Anisotropic mesh adaption based on a posteriori estimates and optimisation of node positions

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Fakultät für Mathematik

TECHNISCHE UNIVERSITÄT CHEMNITZ

Motivation

- The solution features attempted to be captured by adaptive mesh refienement are often of *anisotropic* character.
- In these cases the most efficient approximation is achieved by using *anisotropic* meshes.
- For simplicity of usage the adaptive meshes should be generated automatically, based on *a posteriori* error estimates.

Overview

- Idea: minimise *a posteriori* error estimate η by moving node positions s in the mesh appropriately.
- Discrete Adjoint Technique makes this feasible:
  - once u_h and η are computed it allows to compute $\nabla\eta$ at cost comparable to only <u>one</u> additional evaluation of u_h and η, independent of $\dim(s)$.
- Constraints required to guarantee mesh remains suitable for FEM and error estimate remains reliable.
- Results in nonlinear optimisation problem with inequality constraints, $\nabla\eta$ available. $\Rightarrow$ Use SQP with BFGS update formula to approximate Hessian.
- Approach comprises a module for mesh adaption algorithms, can be combined with other approaches, i.e. standard adaptive isotropic refinement.
- Numerical examples are presented, demonstrating the feasibility of the approach.

Discrete Adjoint Technique

Consider a scalar valued function

$$I(s) = \widetilde{I}(u(s), s), \quad \text{with vector } u(s) \text{ defined by}$$
$$0 = R(u(s), s).$$

Using the solution Ψ of the adjoint equation

$$\left[\frac{\partial R}{\partial u}\right]^T \Psi = \left[\frac{\partial \widetilde{I}}{\partial u}\right]$$

the gradient can be evaluated as

$$\frac{DI}{Ds} = \frac{\partial \widetilde{I}}{\partial s} - \Psi^T \frac{\partial R}{\partial s}.$$

Error estimator and optimisation

- The error estimate η should be of type

$$\eta = \left(\sum_{T \in \mathcal{T}} \eta_T^2 \right)^{1/2} .$$

  Element contributions η_T have to be differentiable w.r.t. the node positions.
- Least squares problem $\Rightarrow$ approximate equidistribution of error.
- Robustness w.r.t. deformed meshes extremely important.
- E.g. hierachical error estimator [Ainsworth Oden, 2000].

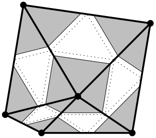
Optimisation Problem

Problem 1:

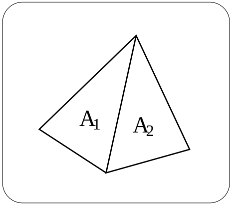
Minimise η^2, with respect to the node positions s, subject to:

1. the mesh approximates the boundaries of the domain,
2. the mesh is non-self-overlapping (NSO),
3. interior angles of the triangles staying bounded well bellow π, and
4. the aspect ratio of triangles varies smoothly (i.e. changes in the aspect ratio of neighbouring elements must be bounded).

- NSO:



- aspect ratio changes:



$$c_5 A_1 - A_2 \geq 0 \quad \text{and} \quad c_5 A_2 - A_1 \geq 0.$$

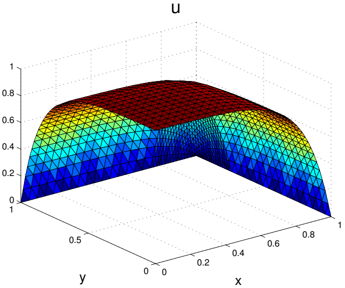
Numerical example (Definition)

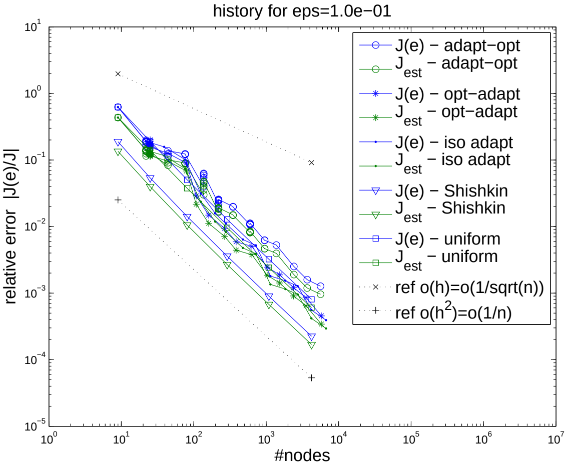
Singularly perturbed reaction diffusion problem

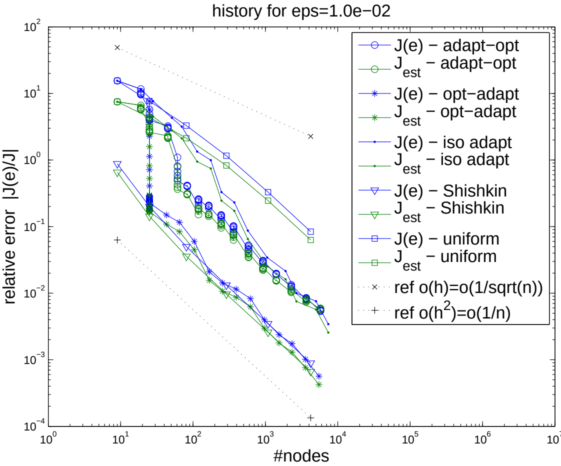
$$\begin{aligned} -\Delta u + \tfrac{1}{\varepsilon^2} u &= \tfrac{1}{\varepsilon^2} && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma_D \\ \tfrac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_N \end{aligned}$$

$$\begin{aligned} \eta^2 &:= \sum_{T \in \mathcal{T}} J_{e,T}^2 && \text{local DWR estimate for } J(e) \\ J(u) &:= \int_{\Gamma_D} \frac{\partial u}{\partial n} \end{aligned}$$

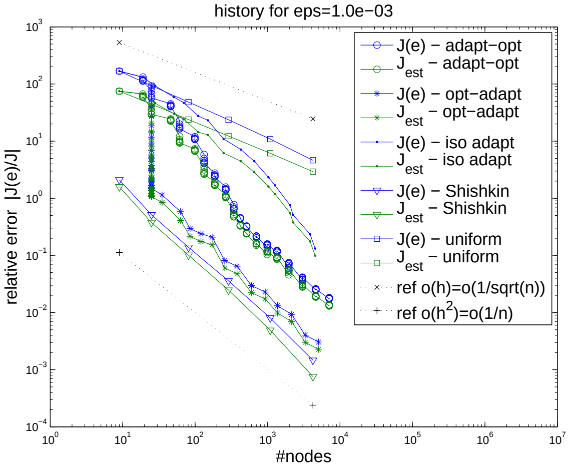
Numerical example 1 (Domain, Solution)



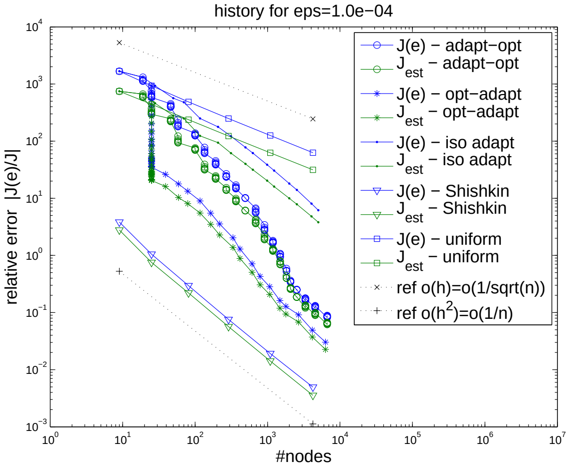

history for eps=1.0e−01
relative error |J(e)/J|
#nodes
J(e) − adapt−opt
J_est − adapt−opt
J(e) − opt−adapt
J_est − opt−adapt
J(e) − iso adapt
J_est − iso adapt
J(e) − Shishkin
J_est − Shishkin
J(e) − uniform
J_est − uniform
ref o(h)=o(1/sqrt(n))
ref o(h^2)=o(1/n)


history for eps=1.0e−02
relative error |J(e)/J|
#nodes
J(e) − adapt−opt
J_est − adapt−opt
J(e) − opt−adapt
J_est − opt−adapt
J(e) − iso adapt
J_est − iso adapt
J(e) − Shishkin
J_est − Shishkin
J(e) − uniform
J_est − uniform
ref o(h)=o(1/sqrt(n))
ref o(h^2)=o(1/n)
10^2
10^1
10^0
10^−1
10^−2
10^−3
10^−4
10^0
10^1
10^2
10^3
10^4
10^5
10^6
10^7

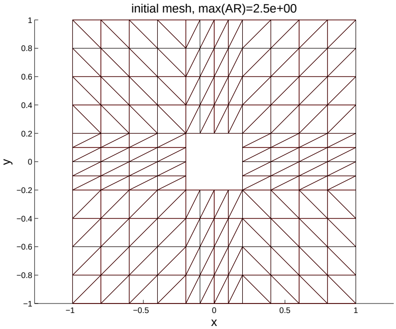
Numerical example 1 (Convergence)

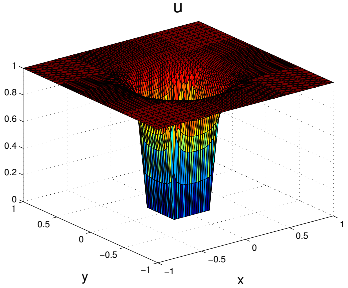


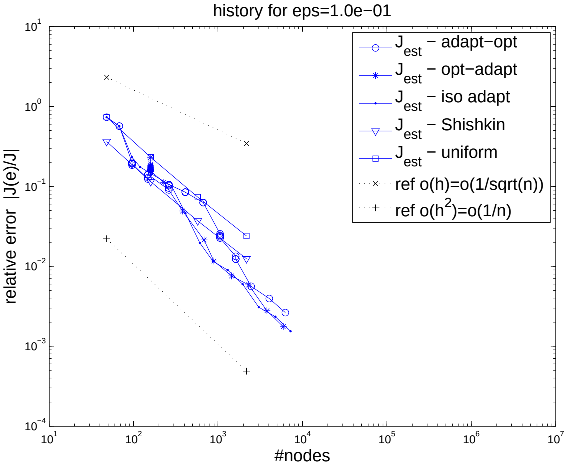
Numerical example 1 (Convergence)

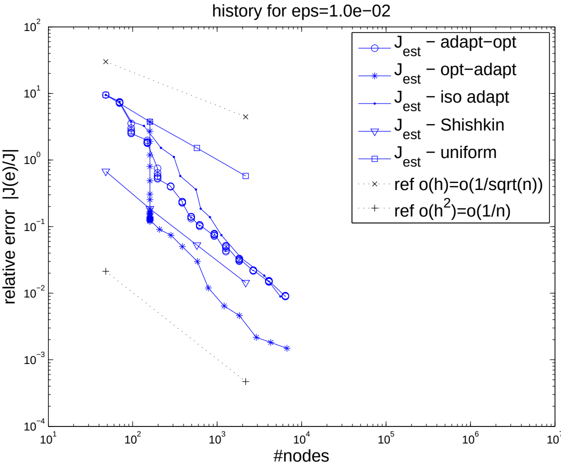


Numerical example 2 (Domain, Solution)

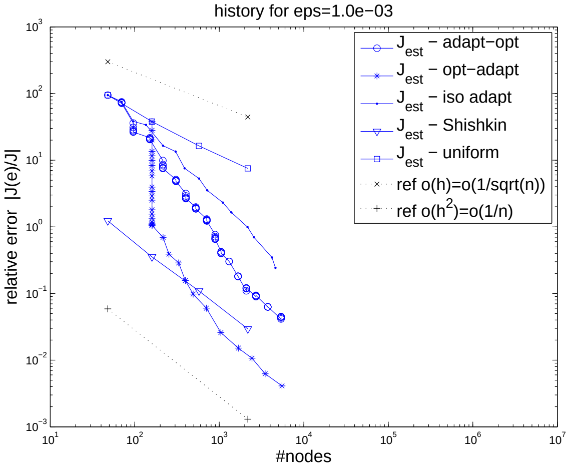




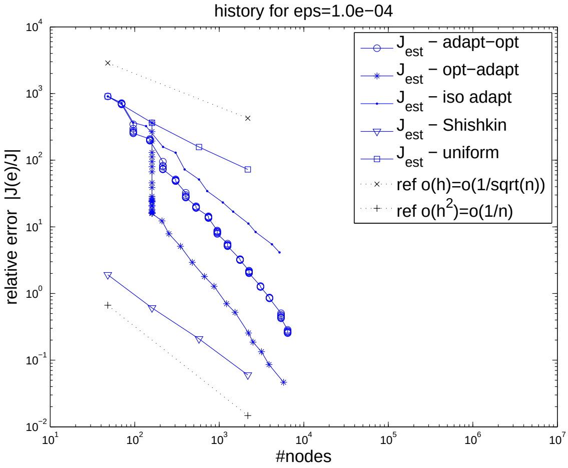

history for eps=1.0e−01
relative error |J(e)/J|
#nodes
J_est − adapt−opt
J_est − opt−adapt
J_est − iso adapt
J_est − Shishkin
J_est − uniform
ref o(h)=o(1/sqrt(n))
ref o(h^2)=o(1/n)
10^1
10^0
10^−1
10^−2
10^−3
10^−4
10^1
10^2
10^3
10^4
10^5
10^6
10^7


history for eps=1.0e−02
relative error |J(e)/J|
#nodes
10^2
10^1
10^0
10^−1
10^−2
10^−3
10^−4
10^1
10^2
10^3
10^4
10^5
10^6
10^7
J_est − adapt−opt
J_est − opt−adapt
J_est − iso adapt
J_est − Shishkin
J_est − uniform
ref o(h)=o(1/sqrt(n))
ref o(h^2)=o(1/n)

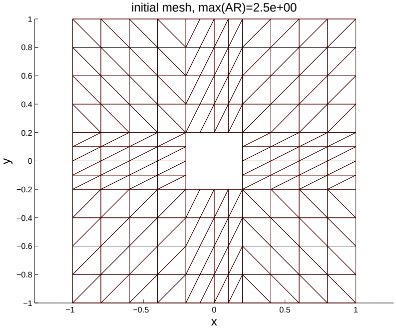
Numerical example 2 (Convergence)

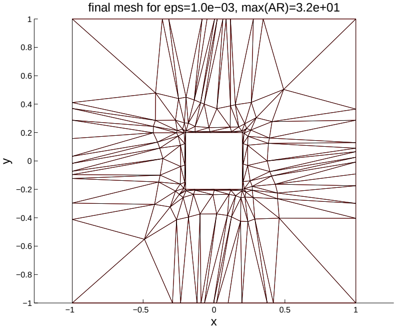


Numerical example 2 (Convergence)



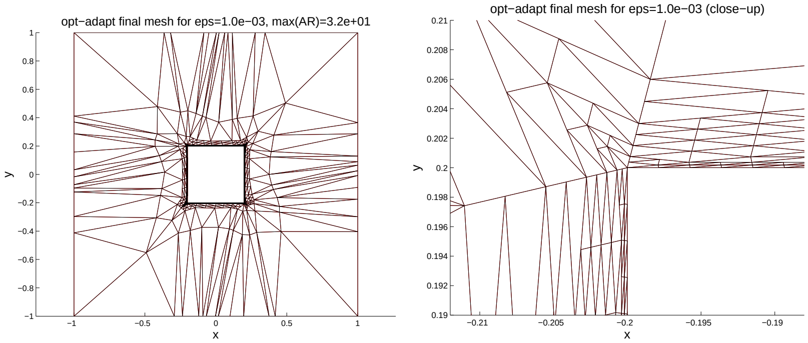
initial and optimised coarse mesh



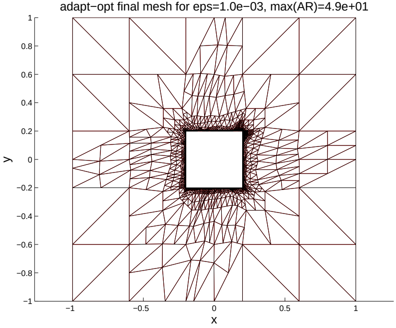


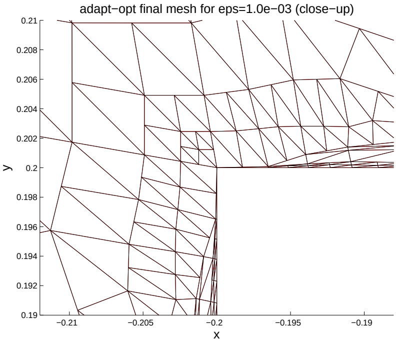
Numerical example 2 (Meshes, $\varepsilon = 10^{-3}$)

opt-adapt



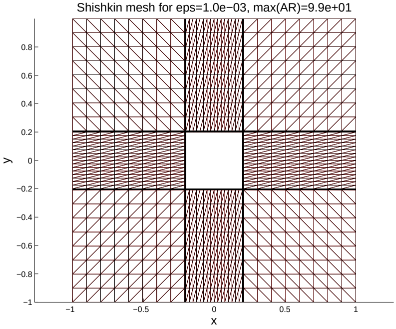
adapt-opt

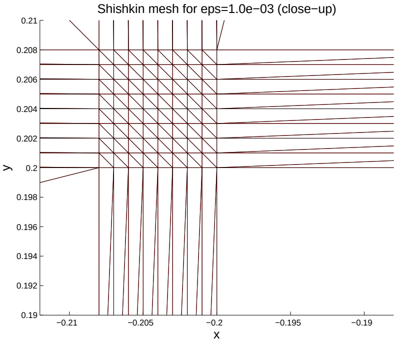

adapt−opt final mesh for eps=1.0e−03, max(AR)=4.9e+01
y
x
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.5
0
0.5
1


adapt−opt final mesh for eps=1.0e−03 (close−up)
y
x
0.21
0.208
0.206
0.204
0.202
0.2
0.198
0.196
0.194
0.192
0.19
−0.21
−0.205
−0.2
−0.195
−0.19

Numerical example 2 (Meshes, $\varepsilon = 10^{-3}$)

Shishkin mesh





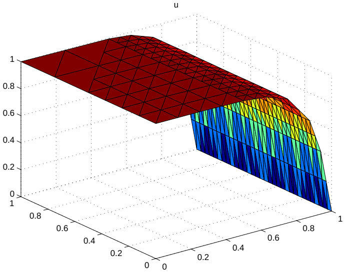
Numerical example 3 (Definition)

Singularly perturbed convection diffusion problem

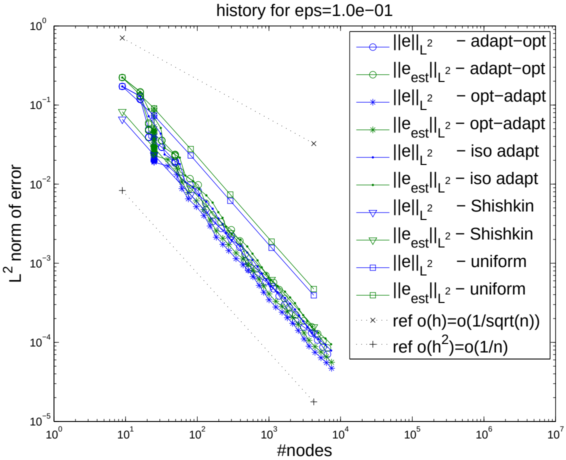
$$\begin{aligned} -\varepsilon \Delta u + b^T \nabla u &= 0 && \text{in } \Omega \\ u &= g && \text{on } \Gamma \end{aligned}$$

$$\begin{aligned} \eta^2 &:= ||e_{2h} - e_h||_{L_2(\Omega)} \\ \delta &= \frac{h}{2\,||b||}\left(\coth Pe - \frac{1}{Pe}\right) && \text{parameter for SUPG} \end{aligned}$$

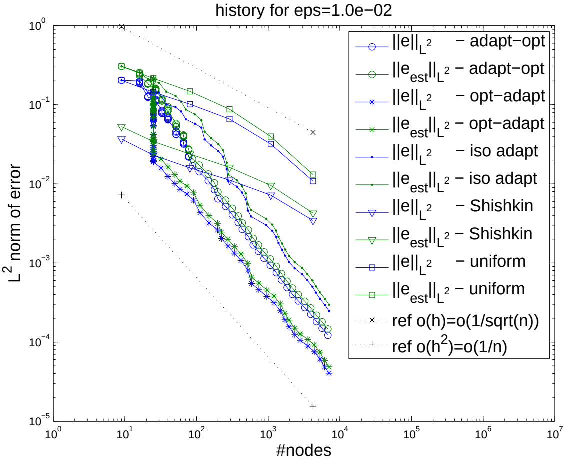
$$b = [1, 0]^T$$



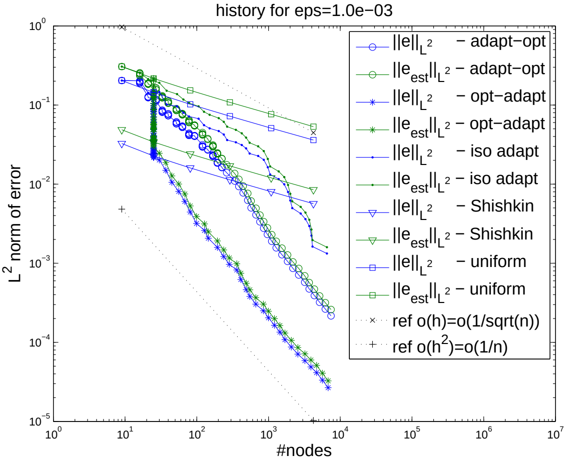
Numerical example 3 (Convergence)



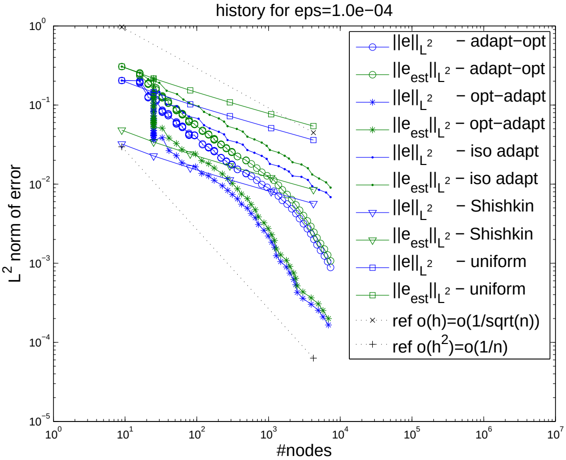
Numerical example 3 (Convergence)



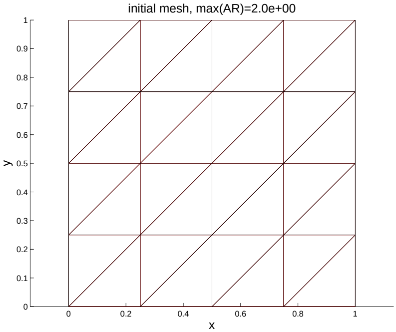
Numerical example 3 (Convergence)

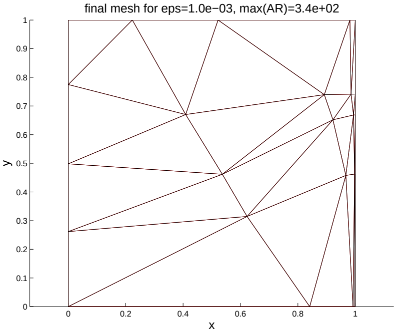


Numerical example 3 (Convergence)

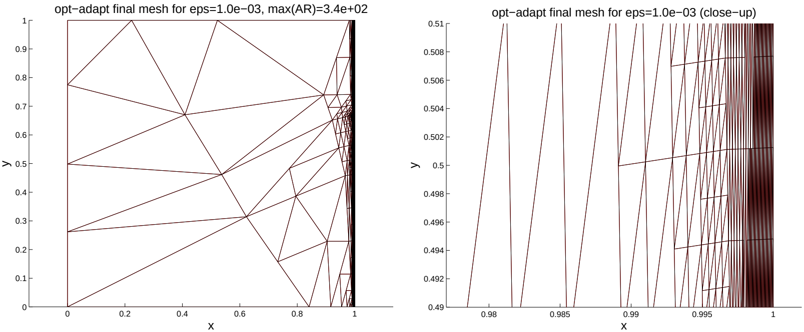


initial and optimised coarse mesh

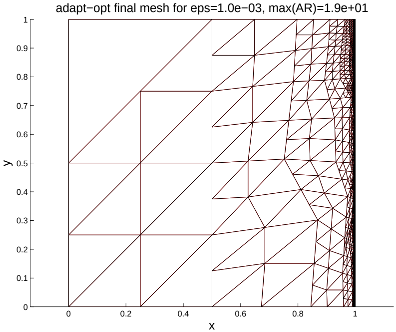


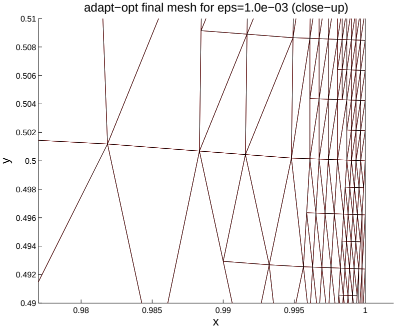


opt-adapt


opt−adapt final mesh for eps=1.0e−03, max(AR)=3.4e+02
opt−adapt final mesh for eps=1.0e−03 (close−up)
x
y

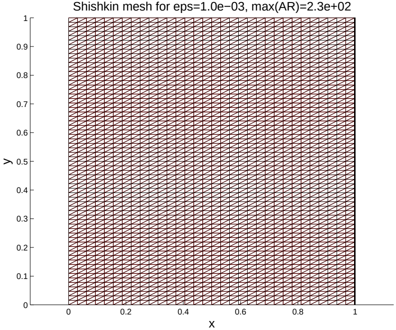
adapt-opt

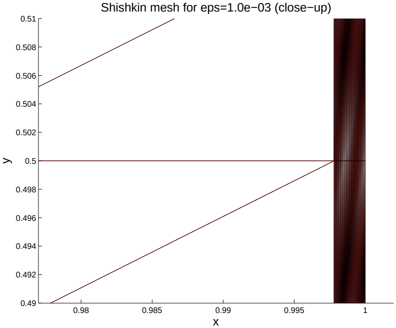

adapt−opt final mesh for eps=1.0e−03, max(AR)=1.9e+01
y
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1


adapt−opt final mesh for eps=1.0e−03 (close−up)
y
x
0.49
0.492
0.494
0.496
0.498
0.5
0.502
0.504
0.506
0.508
0.51
0.98
0.985
0.99
0.995
1

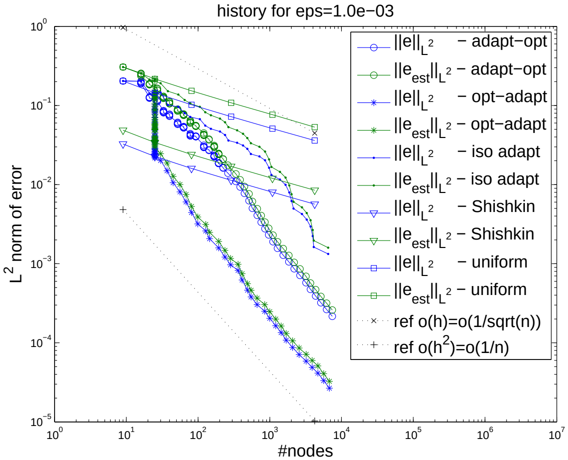
Numerical example 3 (Meshes, $\varepsilon = 10^{-3}$)

Shishkin mesh



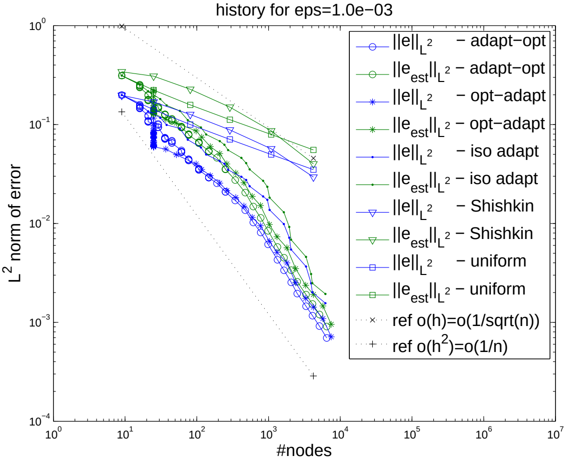


$b = [1, 0]^T$


history for eps=1.0e−03
L^2 norm of error
#nodes
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^0
10^1
10^2
10^3
10^4
10^5
10^6
10^7
$||e||_{L^2}$ − adapt−opt
$||e_{est}||_{L^2}$ − adapt−opt
$||e||_{L^2}$ − opt−adapt
$||e_{est}||_{L^2}$ − opt−adapt
$||e||_{L^2}$ − iso adapt
$||e_{est}||_{L^2}$ − iso adapt
$||e||_{L^2}$ − Shishkin
$||e_{est}||_{L^2}$ − Shishkin
$||e||_{L^2}$ − uniform
$||e_{est}||_{L^2}$ − uniform
ref o(h)=o(1/sqrt(n))
ref o(h^2)=o(1/n)

$b = [1, 1/3]^T$

$$b = [1, 0]^T$$
$$b = [1, 1/3]^T$$


history for eps=1.0e−03
L^2 norm of error
#nodes
||e||_L2 − adapt−opt
||e_est||_L2 − adapt−opt
||e||_L2 − opt−adapt
||e_est||_L2 − opt−adapt
||e||_L2 − iso adapt
||e_est||_L2 − iso adapt
||e||_L2 − Shishkin
||e_est||_L2 − Shishkin
||e||_L2 − uniform
||e_est||_L2 − uniform
ref o(h)=o(1/sqrt(n))
ref o(h^2)=o(1/n)

Uniform behaviour?

- Unfortunately not:
  1. Constraints put bound on maximal aspect ratio.
  2. Ill-conditioning of Hessian matrices grows:

ε	$\kappa(H_{BFGS})$
1e−1	1.7e+03
1e−2	1.3e+07
1e−3	7.1e+13
1e−4	3.0e+17

Conclusions

- New approach to anisotropic refinement, based on minimisation of *a posteriori* estimates w.r.t. node positions.
- Aim not to achieve better asymptotic convergence rate, but to provide an initial mesh for which the constants for the asymptotic behaviour are better and/or for which the asymptotic behaviour sets in earlier. i.e. with less degrees of freedom.
- Demonstrated feasible for a set of model problems, for details, see

[R. Schneider.
Applications of the Discrete Adjoint Method in Computational Fluid Dynamics.
PhD Thesis. University of Leeds, 2006.
`http://www.comp.leeds.ac.uk/cgi-bin/sis/ext/rs_pub.cgi?cmd=listtheses#2006`

- Need to consider a wider class of test problems and alternative error estimators.