Edge Elements and Coercivity

Ralf Hiptmair

Seminar für Angewandte Mathematik
ETH Zürich

(e–mail: hiptmair@sam.math.ethz.ch)

(Homepage: http://www.sam.math.ethz.ch/˜hiptmair)

Variational Problems

Maxwell Boundary Value Problem

Bounded Lipschitz cavity $\Omega \subset \mathbb{R}^3$ with PMC walls

- Electric wave equation

$$\begin{aligned} \mathrm{curl}\, \mu_r^{-1} \mathrm{curl}\, \mathsf{E} - \kappa^2 \epsilon_r \mathsf{E} &= i\kappa \mathsf{j}_0 \quad && \text{in } \Omega\,, \\ \mu_r^{-1} \mathrm{curl}\, \mathsf{E} \times \mathsf{n} &= 0 && \text{on } \partial\Omega\,. \end{aligned}$$

j_0 : exciting current

κ : wavenumber, $\kappa := \omega\sqrt{\epsilon_0 \mu_0} L$

ϵ_r : rel. dielectric constant

μ_r : relative permeability

Assumption: ϵ_r, μ_r uniformly positive, piecewise smooth

Variational formulation

Seek $\mathsf{E} \in \boldsymbol{H}(\mathrm{curl}; \Omega)$ such that

$$\underbrace{\left(\mu_r^{-1} \mathrm{curl}\, \mathsf{E}, \mathrm{curl}\, \mathsf{v}\right)_0 - \kappa^2 (\epsilon_r \mathsf{E}, \mathsf{v})_0}_{=: a(\mathsf{E},\mathsf{v})} = i\kappa\, (\mathsf{j}_0, \mathsf{v})_0 \quad \forall \mathsf{v} \in \boldsymbol{H}(\mathrm{curl}; \Omega)\,.$$

Coercivity

V Banach space, sesqui-linear form $a(\cdot,\cdot) : V \times V \mapsto \mathbb{C}$ satisfies generalized Gårding inequality, if

$$\exists c > 0 : \quad |a(u, Xu) + \langle Ku, \overline{u}\rangle_{V'\times V}| \geq c\,\|u\|_V^2 \quad \forall u \in V \; .$$

for some isomorphism $X : V \mapsto V$, compact $K : V \mapsto V'$.

plus a injective: $\quad a(u,v) = 0 \quad \forall v \in V \quad \Rightarrow \quad u = 0$

Fredholm alternative $\rightarrow$ $\Downarrow$

$$\forall f \in V' : \quad \exists_1 u \in V : \quad a(u,v) = \langle f, v\rangle_{V'\times V} \quad \forall v \in V \; .$$

Example: Helmholtz equation $-\Delta u - \kappa^2 u = f$ with $V = H^1(\Omega)$:

$$a(u,v) := \boxed{(\operatorname{grad} u, \operatorname{grad} v)_0} - \boxed{\kappa^2 (u,v)_0} \, , \quad u, v \in H^1(\Omega) \; .$$

principal part $\qquad$ compact perturbation $\Rightarrow K$ $\Big\}$ $\Rightarrow \quad X = Id$

Splitting Idea

Maxwell Challenge (I)

Acoustic waves

- Helmholtz equation

$$-\Delta\rho - \kappa^2\rho = 0$$

Potential "energy" : $\int |\frac{1}{\kappa}\,\mathrm{grad}\,\rho|^2$

Kinetic "energy" : $\int |\rho|^2$

Kinetic energy is compact perturbation of potential energy

⇕

Strong ellipticity

Electromagnetic waves

- Electric wave equation

$$\mathrm{curl}\,\mathrm{curl}\,\mathsf{E} - \kappa^2\mathsf{E} = 0$$

Magnetic "energy" : $\int |\frac{1}{\kappa}\,\mathrm{curl}\,\mathsf{E}|^2$

Electric "energy" : $\int |\mathsf{E}|^2$

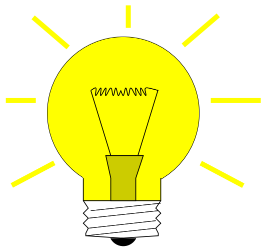
[Perfect symmetry of E and H!]

Electric energy is no compact perturbation of magnetic energy

⇕

Lack of strong ellipticity

Splitting Idea



Idea:
Split E into predominantly electric and predominantly magnetic components.

Example: L^2-orthogonal Helmholtz decomposition of electric field:

$$\mathsf{E} = \mathsf{grad}\,\Phi + \mathsf{curl}\,\mathsf{A}$$

Electric component (curl-free)	Magnetic component (divergence-free)
$\Downarrow$	$\Downarrow$
No magnetic energy	Magnetic energy dominates

- Recover (strong) ellipticity by restricting electric wave equation to components of Helmholtz decomposition
- Generalization: Stability sufficient, orthogonality not required

Regular Decomposition

Lemma (Girault, Raviart): ($\beta_2(\Omega) = 0$)

There is a continuous operator $\mathsf{L} : \boldsymbol{H}(\mathrm{div}\, 0; \Omega) \mapsto \boldsymbol{H}^1(\Omega)$ with

$$\mathrm{curl}\, \mathsf{L}\mathsf{u} = \mathsf{u}\,, \;\; \mathrm{div}\, \mathsf{L}\mathsf{u} = 0\,, \;\; \|\mathsf{L}\mathsf{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\, \|\mathsf{u}\|_{\boldsymbol{L}^2(\Omega)}\;.$$

- Projections: $\mathsf{R} := \mathsf{L} \circ \mathrm{curl} \quad , \quad \mathsf{Z} := Id - \mathsf{R}$

$$[\mathsf{R}^2 = R, \mathsf{Z}^2 = \mathsf{Z}, \mathsf{R} \circ \mathsf{Z} = \mathsf{Z} \circ \mathsf{R} = 0]$$

- Stable direct splitting: $\boldsymbol{H}(\mathrm{curl}; \Omega) = \boldsymbol{\mathcal{X}}(\Omega) \oplus \boldsymbol{\mathcal{N}}(\Omega)$

$$\boldsymbol{\mathcal{X}}(\Omega) := \mathsf{R}(\boldsymbol{H}(\mathrm{curl}; \Omega)) \subset \boldsymbol{H}^1(\Omega)\;,$$
$$\boldsymbol{\mathcal{N}}(\Omega) := \mathsf{Z}(\boldsymbol{H}(\mathrm{curl}; \Omega)) = \mathrm{Ker}(\mathrm{curl})\;.$$

- Compact embedding: $\boldsymbol{\mathcal{X}}(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$

- Stability: $\|\mathsf{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\, \|\mathrm{curl}\, \mathsf{u}\|_{\boldsymbol{L}^2(\Omega)}, \forall \mathsf{u} \in \boldsymbol{\mathcal{X}}(\Omega)$

Coercivity

Split variational problem:

Use $\mathsf{E} = \mathsf{E}^\perp + \mathsf{E}^0$, $\mathsf{v} = \mathsf{v}^\perp - \mathsf{v}^0$, $\mathsf{E}^\perp, \mathsf{v}^\perp \in \mathcal{X}(\Omega)$, $\mathsf{E}^0, \mathsf{v}^0 \in \mathcal{N}(\Omega)$

$$\begin{aligned} \left(\tfrac{1}{\mu_r}\operatorname{curl}\mathsf{E}^\perp, \operatorname{curl}\mathsf{v}^\perp\right)_0 - \kappa^2\left(\epsilon_r\mathsf{E}^\perp, \mathsf{v}^\perp\right)_0 &- \kappa^2\left(\epsilon_r\mathsf{E}^0, \mathsf{v}^\perp\right)_0 &= i\kappa\left(\mathsf{j}_0, \mathsf{v}^\perp\right)_0 \quad \forall \mathsf{v}^\perp, \\ \kappa^2\left(\epsilon_r\mathsf{E}^\perp, \mathsf{v}^0\right)_0 &+ \kappa^2\left(\epsilon_r\mathsf{E}^0, \mathsf{v}^0\right)_0 &= i\kappa\left(\mathsf{j}_0, \mathsf{v}^0\right)_0 \quad \forall \mathsf{v}^0 . \end{aligned}$$

Note: red terms are compact !

- Introduce “sign flipping isomorphism”:

$$\mathsf{X} := \mathsf{R} - \mathsf{Z} : \boldsymbol{H}(\operatorname{curl};\Omega) \mapsto \boldsymbol{H}(\operatorname{curl};\Omega)$$

- Generalized Gårding inequality:

$\exists$ compact sesqui-linear form k on $\boldsymbol{H}(\operatorname{curl};\Omega)$ and $c > 0$

$$|a(\mathsf{u}, \mathsf{X}\mathsf{u}) + k(\mathsf{u},\mathsf{u})| \geq c\,\|\mathsf{u}\|^2_{\boldsymbol{H}(\operatorname{curl};\Omega)} \quad \forall \mathsf{u} \in \boldsymbol{H}(\operatorname{curl};\Omega) .$$

- Existence & uniqueness provided that $\kappa \neq$ resonant frequency

Galerkin Discretization

Abstract Theory (I)

Setting:
- V Hilbert space, $a : V \times V \mapsto \mathbb{C}$ continuous sesqui-linear form
- a satisfies generalized Gårding inequality & is injective
- sequence $V_h \subset V$, $h \in \mathbb{H}$, of finite-dimensional subspaces, asymptotically dense:

$$\forall u \in V : \quad \lim_{h \to 0} \inf_{v_h \in V_h} \|u - v_h\|_V = 0 \,.$$

Goal: Asymptotic *discrete inf-sup condition*

$$\exists \tilde{c} > 0 : \quad \sup_{v_h \in V_h} \frac{|a(u_h, v_h)|}{\|v_h\|_V} \geq \tilde{c}\, \|u_h\|_V \quad \forall u_h \in V_h, \; \boxed{\forall h < h_0} \,.$$

Existence & quasi-optimality of Galerkin solution u_h

$$\|u - u_h\|_V \leq \frac{\|a\|}{\tilde{c}} \inf_{v_h \in V_h} \|u - v_h\|_V \,.$$

Abstract Theory (II)

Define $S : V \mapsto V$ compact : $a(v, Su) = \langle Kv, \overline{u}\rangle_{V'\times V} \quad \forall u, v \in V$,
$P_h : V \mapsto V_h$ V-orthogonal projections .

$$P_h \xrightarrow{n\to\infty} Id \quad \text{pointwise} \quad \overset{S \text{ compact}}{\Longrightarrow} \quad (Id - P_h)S \to 0 \quad \text{uniformly}$$

- Yields discrete inf-sup condition for $X = Id$!

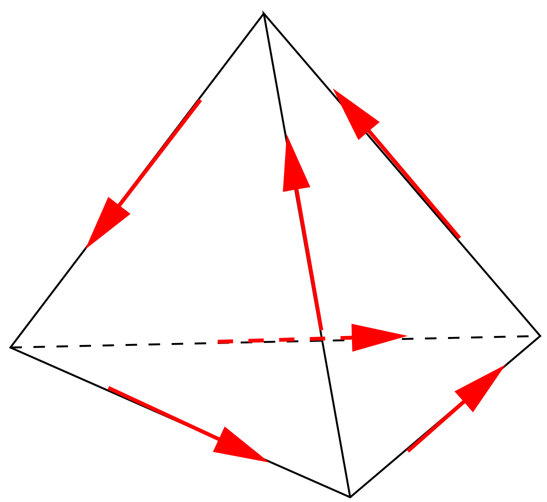
! If $X = \mathsf{R} - \mathsf{Z}$, $X(V_h) \not\subset V_h$ ➢ Need projector $P_h^X : \mathcal{X}(\Omega) \mapsto V_h$ satisfying

$$\exists\{\epsilon_h\} \subset \mathbb{R}_+^{\mathbb{N}}, \lim_{h\to 0} e_h = 0: \|(Id - P_h^Z)\mathsf{R}u_h\|_V \leq \epsilon_h \|u_h\|_V \quad \forall u_h \in V_h, \forall h \in \mathbb{H}$$

$$|a(u_h, (P_h^X X + P_h S)u_h)| =$$
$$|a(u_h, Xu_h) + \langle Ku_h, \overline{u}_h\rangle_{V'\times V} - a((Id - P_h^X)2\mathsf{R}u_h, u_h) - a((Id - P_h)Su_h, u_h)|$$
$$\geq (c - \|a\|(2\epsilon_h + \|(Id - P_h)S\|)) \|u_h\|_V^2$$

- Discrete inf-sup condition for sufficiently small h

Edge Elements



Local space on tetrahedron:

$$\mathcal{E}(T) := \{\mathbf{x} \mapsto \mathbf{a} + \mathbf{b} \times \mathbf{x}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\}$$

6 local degrees of freedom: "edge voltages"

Local shape functions (λ_i = barycentric coord.)

$$\mathbf{b}_{ij} = \lambda_i \,\mathbf{grad}\, \lambda_j - \lambda_j \,\mathbf{grad}\, \lambda_i$$

➤ Edge FE space on tetrahedral mesh $\mathcal{M}_h$: $\mathcal{E}_h$

Π_h = FE interpolation onto $\mathcal{E}_h$

I_h = Nodal interpolation onto $\mathcal{S}_h$
($\mathcal{S}_h$ = p.w. linear continuous FE on $\mathcal{M}_h$)

J_h = Face flux interpolation onto $\mathcal{F}_h$
($\mathcal{F}_h$ = $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming face elements)

➤ Commuting diagram properties

$\Pi_h \circ \mathbf{grad} = \mathbf{grad} \circ \mathrm{I}_h$
$\mathsf{J}_h \circ \mathbf{curl} = \mathbf{curl} \circ \Pi_h$

Note: Π_h unbounded even on $\boldsymbol{H}^1(\Omega)$!

But $\|u - \Pi_h u\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \lesssim h(\|u\|_{H^1(\Omega)} + \|\mathbf{curl}\, u\|_{H^1(\Omega)})$

Z-Projection (I)

Goal: Find projection $P_h^X : \mathsf{R}(\mathcal{E}_h) \mapsto \mathcal{E}_h$ such that

$$\left\| (Id - P_h^X)\mathsf{R}\mathsf{u}_h \right\|_{\boldsymbol{H}(\mathsf{curl};\Omega)} \lesssim h \left\| \mathsf{curl}\, \mathsf{u}_h \right\|_0 \quad \forall \mathsf{u}_h \in \mathcal{E}_h \; .$$

Surprise: $P_h^Z := \Pi_h$ is eligible !

① $\mathsf{curl}\,\mathsf{R}\mathsf{u} = \mathsf{curl}\,\mathsf{u}$ ➢ If $\mathsf{u}_h \in \mathcal{E}_h$, then $\mathsf{curl}\,\mathsf{R}\mathsf{u}_h \subset \mathcal{F}_h$ p.w. constant

② Poincaré mapping $(\mathfrak{L}\mathsf{w})(\mathsf{x}) = \int_0^1 t\,(\mathsf{w}(t\mathsf{x})) \times \mathsf{x}\,\mathsf{d}t$ satisfies

- $\mathsf{curl} \circ \mathfrak{L} \circ \mathsf{curl} = \mathsf{curl}$,
- $\left\| \mathfrak{L}\mathsf{w} \right\|_{L^2(\Omega)} \leq \mathrm{diam}(\Omega) \left\| \mathsf{w} \right\|_{L^2(\Omega)}$,
- if $\mathsf{w} \equiv \mathrm{const}$ on tetrahedron T, then $\mathfrak{L}\mathsf{w} \in \mathcal{E}(T)$.

③ Pick tetrahedron $T \in \mathcal{M}_h$

If $\mathsf{u} \in (\mathsf{R}\mathcal{E}_h)|_T$ ➢ $\mathsf{curl}(\mathsf{u} - \mathfrak{L}(\mathsf{curl}\,\mathsf{u})) = 0$

➢ $\mathsf{u} - \mathfrak{L}(\mathsf{curl}\,\mathsf{u}) = \mathsf{grad}\,\varphi$ for some $\varphi \in H^2(T)$

Z-Projection (II)

④ Use local inverse inequality & $\|\mathfrak{L}\mathsf{w}\|_{L^2(T)} \leq h_T \|\mathsf{w}\|_{L^2(T)}$

$$
\begin{aligned}
|\varphi|_{H^2(T)} &\lesssim |\mathsf{u}|_{H^1(T)} + h_T^{-1} \|\mathfrak{L}\,\mathsf{curl}\,\mathsf{u}\|_{L^2(T)} \\
&\lesssim |\mathsf{u}|_{H^1(T)} + \|\mathsf{curl}\,\mathsf{u}\|_{L^2(T)}
\end{aligned}
$$

⑤ Use commuting diagram property $\Pi_h \circ \mathsf{grad} = \mathsf{grad} \circ \mathrm{I}_h$

$$
\mathsf{u} - \Pi_h \mathsf{u} = \underbrace{\mathfrak{L}(\mathsf{curl}\,\mathsf{u}) - \Pi_h \mathfrak{L}(\mathsf{curl}\,\mathsf{u})}_{=0} + \mathsf{grad}(\varphi - \mathrm{I}_h \varphi)
$$

▶ $\|\mathsf{u} - \Pi_h \mathsf{u}\|_{L^2(T)} = |(Id - \mathrm{I}_h)\varphi|_{H^1(T)} \lesssim h_T |\varphi|_{H^2(T)} \lesssim h_T |\mathsf{u}|_{H^1(T)}$.

⑤ By commuting diagram property $\mathsf{J}_h \circ \mathsf{curl} = \mathsf{curl} \circ \Pi_h$

$$
\mathsf{curl}(\mathsf{R}\mathsf{u}_h - \Pi_h \mathsf{R}\mathsf{u}_h) = (Id - \mathsf{J}_h)\,\mathsf{curl}\,\mathsf{R}\mathsf{u}_h = 0 \,.
$$

Another Application: EFIE

EFIE = simplest boundary integral equation for electromagnetic scattering

$$a_\Gamma(\boldsymbol{\eta}, \boldsymbol{\mu}) = \frac{1}{\kappa^2} \boxed{\langle V_\kappa(\mathrm{div}_\Gamma \boldsymbol{\eta}), \mathrm{div}_\Gamma \boldsymbol{\mu} \rangle_{\boldsymbol{\tau}}} - \boxed{\langle V_\kappa \boldsymbol{\eta}, \boldsymbol{\mu} \rangle_{\boldsymbol{\tau}}}, \quad \boldsymbol{\eta}, \boldsymbol{\mu} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) .$$

Term of order 1 — Term of order -1

Single layer BI-Op: $V_\kappa : \begin{cases} H^{-\frac{1}{2}}(\Gamma) & \mapsto & H^{\frac{1}{2}}(\Gamma) \\ \psi & \mapsto & \int_\Gamma \frac{e^{i\kappa|Vx-\mathrm{y}|}}{4\pi|\mathrm{x}-\mathrm{y}|} \psi(\mathrm{y})\, dS(\mathrm{y}) \end{cases}$ coercive

Tangential trace space of $\boldsymbol{H}(\mathrm{curl};\Omega)$

Again:

- Use trace induced "Hodge-type" regular deomposition of $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$

➤ Generalized Gårding inequality for $a_\Gamma : \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \times \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \mapsto \mathbb{C}$

- P_h^Z-projection = FE interpolation for $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$-conforming BEM

R. Hiptmair and C. Schwab, *Natural boundary element methods for the electric field integral equation on polyhedra*, SIAM J. Numer. Anal., 40 (2002), pp. 66–86.

References

☞ Survey of numerical analysis results for edge elements:

R. HIPTMAIR, *Finite elements in computational electromagnetism*, Acta Numerica, (2002), pp. 237–339.

☞ Coecivity arguments for edge boundary elements:

A. BUFFA AND R. HIPTMAIR, *Galerkin boundary element methods for electromagnetic scattering*, in Topics in Computational Wave Propagation. Direct and inverse Problems, M. Ainsworth, P. Davis, D. Duncan, P. Martin, and B. Rynne, eds., vol. 31 of Lecture Notes in Computational Science and Engineering, Springer, Berlin, 2003, pp. 83–124.

☞ Presentation of abstract framework:

A. BUFFA, *Remarks on the discretization of some non-positive operators with application to heterogeneous Maxwell problems*, SIAM J. Numer. Anal., 43 (2005), pp. 1–18.