Convergence of the Adaptive Finite Element Method

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Chemnitz FEM Symposium

Outline

- Introduction: UFEM&AFEM, References
- AFEM: Algorithm for Energy Minimization
- Short History of Arguments to Prove Error Reduction
- Convergence Theory for Convex Minimisation
- Applications and Examples

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Introduction

UFEM

1. Universal algorithm (uniform for all data: RHSs etc.)
2. $\mathcal{T}_{\ell+1} := red(\mathcal{T}_\ell)$
3. Convergence from $\lim_{\ell\to\infty} \|h_\ell\|_{L^\infty(\Omega)} = 0$, but convergence can be arbitrarily bad (There is always some disaster RHS)

AFEM

1. Specialised "adapted" feedback algorithm (for ONE set of data)
2. $\mathcal{T}_{\ell+1}$ generated from $\mathcal{T}_\ell$, u_ℓ, all data, plus extra computations
3. Convergence is open since $\|h_\ell\|_{L^\infty(\Omega)} \not\to 0$ is not guaranteed a priori! Aims at error reduction or some other kind of convergence control

Books on a posteriori FE error control

- Eriksson-Estep-Hansbo-Johnson (1995)
- Verfürth (1996)
- Ainsworth-Oden (2000)
- Babuska-Strouboulis (2001)
- Bangerth-Rannacher (2003)
- Neittaanmäki-Repin (2004)

... and inside no theorem on AFEM convergence!

(Except early results by Babuska et al. in 1D.)

Papers on Convergence of AFEM in 2D

- *W.* **Dörfler***:* A convergent adaptive algorithm for Poisson's equation. *SIAM Journal on Numerical Analysis* **33** (1996) 1106–1124.
- *P.* **Morin**, *RH.* **Nochetto**, *and KG.* **Siebert***:* Local problems on stars: a posteriori error estimation, convergence, and performance. *Mathematics of Computation* **72** (2003) 1067–1097 AND Convergence of adaptive finite element methods. *SIAM Review* **44** (2003) 631–658.
- *A.* **Veeser** *(2002)*: *Convergent adaptive finite elements for the nonlinear Laplacian.* Numer. Math., **92**, 4, 743–770.
- *P.* **Binev**, *W.* **Dahmen**, *and R.* **DeVore**: *Adaptive Finite Element methods with Convergence Rates.* Num. Math., **97**(2) 219–268, (2004).
- *R.* **Stevenson**: Optimality of AFEM, preprint 2005.
- **C**, **C-Hoppe**, **Braess-C-Hoppe**

(AFEM)

Input: coarse mesh $\mathcal{T}_0$

```
For $\ell = 0, 1, 2, \dots$        SOLVE
                                   ESTIMATE
                                   MARK
                                   REFINE
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Output: Sequence of nested discrete spaces

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq \bigcup_{\ell=0}^{\infty} V_\ell \subseteq V = W_0^{1,p}(\Omega; \mathbb{R}^m)$$

with associated stress approximations $(\sigma_\ell)_{\ell \in \mathbb{N}_0}$.

Task: Design an (AFEM) with $\lim_{\ell \to \infty} \|\sigma - \sigma_\ell\|_{L^q} = 0$.

SOLVE

Given shape regular triangulation $\mathcal{T}_\ell$, define $V_\ell := \mathcal{P}_1(\mathcal{T}_\ell;\mathbb{R}^m) \cap V$, compute some minimizer with Newton-Raphson scheme for

$$E(v_\ell) := \int_\Omega W(Dv_\ell)\,dx - \int_\Omega f \cdot v_\ell\,dx \quad \text{for all } v_\ell \in V_\ell.$$

Compute discrete stress $\sigma_\ell := DW(Du_\ell) \in P_0(\mathcal{T}_\ell;\mathbb{R}^{m\times n})$.

$W : \mathbb{R}^{m\times n} \to \mathbb{R}$ with pth order growth causes $V = W_0^{1,p}(\Omega;\mathbb{R}^m)$, i.e., gradients in L^p and stresses in dual $L^{p'}$, $1/p + 1/p' = 1$.

Remark: For class of degenerate convex minimization problems with $p, p', q, r, s, t,\quad u$ and u_ℓ are non-unique, σ and σ_ℓ are unique s.t.

$$\|\sigma - \sigma_\ell\|_{L^{r/t}} \lesssim \min_{v_\ell \in V_\ell} \|u - v_\ell\|_{W^{1,p}}.$$

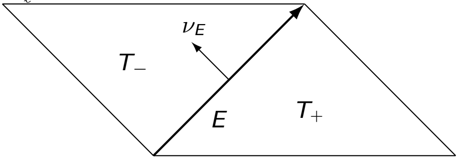
ESTIMATE

Given interior edge $E = \partial T_+ \cap \partial T_- \in \mathcal{E}_\ell$, compute $[\sigma_\ell] := \sigma_\ell|_{T_+} - \sigma_\ell|_{T_-}$,

$$\eta_E^{(\ell)} := \operatorname{diam}(E)^{1/p'} \|[\sigma_\ell] \cdot \nu_E\|_{L^{p'}(E)}$$

and set

$$\eta_\ell := \Big(\sum_{E \in \mathcal{E}_\ell} \eta_E^{(\ell)\,p'} \Big)^{1/p'} .$$



Theorem (C 2006). There holds

$$\|\sigma - \sigma_\ell\|^r_{L^{r/t}(\Omega)} \lesssim \eta_\ell + \operatorname{osc}_\ell$$

for oscillation

$$\operatorname{osc}_\ell^{p'} := \sum_{z \in \mathcal{K}_\ell} \operatorname{osc}(f, \omega_z)^{p'} .$$

For each inner node z with nodal basis function $\varphi_z \in V_\ell$, the patch reads $\omega_z := \{x \in \Omega : 0 < \varphi_z(x)\}$ and $\operatorname{osc}(f, \omega_z) := \operatorname{diam}(\omega_z)\|f - f_{\omega_z}\|_{L^{p'}(\omega_z)}$.

MARK

Bulk criterion with greedy algorithm.

Sort $\mathcal{E}_\ell = \{E_1, \dots, E_N\}$ in list $(E_1, \dots, E_N)$ s.t. $\eta_{E_1}^{(\ell)} \le \eta_{E_2}^{(\ell)} \le \dots \le \eta_{E_N}^{(\ell)}$ and set k maximal with

$$\Theta\, \eta_\ell^{p'} \le \eta_{E_k}^{(\ell)^{p'}} + \dots + \eta_{E_N}^{(\ell)^{p'}}.$$

Then, for fixed $0 < \Theta \le 1$,

$$\mathcal{M}_\ell := \{E_k, E_{k+1}, \dots, E_N\}$$

satisfies

$$\eta_\ell^{p'} \lesssim \sum_{E \in \mathcal{M}_\ell} \eta_E^{(\ell)^{p'}}.$$

Monitor $\mathrm{osc}_{\ell+1}$ to achieve at least $\lim_{\ell\to\infty} \mathrm{osc}_\ell = 0$.

REFINE

Use concept of reference edge (i.e. opposite side of newest local vertex) $E(T)$ for $T = \operatorname{conv}\{E, F, G\}$ and $E, F, G \in \mathcal{E}_\ell$.

Closure algorithm: Given shape-regular triangulation $\mathcal{T} := \mathcal{T}_\ell$ and subset $\mathcal{M} := \mathcal{M}_\ell$ of edges $\mathcal{E}_\ell$ in $\mathcal{T}_\ell$, repeat (a)-(b) until $\mathcal{T} = \emptyset$:

(a) Choose $T \in \mathcal{T}$ with $\big(\mathcal{E}(T) \cap \mathcal{M} \neq \emptyset \quad \text{AND} \quad E(T) \notin \mathcal{M}\big)$ and stop if there is no such T.

(b) While $\big(T \neq \emptyset \quad \text{AND} \quad E(T) \notin \mathcal{M}\big)$ do $(\mathcal{M} := \mathcal{M} \cup \{E(T)\}, \mathcal{T} := \mathcal{T} \setminus \{T\}, T := N(T))$.

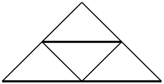
Then $\mathcal{M} := \operatorname{closure}(\mathcal{T}_\ell, \mathcal{M}_\ell)$.

Theorem (Bolte-C 2005^+). $\mathcal{M}$ is minimal with

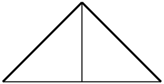
$$\mathcal{M}_\ell \subseteq \mathcal{M} \quad \text{and} \quad \forall E \in \mathcal{M}\, \forall T \in \mathcal{T}_\ell(E) \quad E(T) \in \mathcal{M}.$$

Red-Green-Blue Refinement with Reference Edge

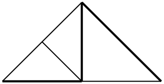
Reference edge $E(T)$ is bottom line and all $\mathcal{M}(T)$ are bisected via



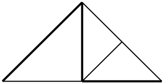
red(T)



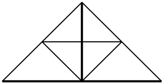
green(T)



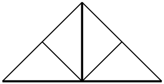
blue$_{left}$



blue$_{right}$



bisect5(T)



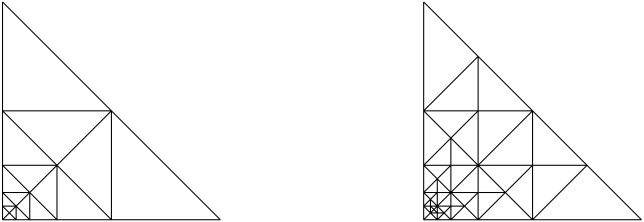
bisec3(T)

Inner Node Property achieved via bisect5(T) for at least on neighbouring triangle of each edge in $\mathcal{M}_\ell$.

Mesh-Refinement

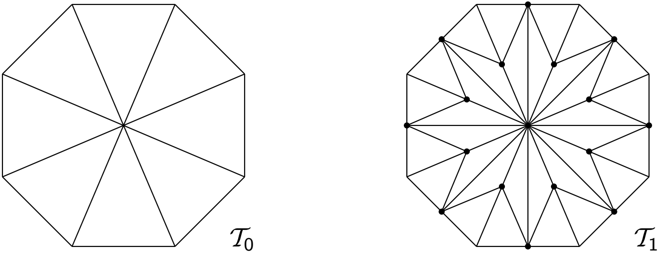
- Maintain shape regularity.
- On coarse $K \in \mathcal{T}_0$, $\mathcal{T}_\ell|_K$ is affine picture of reference triangle with solely right isosceles triangles.
- $\mathcal{T}_\ell$ allows for ℓ-independent H^1-stable L^2-projections (C 2004).
- Binev-Dahmen-DeVore 2004, Bolte-C 2005$^+$ show

$$\operatorname{card}(\mathcal{T}_\ell \setminus \mathcal{T}_0) \lesssim \sum_{j=1}^{\ell} \operatorname{card}(\mathcal{M}_j).$$



Error Reduction Without Inner Node Property?

A counterexample to error reduction for $W = \psi(|\cdot|)$ and nearest-vertex-bisection on regular polygon:



Theorem (Bartels-C 2006^+). Suppose $\overline{\Omega} = \bigcup \mathcal{T}_0 = \bigcup \mathcal{T}_1$ is regular polygon, decomposed in regular triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$. Then, $\sigma_0 = \sigma_1$.

History of Arguments to Prove Error Reduction

Linear elliptic PDE with energy norm $\|\!|\cdot|\!\|$ and solution $u \in W_0^{1,2}(\Omega;\mathbb{R}^m)$ (and stress field σ) respective discrete solution u_ℓ (and σ_ℓ):

(a) Reliability of error estimator [Rodriguez 94, C-Verfürth SINUM 99]

$$|\!|\!| u - u_\ell |\!|\!|^2 \lesssim \sum_{E\in\mathcal{E}_\ell} h_E \int_E |[Du_\ell]|^2 \, ds + \operatorname{osc}(f;\mathcal{T}_\ell)^2$$

(b) Bulk criterion [Dörfler SINUM 96]

$$|\!|\!| u - u_\ell |\!|\!|^2 \lesssim \sum_{E\in\mathcal{M}_\ell} h_E \int_E |[Du_\ell]|^2 \, ds + \operatorname{osc}(f;\mathcal{T}_\ell)^2$$

(c1) Discrete local efficiency [Dörfler SINUM 96] for $E \in \mathcal{M}_\ell$

$$h_E \int_E |[Du_\ell]|^2 \, ds \lesssim |\!|\!| u_{\ell+1} - u_\ell |\!|\!|(\omega_E)^2 + h_E^2 \|f\|_{L^2(\omega_E)}^2$$

Cont. Arguments to Prove Error Reduction

(c2) Refined discrete local efficiency [Nochetto et al., Veeser]

$$h_E \int_E |[Du_\ell]|^2 \, ds \lesssim |||u_{\ell+1} - u_\ell|||(\omega_E)^2 + h_E^2 \|f - f_E\|_{L^2(\omega_E)}^2$$

(d) Finite overlap in (b)&(c2) yields

$$|||u - u_\ell|||^2 \leq C_1 \, |||u_{\ell+1} - u_\ell|||^2 + C_2 \operatorname{osc}(f; \mathcal{T}_\ell)^2$$

(e) Galerkin orthogonality

$$|||u_{\ell+1} - u_\ell|||^2 = |||u - u_\ell|||^2 - |||u - u_{\ell+1}|||^2$$

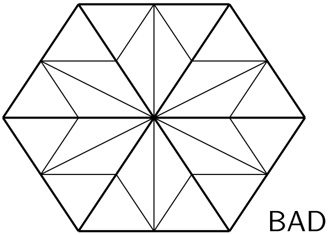
(f) Finish by rearranging

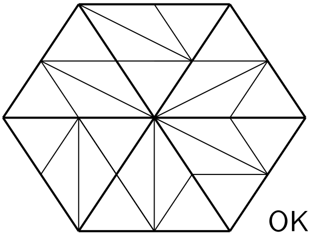
$$C_1 |||u - u_{\ell+1}|||^2 \leq (C_1 - 1) \, |||u - u_\ell|||^2 + C_2 \operatorname{osc}(f; \mathcal{T}_\ell)^2$$

and division by C_1

$$|||u - u_{\ell+1}|||^2 \leq (1 - C_1^{-1}) \, |||u - u_\ell|||^2 + C_1^{-1} C_2 \operatorname{osc}(f; \mathcal{T}_\ell)^2$$

Substitute of Inner Node Property





Suppose free node z in $\mathcal{T}_\ell$ with patch ω_z and edges $\mathcal{E}(z)$ in $\mathcal{T}_\ell|_{\omega_z}$. Assume all edges in $\mathcal{E}(z)$ are bisected but NOT all triangles in $\mathcal{T}_\ell(\omega_z)$ with bisec3 and reference edge on $\partial\omega_z$. Then

$$\sum_{E\in\mathcal{E}(z)} \eta_E^2 \lesssim |||u_{\ell+1}-u_\ell|||^2(\Omega_z) + \sum_{y\in\overline{\omega}_z\cap\mathcal{K}} \operatorname{osc}(f,\omega_y)^2$$

Adaptive Nonstandard FEM

Difficulty: Galerkin orthogonality fails for MFEM, but generalised Galerkin orthogonality holds for fluxes p and p_ℓ

$$\|p_{\ell+1} - p_\ell\|^2 \lesssim \|p - p_\ell\|^2 - \|p - p_{\ell+1}\|^2 + \operatorname{osc}(f;\mathcal{T}_\ell)^2$$

and eventually allows for error reduction property [C-Hoppe (2006)]

$$\|p - p_{\ell+1}\|^2 \le \varrho\,\|p - p_\ell\|^2 + C\operatorname{osc}(f;\mathcal{T}_\ell)^2.$$

Extra difficulty with nonconforming FEM: Balance 2 discrete equations!

C-Hoppe: Convergence analysis of an adaptive edge finite element method for the 2D eddy current equations, 2005, *J. Num. Math.*

C-Hoppe: Error reduction and convergence for an adaptive mixed finite element method, 2005, *to appear in Math. Comp.*

C-Hoppe: Convergence analysis of an adaptive nonconforming finite element method, 2006, *Numer. Math.* 103: 251-266

Convergence Rates by Rob Stevenson

1. Given $0 < \lambda < 1$ and $\delta := \lambda \|\!|u - u_\ell|\!\|$ there exists mesh $\mathcal{T}_0^\delta$ with FE solution u_0^δ and $\|\!|u - u_0^\delta|\!\| \leq \delta$ plus

$$\operatorname{card}(\mathcal{T}_0^\delta) \leq \delta^{-1/s}\, |u|_{\mathcal{A}^s}^{1/s}$$

2. Consider mesh $\mathcal{T}_\ell^\delta := \mathcal{T}_0^\delta \cup \mathcal{T}_\ell$ coarsest refinement of $\mathcal{T}_0^\delta$ and $\mathcal{T}_\ell$ with set of edges $\mathcal{E}_\ell^\delta$ with FE solution u_ℓ^δ
3. Lemma A: $\operatorname{card}(\mathcal{E}_\ell \setminus \mathcal{E}_\ell^\delta) \lesssim \operatorname{card}(\mathcal{T}_0^\delta)$
4. Lemma B: $\|\!|u_\ell^\delta - u_\ell|\!\| \lesssim \eta_\ell(\mathcal{E}_\ell \setminus \mathcal{E}_\ell^\delta) + \operatorname{osc}(f, \mathcal{T}_\ell)$
5. Lemma C: For $\mu_\ell := \operatorname{osc}(f, \mathcal{T}_\ell)/\|\!|u - u_\ell|\!\|$ bdd and Θ sufficiently small, $\mathcal{E}_\ell \setminus \mathcal{E}_\ell^\delta$ satisfies bulk criterion with Θ, whence

$$\operatorname{card}(\mathcal{M}_\ell) \leq \operatorname{card}(\mathcal{E}_\ell \setminus \mathcal{E}_\ell^\delta)$$

6. Refinement with $\operatorname{card}(\mathcal{T}_L) \lesssim \sum_{\ell=0}^{L} \operatorname{card}(\mathcal{M}_\ell)$

Convergence Theory for Convex Minimisation

Given strongly convex $W : \mathbb{R}^{m\times n} \to \mathbb{R}$ with quadratic growth and DW Lipschitz and RHS $f \in L^2(\Omega;\mathbb{R}^m)$, minimise

$$E(v) := \int_\Omega W(Dv(x))\,dx - \int_\Omega f\cdot v\,dx \quad \text{for } v \in V := W_0^{1,2}(\Omega;\mathbb{R}^m).$$

Theorem (C 2006^+). (a)-(c) are equivalent:
(a) energy-reduction property

$$\delta_{\ell+1} \le \rho\delta_\ell + \text{h.o.t.} \quad \text{for some } \rho < 1;$$

(b) discrete residual control

$$\kappa\|R_\ell\|_{V^*} \le \|R_\ell\|_{V^*_{\ell+1}} + \text{h.o.t.} \quad \text{for some } \kappa > 0;$$

(c) reliability of an hierarchical error estimator

$$\|R_\ell\|_{V^*} \le C\eta_H + \text{h.o.t.} \quad \text{for some } C > 0.$$

Notation

There exist unique exact $u \in V$ resp. $u_\ell \in V_\ell$ discrete minimizer with error

$$\|u - u_\ell\|_V^2 \approx \delta_\ell := E(u_\ell) - E(u) \quad \text{and} \quad R_\ell := -DE(u_\ell).$$

Let $(\varphi_m : m \in \mathcal{M})$ be basis of complement $W_{\ell+1}$ of V_ℓ in

$$V_{\ell+1} = V_\ell \oplus W_{\ell+1}, \quad W_{\ell+1} := \operatorname{span}(\varphi_m : m \in \mathcal{M}).$$

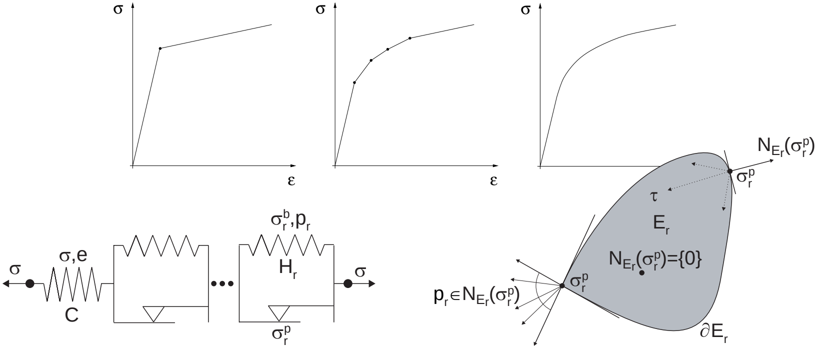
Define hierarchical estimator

$$\begin{aligned} \eta_H &= \Big(\sum_{m\in\mathcal{M}} \eta_m^2\Big)^{1/2} && \text{with} \\ \eta_m &= R_\ell(\varphi_m)/\|\varphi_m\|_V && \text{for each } m \in \mathcal{M}. \end{aligned}$$

$\mathcal{M}$ are marked edges and φ_m is (modified) nodal basis function of new node mid(E).

Multiyield Plasticity

Generalisation of standard infinite elastoplasticity with different hardening mechanisms from C-Orlando-Valdmann (2005)

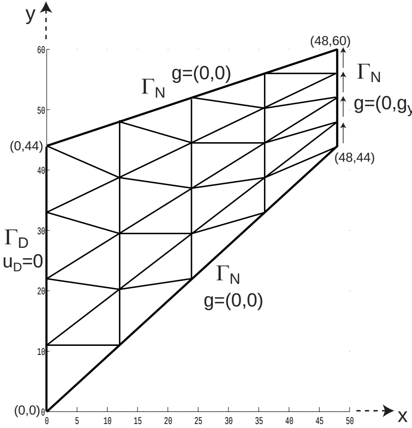


Rheological model for multisurface plasticity with hardening.

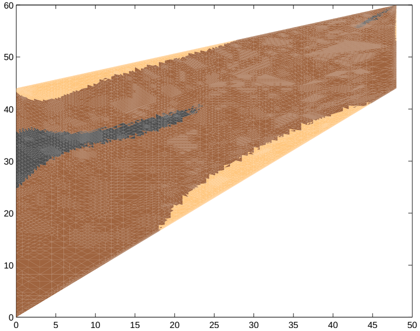
Normal cones in evolution law on set of admissible stresses.

- AFEM with linear convergence of energy and stress errors

Multiyield Plasticity

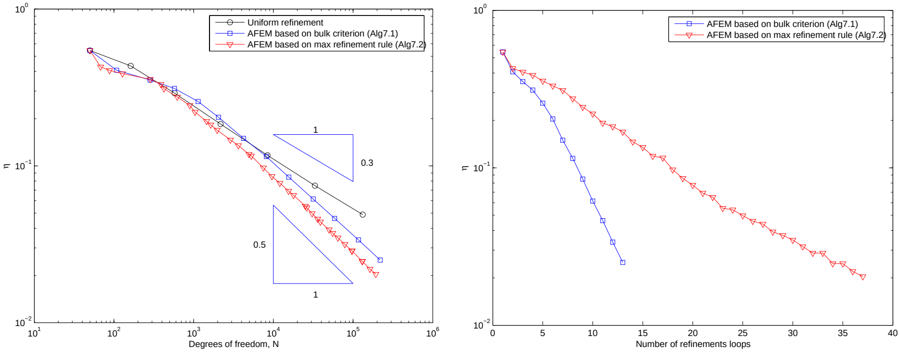


Cooks membrane with $\mathcal{T}_0$.



$\mathcal{T}_9$ with 3 different phases.

Convergence History for Multiyield Plasticity



Convergence for (degenerate) Convex Minimization

Given $W : \mathbb{R}^{m\times n} \to \mathbb{R}$ and $f \in L^p(\Omega;\mathbb{R}^m)$, minimize

$$E(v) := \int_\Omega W(Dv(x))\,dx - \int_\Omega f\cdot v\,dx \quad \text{for } v \in V := W_0^{1,p}(\Omega;\mathbb{R}^m).$$

Class of degenerate convex energy densities with p, r, s, t for growth conditions

$$|F|^p - 1 \lesssim W(F) \lesssim |F|^p + 1 \quad \text{for all } F \in \mathbb{R}^{m\times n}$$

and for convexity control

$$\begin{aligned}&(1+|A|^s+|B|^s)^{-1}|DW(A)-DW(B)|^r\\ &\quad\lesssim W(B)-W(A)-DW(A;B-A) \quad \text{for all } A,B\in\mathbb{R}^{m\times n}.\end{aligned}$$

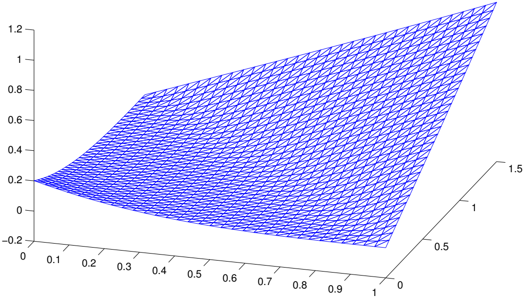
Theorem (C-2006): $\quad \Big(\|\sigma-\sigma_\ell\|_{L^{r/t}}\Big)_{\ell=0,1,2,\dots} \in \ell^r$

Rexaled 2-Well Benchmark Example

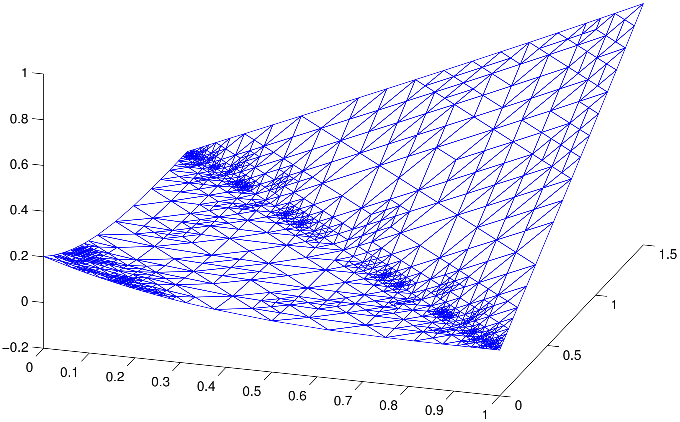
With lower convex envelope to model macroscopic well-posed variables (e.g. stress and deformation)

$$W(F) := \max\{0, |F|^2 - 1\}^2 + 4(|F|^2 - [F_2 \cdot F]^2)$$

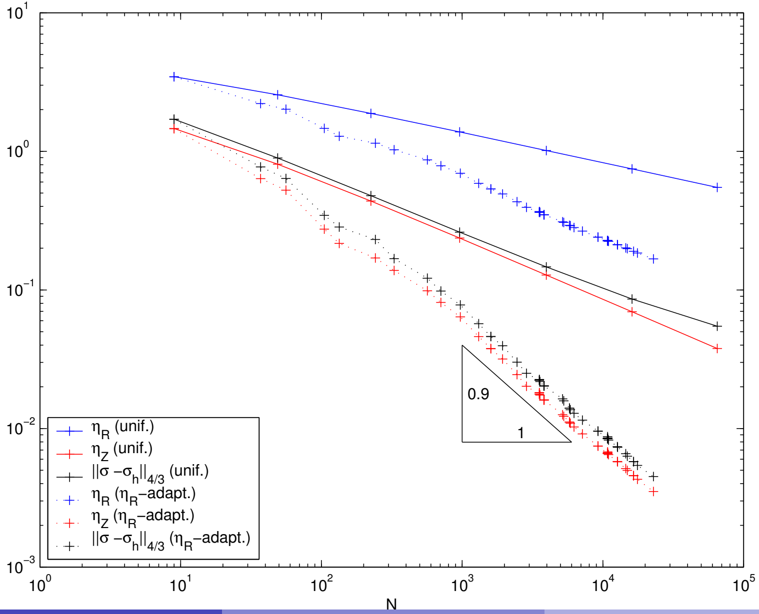
C-Plechac (1997), C-Jochimsen (2003)



Rexaled 2-Well Benchmark Example



AFEM with Reliability-Efficiency-Gap



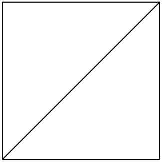
Optimal Design Task

Topology optimisation leads to

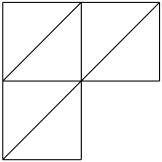
$$\min_{v \in H_0^1(\Omega)} \int_\Omega \psi\big(|Dv|\big)\,dx - \int_\Omega v\,dx$$

for $\mu_1 = 1$, $\mu_2 = 2$, $2t_1 = t_2$, and

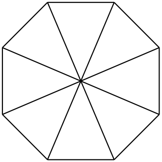
$$\psi'(t) := \begin{cases} \mu_2 t & \text{for } 0 \le t \le t_1, \\ t_1\mu_2 = t_2\mu_1 & \text{for } t_1 \le t \le t_2, \\ \mu_1 t & \text{for } t_2 \le t. \end{cases}$$



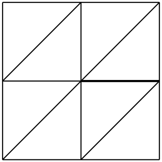
$\lambda = 0.0084$
Square



$\lambda = 0.0145$
L-Shape



$\lambda = 0.0284$
Stop Sign



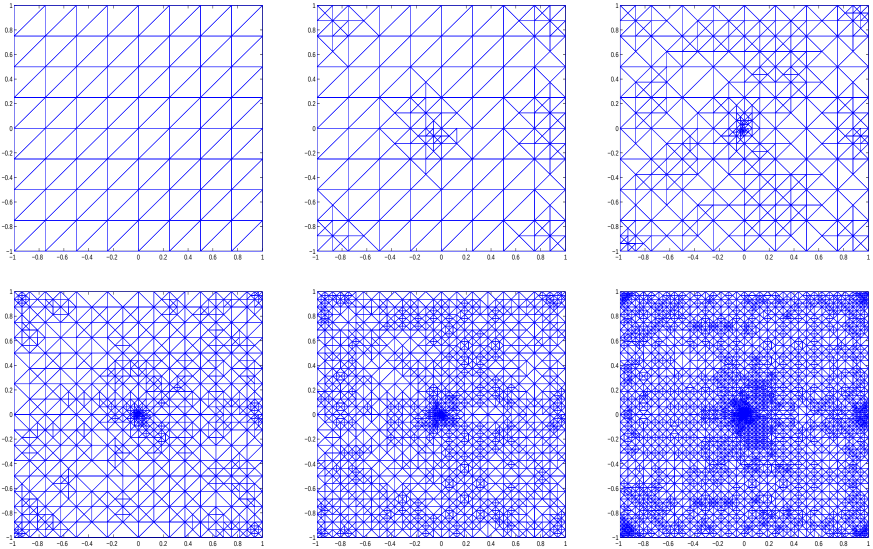
$\lambda = 0.0163$
Slit

Results from Bartels-C 2006

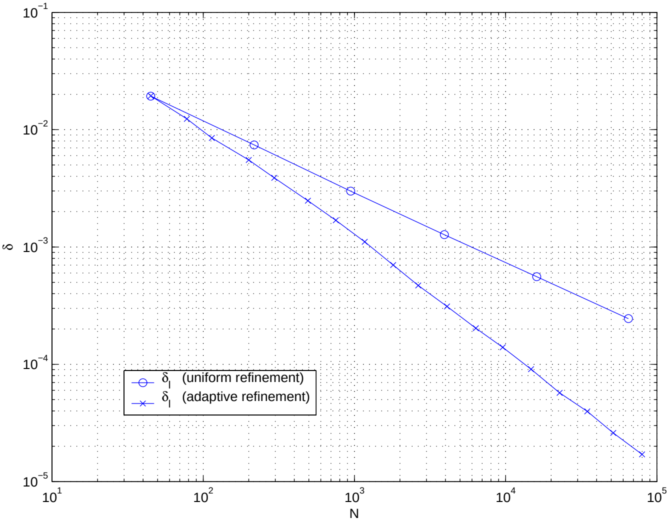
- No oscillations
- $\delta_\ell := E(u_\ell)$ - E(u) satisfies

$$\kappa 2^{-4} \|\sigma - \sigma_\ell\|_{L^2}^4 + \delta_{\ell+1} \leq (1 - \kappa\delta_\ell)\delta_\ell$$

  for $\ell = 0, 1, 2, \ldots$ and fixed $0 < \kappa \leq 1$.
- R-linear convergence in pre-asymptotic rate
- $\left(\|\sigma - \sigma_\ell\|_{L^2}\right)_{\ell=0,1,2,\ldots} \in \ell^4$ and $(\delta_\ell)_{\ell=0,1,2,\ldots} \in \ell^2$



Adaptively generated $\mathcal{T}_0, \mathcal{T}_2, \ldots, \mathcal{T}_{10}$ of $\Omega_4 = (-1,1)^2 \setminus [0,1) \times \{0\}$.



Decay of energy difference δ_ℓ on the sequence of adaptively generated triangulations $\mathcal{T}_0, \mathcal{T}_1, \dots \mathcal{T}_{17}$ of $\Omega_4 = (-1,1)^2 \setminus [0,1) \times \{0\}$.

Papers at website http://www.math.hu-berlin.de/˜cc/

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