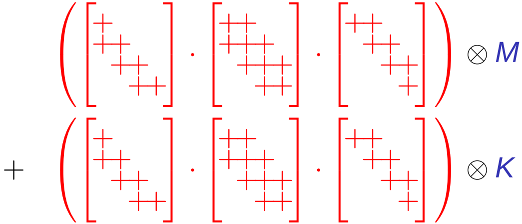
$$\begin{aligned}
a(\,(v,\widetilde{V})\,,\,(u,\widetilde{U})\,) &= \int_{\Omega_{\text{int}}} \nabla\bar{v}\nabla u\,dx \\
&+ \frac{s_0^2}{2\pi}\int_\Gamma\int_{S^1}[\ldots]\,\widetilde{D}_a^{(d-1)}\left[cu|_{S_a^{d-1}}(\hat{x}) + (z+1)\widetilde{U}(z,\hat{x})\right]|dz|d\hat{x} \\
&+ \frac{1}{2\pi}\int_\Gamma\int_{S^1}\nabla_{\hat{x}}[\ldots]\,\widetilde{I}_a^{(3-d)}\nabla_{\hat{x}}\left[cu|_{S_a^{d-1}}(\hat{x}) + (z-1)\widetilde{U}(z,\hat{x})\right]|dz|d\hat{x}, \\
b(\,(v,\widetilde{V})\,,\,(u,\widetilde{U})\,) &= \int_{\Omega_{\text{int}}} \bar{v}\,u\,dx \\
&+ \frac{1}{2\pi}\int_\Gamma\int_{S^1}[\ldots]\,\widetilde{D}_a^{(d-1)}\left[cu|_{S_a^{d-1}}(\hat{x}) + (z-1)\widetilde{U}(z,\hat{x})\right]|dz|d\hat{x}.
\end{aligned}$$

Hardy space infinite elements

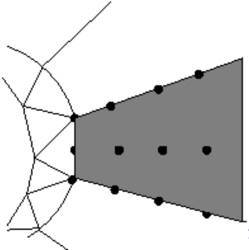
Galerkin discretization:

$\mathrm{span}\{z^0, \ldots, z^N\} \otimes \mathcal{P}^p(\Gamma_h) \subset \mathcal{H}^2_-(S^1) \otimes H^{1/2}(\Gamma)$

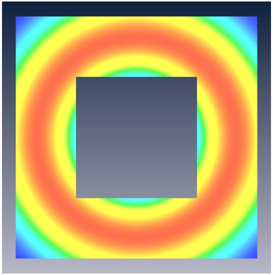
$\rightsquigarrow$ "Hardy–space infinite elements". local element matrices:

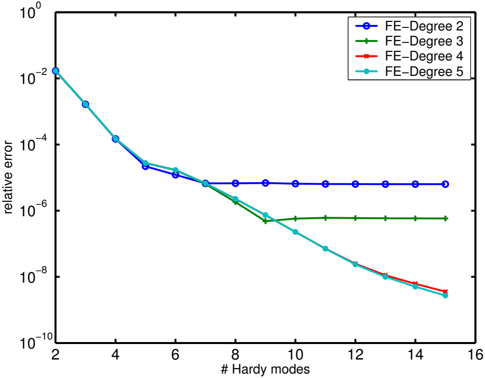


M	boundary mass matrix corresponding to $\int_\Gamma uv\, ds$
K	boundary stiffness matrix corresponding to $\int_\Gamma \nabla_{\hat{x}} u \nabla_{\hat{x}} v\, ds$



numerical convergence





Hardy space infinite elements in space domain

In the space domain our ansatz functions $\widetilde{U}(z) = \sum_{n=0}^{N} \alpha_n z^n$ are given by

$$u_a(r) = e^{s_0 r} \left\{ u_0 - 2\sqrt{-2s_0} \sum_{n=0}^{N} \alpha_n \sum_{j=0}^{n} \binom{n}{j} \frac{(2s_0 r)^{j+1}}{(j+1)!} \right\}.$$

$\rightsquigarrow$ bad convergence in the far field!

separation of variables

For given k consider the exterior boundary value problem

$$\Delta u + k^2 u = 0 \qquad \text{in } \{x : |x| > a\},$$
$$u = u_0 \qquad \text{on } \Gamma,$$

which we solve using the Hardy space method.
Let $\{\varphi_j : j \in \mathbb{N}\} \subset L^2(\Gamma)$ be a complete orthonormal system of the Laplace-Beltrami operator $\Delta_{\hat{x}}$ on Γ (e.g. trigonometric monomials). Then the equations of the Hardy space method can be separated into

$$A_j \tilde{U}_j = F_j(u_0), \qquad j \in \mathbb{N}$$

with operators $A_j \in L(\mathcal{H}^2_-(S^1))$ and right hand sides $F_j(u_0) \in \mathcal{H}^2_-(S^1)$.

Toeplitz operators

Definition

Let $f : S^1 \to \mathbb{C}$ be continuous, and $P : L^2(S^1) \to H^2_-(S^1)$ the orthogonal projection. The Toeplitz operator $T_f : \mathcal{H}^2_-(S^1) \to \mathcal{H}^2_-(S^1)$ with symbol f is defined by

$$T_f\varphi := P(f \cdot \varphi), \qquad \varphi \in \mathcal{H}^2_-(S^1).$$

Theorem

If $f(z) \neq 0$ for all $z \in S^1$, then T_f is a Fredholm operator with $\mathrm{index}(T_f) = -\mathrm{wn}(f)$, where $\mathrm{wn}(f)$ is the winding number of f around 0.

Lemma

The operators A_j are complex perturbations of the Toeplitz operator with symbol $f(z) = -s_0^2|z+1|^2 + k^2|z-1|^2$. Moreover, they are one-to-one. Hence, A_j is boundedly invertible.

convergence theorem

Let $P_n : \mathcal{H}^2_-(S^1) \to \mathrm{span}\{z^0, \dots, z^n\}$ denote the orthogonal projection. We approximate the exact equation by

$$P_n A_j P_n \tilde{U}_j^{(n)} = P_n F_j(u_0).$$

Theorem (Hohage, Nannen)
There exists n_0 such that the discrecte equation has a unique solution $\tilde{U}_j^{(n)}$ for all $n \geq n_0$, and $\left\| \tilde{U}_j^{(n)} - \tilde{U}_j \right\|_{L^2}$ converges super–algebraically to 0 for $n \to \infty$.

Sketch of the proof.

- stability: $\|(P_n A_j P_n)^{-1}\| \leq C$ follows from results in Prössdorf & Silbermann, 1991 on shift-operators.
- $\tilde{U}_j \in C^\infty(S^1)$ due to results in Hohage, Schmidt & Zschiedrich, 2003. This entails super–algebraic convergence of the approximation error $\inf_{v \in R(P_n)} \|v - \tilde{U}_j\|$.

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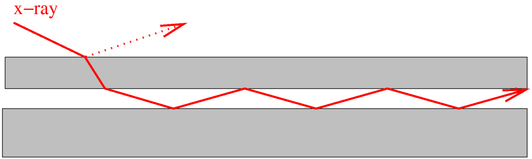
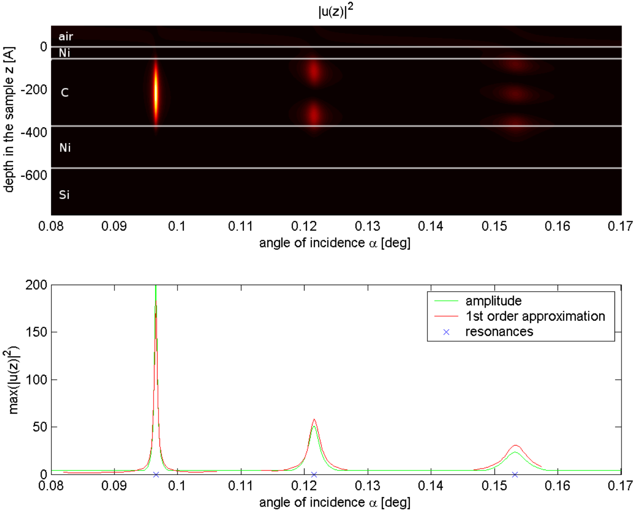
application in x-ray physics

cooperation with Prof. Tim Salditt
Institute for x-ray physics, Univ. Göttingen

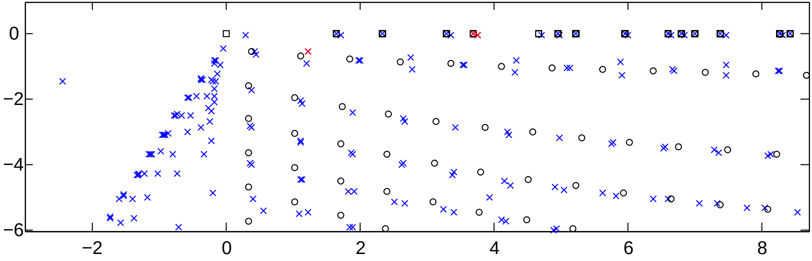


aim: Design the layers to achieve maximal field enhancement under the restriction that refractive indices for x-rays are close to 1.

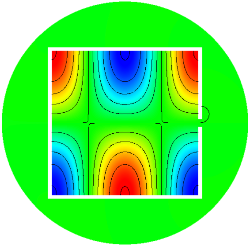
scattering solutions and resonances



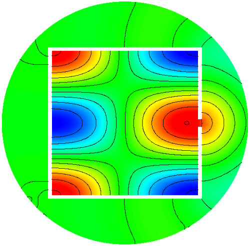
resonances of an open square



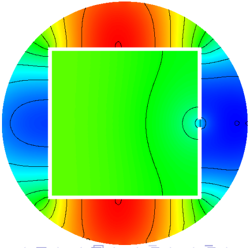
QF≈ $1,98 \cdot 10^7$



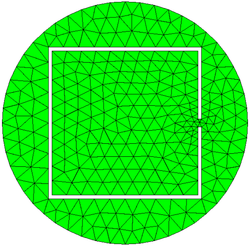
QF≈ 199, 6



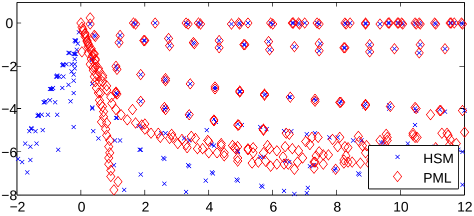
QF≈ 2, 24

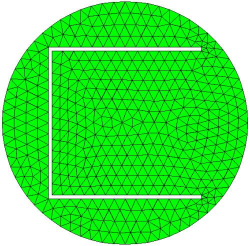


comparison HSM - PML

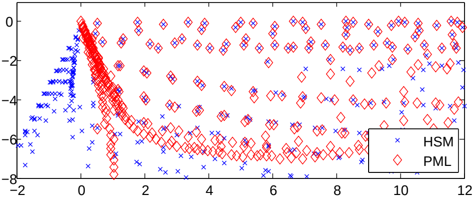


resonances of an open square



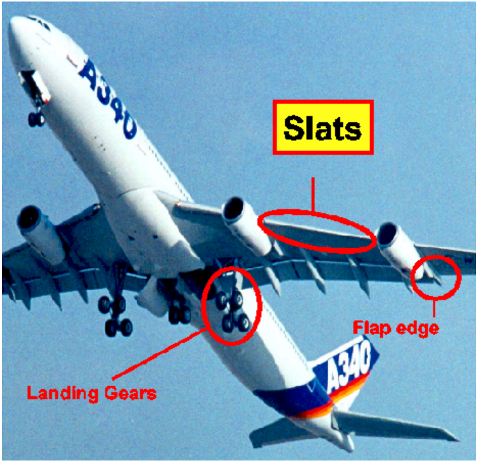


resonances of an one−sided open square



major sources of airframe noise

collaboration with W. Koch and S. Hein, DLR, Göttingen
and J. Schöberl, RWTH Aachen



Source: U. Michel *International Symposium Arcachon, France* (2002)

resonances of the high lift configuration