An interlacing property of the signless Laplacian of threshold graphs

Christoph Helmberg* Guilherme Porto† Guilherme Torres‡
Vilmar Trevisan§

August 24, 2023

Abstract

We show that for threshold graphs, the eigenvalues of the signless Laplacian matrix interlace with the degrees of the vertices. As an application, we show that the signless Brouwer conjecture holds for threshold graphs, i.e., for threshold graphs the sum of the k largest eigenvalues is bounded by the number of edges plus $k+1$ choose 2.
Keywords: threshold graphs, signless Laplacian spectrum, Brouwer conjecture
MSC 2020: 05C50, 15A18

1 Introduction and main result

Given an undirected graph $G = (N, E)$ on node set $N = [n] := \{1, \dots, n\}$ for some $n \in \mathbb{N}$ and edge set $E \subseteq \binom{N}{2} := \{\{i, j\} : i, j \in N, i \neq j\}$, the signless Laplacian is the symmetric $n \times n$ matrix $Q(G) = \sum_{\{i,j\} \in E} (e_i + e_j)(e_i + e_j)^\top$, where e_i denotes the i-th column of the $n \times n$ identity matrix I_n. $Q(G)$ may also be written as $Q(G) = D + A$, where A is the adjacency matrix and D the diagonal degree matrix. Thus, $Q(G)$ is positive semidefinite and its eigenvalues may be ordered as

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Spectral properties of the signless Laplacian $Q(G)$ of a graph G were collected and developed in a series of papers by Cvetković and Simić [5, 7, 6]. Nevertheless, it seems that the spectrum of this matrix is far less understood than that of the combinatorial Laplacian $L(G) = D - A$, for example.

In this note, we show an interlacing property of the eigenvalues of $Q(G)$ when G is a threshold graph. Essentially, our result says that the eigenvalues of a threshold graph interlace with the degrees of the vertices.

In order to more precisely state the result, we first review some facts about threshold graphs. This class of graphs has been discovered independently by several authors in many distinct contexts since the 1970's. They are an important class of graphs because of their

*Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany. helmberg@mathematik.tu-chemnitz.de

†Instituto Federal de Educação, Ciência e Tecnologia Farroupilha, Campus São Borja. guilherme.porto@iffar.edu.br

‡In Memory: During the preparation of this note, Guilherme Torres tragically passed away.

§Instituto de Matemática, Universidade Federal do Rio Grande do Sul, Porto Alegre. Brazil. trevisan@mat.ufrgs.br

numerous applications in diverse areas which include computer science, social sciences and psychology. See, for example, [15] for a more detailed account. A threshold graph can be characterized in many ways. We are going to view threshold graphs as given through an iterative process steered by a binary string which starts with an isolated vertex (for the initial digit 0 or 1), and where, at each step, either a new isolated vertex is added (digit 0), or a dominating vertex (adjacent to all previous vertices, digit 1) is added.

Building on the notation in [10], this construction is encoded in a binary sequence $b_1^{q_1} \dots b_r^{q_r}$ of length $n = \sum_{k=1}^r q_k$ with $b_k \in \{0,1\}$ and $q_k \in \mathbb{N}$ giving this digit's repetitions in the sequence for $k = 1, \dots, r$. For convenience, let $n_k = \sum_{h=1}^k q_k$ for $k = 0, \dots, r$, so $n = n_r$ and $n_0 = 0$. Then the corresponding threshold graph $G = (N = [n], E)$ has edge set E so that for $1 \le i < j \le n$

$$ij \in E \quad \Leftrightarrow \quad \exists k \in \{1, \dots, r\} \ (n_{k-1} < j \le n_k \wedge b_k = 1). \tag{1}$$

All nodes of block k have the same degree. Their values are

$$p_k = \begin{cases} \sum_{h=k}^r b_h q_h & \text{for } b_k = 0, \\ \sum_{h=1}^k q_h - 1 + \sum_{h=k+1}^r b_h q_h & \text{for } b_k = 1, \end{cases} \qquad k = 1, \dots, r. \tag{2}$$

We may and will assume the sequence of b_k to be alternating, *i. e.*, $b_k + b_{k+1} = 1$ for $k = 1, \dots, r-1$. Because G is independent of the choice of the first digit, there is no loss in generality in requiring $q_1 \ge 2$. It is known (see, for example, [4, 10]) that $p_k - b_k$ is an eigenvalue of $Q(G)$ of multiplicity at least $q_k - 1$, for $k = 1, \dots, r$. As a direct consequence of the assumptions, the degrees and corresponding eigenvalues satisfy

$$\begin{gathered} p_{r-b_r} < p_{r-b_r-2} < \cdots < p_{1+b_1+2} < p_{1+b_1} \le p_{2-b_1} - 1, \\ p_{2-b_1} - 1 < p_{4-b_1} - 1 < p_{6-b_1} - 1 < \cdots < p_{r-(1-b_r)} - 1. \end{gathered} \tag{3}$$

The results of [10] imply that the remaining r eigenvalues are those of the following *condensed signless Laplacian* $r \times r$ matrix

$$C(G) = \begin{bmatrix} p_1 + b_1(q_1 - 1) & b_2\sqrt{q_1 q_2} & b_3\sqrt{q_1 q_3} & \cdots & b_r\sqrt{q_1 q_r} \\ b_2\sqrt{q_1 q_2} & p_2 + b_2(q_2 - 1) & b_3\sqrt{q_2 q_3} & \cdots & b_r\sqrt{q_2 q_r} \\ b_3\sqrt{q_1 q_3} & b_3\sqrt{q_2 q_3} & p_3 + b_3(q_3 - 1) & \cdots & b_r\sqrt{q_3 q_r} \\ \vdots & & & & \vdots \\ b_r\sqrt{q_1 q_r} & b_r\sqrt{q_2 q_r} & b_r\sqrt{q_3 q_r} & \cdots & p_r + b_r(q_r - 1) \end{bmatrix}.$$

In Theorem 8 we establish that these remaining signless Laplacian eigenvalues of the threshold graph G interlace with $p_k - b_k$. More precisely, for $C = C(G)$ we prove the inequalities

$$\lambda_1(C) \le p_{r-b_r} \le \lambda_2(C) \le p_{r-b_r-2} \le \cdots \le \lambda_{\frac{r-b_r-b_1+1}{2}}(C) \le p_{1+b_1}$$

and

$$p_{2-b_1} - 1 \le \lambda_{\frac{r-b_r-b_1+1}{2}+1}(C) \le p_{4-b_1} - 1 \le \lambda_{\frac{r-b_r-b_1+1}{2}+2}(C) \le \cdots \le p_{r-(1-b_r)} - 1 \le \lambda_r(C).$$

This interlacing property gives rise to the main result Theorem 13 which sheds some light on the distribution of the signless Laplacian spectrum of graphs. The trivial upper bound $2n - 1$ on the largest signless Laplacian eigenvalue of a graph G — called the signless spectral radius of G — is frequently an obstacle for obtaining meaningful upper bounds involving this spectral parameter. For threshold graphs, however, this interlacing property may be used to obtain tighter bounds for the sum of the largest eigenvalues. Indeed, together with a subtle sharpening for a special case, the bounds suffice to establish the signless Brouwer conjecture [1] for threshold graphs in Theorem 19.

The remainder of the paper is organized as follows. The next section is devoted to prove our main result. In Section 3 we interpret the result and give an estimate of the eigenvalues growth with the evolution of a binary sequence defining a threshold graph. In Section 4, as an application of the interlacing property for threshold graphs, we prove that the signless Brouwer conjecture holds for threshold graphs.

2 Proof of the main result

We will make heavy use of the following direct consequence of the Courant-Fischer Theorem.

Theorem 1. *Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$ and $h \in \{1,\dots,n\}$, for any choice of vectors $v_1,\dots,v_{h-1} \in \mathbb{R}^n$ and $u_1,\dots,u_{n-h} \in \mathbb{R}^n$ there holds*

$$\min_{\substack{x\perp v_1,\dots,v_{h-1}\\ \|x\|=1}} x^\top Ax \quad\leq\quad \lambda_h(A) \quad\leq\quad \max_{\substack{x\perp u_1,\dots,u_{n-h}\\ \|x\|=1}} x^\top Ax.$$

If $h-1$ or $n-h$ is zero, x ranges over $\mathbb{R}^n$ in the respective expression.

We recall and use a number of further consequences of the Courant-Fischer Theorem.

Theorem 2 (Weyl Inequalities – [14],Theorem 4.3.1). *Let M and N be two $n\times n$ real symmetric matrices. Then*

$$\lambda_i(M+N) \leq \lambda_{i+j}(M) + \lambda_{n-j}(N),$$

for $1 \leq i \leq n$ and $0 \leq j \leq n-i$, and

$$\lambda_i(M+N) \geq \lambda_{i-j+1}(M) + \lambda_j(N)$$

for $1 \leq i \leq n$ and $1 \leq j \leq i$.

Theorem 3 (Cauchy Interlacing – [14],Theorem 4.3.28). *Let A be a real, symmetric $n\times n$ matrix and let B be a principal submatrix of A with order $m \times m$. Then, for $k = m,\cdots,1$,*

$$\lambda_{k+n-m}(A) \geq \lambda_k(B) \geq \lambda_k(A).$$

Theorem 4 (Interlacing for a rank-one perturbation – [14], Corollary 4.3.9). *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let $z \in \mathbb{R}^n$ be a vector. Then the eigenvalues of A and $A+zz^\top$ satisfy*

$$\begin{gathered}\lambda_h(A) \leq \lambda_h(A+zz^\top) \leq \lambda_{h+1}(A), \quad h=1,2,\dots,n-1,\\ \lambda_n(A) \leq \lambda_n(A+zz^\top).\end{gathered} \tag{4}$$

In order to relate the eigenvalues of the signless Laplacian matrix of a general graph G to those of its complement, note that the signless Laplacian of the complete graph is $Q(K_n) = \sum_{\{i,j\}\in\binom{[n]}{2}}(e_i+e_j)(e_i+e_j)^\top = (n-2)I_n + \mathbf{1}_n\mathbf{1}_n^\top$, where $\mathbf{1}$ denotes the vector of all ones of given or appropriate dimension. With this the signless Laplacian of the complement graph $\bar{G} = (N, \binom{N}{2}\setminus E)$ of G computes to $Q(\bar{G}) = Q(K_n) - Q(G) = (n-2)I_n + \mathbf{1}_n\mathbf{1}_n^\top - Q(G)$.

Lemma 5. *Let $Q = Q(G)$ be the signless Laplacian of a graph G on n nodes and let $\bar{Q} = Q(\bar{G})$ represent its complement, then*

$$\max\{n-2-\lambda_n(\bar{Q}),0\} \le \lambda_1(Q) \le n-2-\lambda_{n-1}(\bar{Q}) \le \lambda_2(Q) \le \cdots$$
$$\cdots \le \lambda_{n-1}(Q) \le n-2-\lambda_1(\bar{Q}) \le \min\{n-2,\lambda_n(Q)\}.$$

Proof. The signless Laplacian is positive semidefinite, so $\lambda_1(Q) \ge 0$ and $\lambda_1(\bar{Q}) \ge 0$. By $Q = \mathbf{1}\mathbf{1}^\top + (n-2)I_n - \bar{Q}$ the statement follows from Theorem 4. □

If now G is a threshold graph, we notice that Equ. (1) establishes that the complement graph $\bar{G}$ of G is defined by the binary sequence $(1-b_1)^{q_1}\dots(1-b_r)^{q_r}$. This sequence is again alternating with $q_1 \ge 2$. The degrees $\bar{p}_i$ of the blocks of the complement graph satisfy

$$\bar{p}_k = n-1-p_k \quad \text{for } k = 1,\dots,r. \tag{5}$$

Up to the specification of the eigenvectors the following result has already been observed in [10].

Lemma 6. *Let $b_1^{q_1}\dots b_r^{q_r}$ be a binary sequence specifying a threshold graph G on $n = \sum_{k=1}^r q_k$ nodes with block degrees p_k and put $n_0 = 0$, $n_k = \sum_{h=1}^k q_h$ for $k = 1,\dots,r$. For $k \in \{1,\dots,r\}$ its signless Laplacian $Q(G)$ has an eigenvalue $p_k - b_k$ of multiplicity at least $q_k - 1$ with an associated orthogonal basis of eigenvectors*

$$v_1^{(k)} = \begin{bmatrix} \mathbf{0}_{n_{k-1}} \\ 1 \\ -1 \\ \mathbf{0}_{n-n_{k-1}-2} \end{bmatrix}, v_2^{(k)} = \begin{bmatrix} \mathbf{0}_{n_{k-1}} \\ \frac{1}{2}\mathbf{1}_2 \\ -1 \\ \mathbf{0}_{n-n_{k-1}-3} \end{bmatrix}, \dots, v_{q_k-1}^{(k)} = \begin{bmatrix} \mathbf{0}_{n_{k-1}} \\ \frac{1}{q_k-1}\mathbf{1}_{q_k-1} \\ -1 \\ \mathbf{0}_{n-n_{k-1}-q_k} \end{bmatrix}.$$

Proof. Direct computation shows $(v_i^{(k)})^\top v_j^{(k)} = 0$ for $1 \le i < j < q_k$. Let v be any of these vectors. By (1) columns $n_{k-1}+1$ to n_k of row $i \le n_{k-1}$ or row $i > n_k$ of Q have the same value b_k, so $(Qv)_i = 0$ for these i. It remains to consider the principal submatrix Q' on indices $i = n_{k-1}+1,\dots,n_k$. The case $b_k = 0$ yields $Q' = p_k I_k$, so $Qv = p_k v$; the case $b_k = 1$ yields $Q' = \mathbf{1}_{q_k}\mathbf{1}_{q_k}^\top + (p_k-1)I_{q_k}$, so $Qv = (p_k-1)v$. □

Lemma 6 describes $n-r$ eigenvalues of Q via its eigenvectors. It is proved in [10] that the remaining r eigenvalues are those of $C = C(G)$. By (5) the condensed signless Laplacian $\bar{C} = C(\bar{G})$ of the complement graph may be computed via

$$\bar{C} = (n-2)I_r + \hat{q}\hat{q}^\top - C \quad \text{with } \hat{q} = (\sqrt{q_1},\dots,\sqrt{q_r})^\top.$$

So the eigenvalues of C and $\bar{C}$ satisfy the same interlacing property as those of Q and $\bar{Q}$.

Lemma 7. *Let C be the condensed signless Laplacian of a threshold graph G specified by a binary sequence $b_1^{q_1}\dots b_r^{q_r}$ on $n = \sum_{k=1}^r q_k$ nodes and let $\bar{C}$ represent its corresponding complement, then*

$$\max\{n-2-\lambda_r(\bar{C}),0\} \le \lambda_1(C) \le n-2-\lambda_{r-1}(\bar{C}) \le \lambda_2(C) \le \cdots$$
$$\cdots \le \lambda_{r-1}(C) \le n-2-\lambda_1(\bar{C}) \le \min\{n-2,\lambda_r(C)\}.$$

Proof. The matrices C and $\bar{C}$ are positive semidefinite, thus $\lambda_1(C) \ge 0$ and $\lambda_1(\bar{C}) \ge 0$. By $C = \hat{q}\hat{q}^\top + (n-2)I_r - \bar{C}$ the statement again follows from Theorem 4. □

We are going to prove the following bounds on the eigenvalues of C.

Theorem 8. *Let $b_1^{q_1} \dots b_r^{q_r}$ be an alternating binary sequence with $q_1 \geq 2$ specifying a threshold graph G on $n = \sum_{k=1}^r q_k$ nodes with block degrees p_k and C its condensed signless Laplacian.*

$$(0 \leq) \quad \lambda_1(C) \leq p_{r-b_r} \leq \lambda_2(C) \leq p_{r-b_r-2} \leq \cdots \leq \lambda_{\frac{r-b_r-b_1+1}{2}}(C) \leq p_{1+b_1}$$

and

$$p_{2-b_1} - 1 \leq \lambda_{\frac{r-b_r-b_1+1}{2}+1}(C) \leq p_{4-b_1} - 1 \leq \lambda_{\frac{r-b_r-b_1+1}{2}+2}(C) \leq \cdots \leq p_{r-(1-b_r)} - 1 \leq \lambda_r(C).$$

We break the proof into four intermediate steps.

Observation 9. *For $i = 1, \dots, \frac{r-b_r-b_1+1}{2}$ there holds $\lambda_i(C) \leq p_{r-b_r-2i+2}$.*

Proof. Choosing the $r-i$ vectors $e_1, e_2, \dots, e_{r-b_r-2i+1}, e_{r-b_r-2i+3}, e_{r-b_r-2i+5}, \dots, e_{r-1+b_r}$ for the right hand side u vectors in Theorem 1 restricts x to the coordinates e_{r-b_r-2i+2}, e_{r-b_r-2i+4}, $\dots$, e_{r-b_r} (all these indices k satisfy $b_k = 0$), so the bound is obtained by the maximum eigenvalue of the submatrix

$$\begin{bmatrix} p_{r-b_r-2i+2} & 0 & \dots & 0 \\ 0 & p_{r-b_r-2i+4} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & p_{r-b_r} \end{bmatrix}.$$

The statement now follows from (3) and Theorem 1. □

Observation 10. *For $i = 1, \dots, \frac{r-1+b_r+b_1}{2}$ there holds $p_{r+b_r+1-2i} - 1 \leq \lambda_{r-i+1}(C)$.*

Proof. Choosing the $r-i$ vectors e_1, e_2, $\dots$, e_{r+b_r-2i}, e_{r+b_r-2i+2}, e_{r+b_r-2i+4}, $\dots$, e_{r-b_r} for the left hand side v vectors in Theorem 1 restricts x to the coordinates e_{r+b_r-2i+1}, e_{r+b_r-2i+3}, $\dots$, e_{r+br-1} (all these indices k satisfy $b_k = 1$), so the bound is obtained by the minimum eigenvalue of the submatrix

$$\bar{q}\bar{q}^\top + \begin{bmatrix} p_{r+b_r-2i+1} - 1 & 0 & \dots & 0 \\ 0 & p_{r+b_r-2i+3} - 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & p_{r+b_r-1} - 1 \end{bmatrix} \text{ with } \bar{q} = \begin{bmatrix} \sqrt{q_{r+b_r-2i+1}} \\ \sqrt{q_{r+b_r-2i+3}} \\ \vdots \\ \sqrt{q_{r+b_r-1}} \end{bmatrix}.$$

Because $\bar{q}\bar{q}^\top$ is positive semidefinite, the smallest diagonal element is certainly a lower bound, thus the statement again follows from (3) and Theorem 1. □

Observation 11. *For $i = 2, \dots, \frac{r-1+b_r+b_1}{2}$ there holds $\lambda_{r-i+1}(C) \leq p_{r+b_r+3-2i} - 1$.*

Proof. Without loss of generality it suffices to consider the case $b_r = 1$. Indeed, in the case $b_r = 0$ we may split off the eigenvector e_r to eigenvalue $p_r = 0$ of C and work with the principal submatrix C' on indices $i = 1, \dots, r-1$. This C' is the condensed matrix of the threshold graph corresponding to the sequence $b_1^{q_1} \dots b_{r-1}^{q_{r-1}}$ and its eigenvalues coincide with the remaining ones of C. Thus let $b_r = 1$ and $p_r = n - 1$ in the following.

For $i = 2$ the statement follows directly from Lemma 7 by $\lambda_{r-1}(C) \leq n - 2$.

For $i > 2$ first choose the $i-2$ vectors $u_1 = e_r, u_2 = e_{r-2}, \ldots, u_{i-2} = e_{r-2(i-3)}$. Orthogonality with respect to these vectors already restricts x to the submatrix

$$\begin{bmatrix} \hat{C} + \sum_{j=1}^{i-2} q_{r+2-2j} I_{r+4-2i} & 0 & \cdots & 0 \\ 0 & p_{r+5-2i} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_{r-1} \end{bmatrix},$$

where $\hat{C}$ is the condensed signless Laplacian of the threshold graph $b_1^{q_1} \ldots 1^{q_{r+4-2i}}$. By the same argument as above Lemma 7 yields $\lambda_{r+4-2i-1}(\hat{C}) \leq \sum_{j=1}^{r+4-2i} q_j - 2$ and

$$\lambda_{r+4-2i-1}(\hat{C} + \sum_{j=1}^{i-2} q_{r+2-2j} I_{r+4-2i}) \leq \sum_{j=1}^{r+4-2i} q_j - 2 + \sum_{j=1}^{i-2} q_{r+2-2j} \overset{(2)}{=} p_{r+4-2i} - 1.$$

Now choose u_{i-1} to hold the Perron vector to the largest eigenvalue of $\hat{C}$ on components 1 to $r+4-2i$ and zero otherwise, then by (3) and Theorem 1 the value $p_{r+4-2i} - 1$ is an upper bound on $\lambda_{r-i+1}(C)$. □

Observation 12. *For $i = 2, \ldots, \frac{r-b_r-b_1+1}{2}$ there holds $p_{r-b_r-2i+4} \leq \lambda_i(C)$.*

Proof. Consider the condensed signless Laplacian $\bar{C}$ of the complement graph $(1-b_i)^{q_1} \ldots (1-b_r)^{q_r}$ with degrees $\bar{p}_i$ satisfying (5). For $i = 2, \ldots, \frac{r-1+(1-b_r)+(1-b_1)}{2}$ Observation 11 proves $\lambda_{r-i+1}(\bar{C}) \leq \bar{p}_{r+(1-b_r)+3-2i} - 1$. Therefore Lemma 7 and (5) establish $\lambda_i(C) \geq n - 2 - \lambda_{r-i+1}(\bar{C}) \geq n - 2 - (n - 1 - p_{r-b_r+4-2i} - 1) = p_{r-b_r+4-2i}$. □

Observations 9–12 prove Theorem 8. Together with Lemma 6 the latter gives rise to the main result, which is more conveniently stated in terms of the degree sequence $d_n(G) \geq \cdots \geq d_1(G)$ of the graph. Furthermore, if $b_1 = 1$ and the number $\bar{k}$ of ones in the binary sequence exceeds q_1, it will be possible to improve the bound on the larger "central" eigenvalue $\lambda_{\frac{r-b_r-b_1+1}{2}+1}(C)$ by one.

Theorem 13. *Let G be a threshold graph with n vertices represented by the binary sequence $b_1^{q_1} \ldots b_r^{q_r}$. Denote the degree sequence of G by $d_n(G) \geq \cdots \geq d_1(G)$ and the number of ones in the binary sequence of G by $\bar{k} = \sum_{i=1}^r b_i q_i$. Then, the eigenvalues of G satisfy*

$$\lambda_n \geq d_n - 1 \geq \lambda_{n-1} \geq \cdots \geq \lambda_{n+1-\bar{k}} \geq d_{n+1-\bar{k}} - 1 \geq d_{n-\bar{k}} \geq \lambda_{n-\bar{k}} \geq \cdots \geq d_1 \geq \lambda_1 \geq 0.$$

Furthermore, if $b_1 = 1$ and $\bar{k} > q_1$, then $\lambda_{n-\bar{k}+q_1} \geq d_{n-\bar{k}+q_1}$.

Proof. Note that $b_r = 0$ results in appending q_r isolated nodes or, equivalently in appending zero rows and columns which does not influence the other eigenvalues and eigenvectors. So for simplifying notation we assume without loss of generality $b_r = 0$. Then according to (2) and (3) the degrees satisfy

$$d_1 = \cdots = d_{q_r} = p_r < d_{q_r+1} = \cdots = d_{q_r+q_{r-2}} = p_{r-2} < \cdots \leq d_{n-\bar{k}} = p_{1+b_1} \leq p_{2-b_1} - 1,$$
$$p_{2-b_1} = d_{n-\bar{k}+1} = \cdots = d_{n-\bar{k}+q_{2-b_1}} < p_{4-b_1} = \cdots < p_{r-1} = d_{n-q_{r-1}+1} = \cdots = d_n.$$

With this Theorem 8 and Lemma 6 yield the general eigenvalue bounds.

Now assume $b_1 = 1$ and $q_1 < \bar{k}$. This implies $q_2 \geq 1$. Then, by (2) and the relation above, $d_{n-\bar{k}} = p_2 = \bar{k} - q_1$, $d_{n-\bar{k}+1} = \cdots = d_{n-\bar{k}+q_1} = \bar{k} - 1$, and the $(\bar{k}+1) \times (\bar{k}+1)$ matrix

$$M = \begin{bmatrix} d_n & 1 & \cdots & 1 & 1 & 1 \\ 1 & d_{n-1} & \cdots & 1 & 1 & 1 \\ \cdots & & & \cdots & & \cdots \\ 1 & 1 & \cdots & \bar{k}-1 & 1 & 0 \\ 1 & 1 & \cdots & 1 & \bar{k}-1 & 0 \\ 1 & 1 & \cdots & 0 & 0 & \bar{k}-q_1 \end{bmatrix}$$

is a principal submatrix of $Q(G)$ for a suitable index reordering. The matrix

$$Q(X) = \begin{bmatrix} \bar{k} & 1 & \cdots & 1 & 1 & 1 \\ 1 & \bar{k} & \cdots & 1 & 1 & 1 \\ \cdots & & & \cdots & & \cdots \\ 1 & 1 & \cdots & \bar{k}-1 & 1 & 0 \\ 1 & 1 & \cdots & 1 & \bar{k}-1 & 0 \\ 1 & 1 & \cdots & 0 & 0 & \bar{k}-q_1 \end{bmatrix}$$

is the signless Laplacian matrix of the graph X obtained from $K_{\bar{k}+1}$ by removing a copy of $K_{q_1,1}$. Hence, the matrix $Q(X)$ may be written as

$$Q(X) = Q(K_{\bar{k}+1}) - Q((\bar{k} - q_1)K_1 \cup K_{q_1,1}).$$

The signless Laplacian $Q(K_{q_1,1})$ has an eigenvalue zero with eigenvector $(\mathbf{1}_{q_1}^\top, -1)^\top$. Therefore the smallest q_1 eigenvalues of the matrix $-Q((\bar{k} - q_1)K_1 \cup K_{q_1,1})$ are its only nonzero eigenvalues. Thus, by the Weyl inequalities of Theorem 2,

$$\begin{aligned} \lambda_{q_1+1}(X) &\geq \lambda_1(K_{\bar{k}+1}) + \lambda_{q_1+1}(-Q((\bar{k} - q_1)K_1 \cup K_{q_1,1})) \\ &= \bar{k} - 1 + 0 \\ &= \bar{k} - 1. \end{aligned}$$

Now, we notice that $M = Q(X) + F$, where

$$F = \mathrm{Diag}(d_n - \bar{k}, d_{n-1} - \bar{k}, \cdots, d_{n-\bar{k}+q_1+1} - \bar{k}, 0, \cdots, 0).$$

Therefore, again by Theorem 2 we obtain

$$\begin{aligned} \lambda_{q_1+1}(M) &\geq \lambda_{q_1+1}(X) + \lambda_1(F) \\ &= \lambda_{q_1+1}(X). \end{aligned}$$

Since M is a principal submatrix of $Q(G)$, we may apply Cauchy's interlacing Theorem 3 to obtain

$$\lambda_{n-\bar{k}+q_1}(G) \geq \lambda_{q_1+1}(M) \geq \lambda_{q_1+1}(X) \geq \bar{k} - 1.$$

Now, the proof follows because when $b_1 = 1$ we have $d_{n-\bar{k}+q_1} = p_1 = \bar{k} - 1$ by (2). □

3 Discussion

We first give an example of the interlacing result and an interpretation based on the Ferrers diagram of a threshold graph.

Example 14. *Let G be the threshold graph with binary sequence $b_1^{q_1} \dots b_8^{q_8} = 0^2 1^2 010101^3$. We have $n = 12$ and $r = 8$. Let λ_i, $i = 1, \dots, 12$, be the eigenvalues of $Q(G)$ and let γ_j, $j = 1, \dots, 8$, be the eigenvalues of $C(G)$ (note that $\gamma_i = \lambda_j$, for some i and j). The signless Laplacian spectrum is (approximately)*

$$\{2.46158, 3.50373, 4.49073, 5.68371, 7, 7, 7.84337, 8.68471, 9.49912, 10, 10, 17.83303\}.$$

As theory predicts, repeated lines within the Ferrers diagram lead to signless Laplacian eigenvalues of G determined by Lemma 6:

$$\lambda_{10} = \lambda_{11} = 10 \quad \text{and} \quad \lambda_5 = \lambda_6 = 7.$$

By the bounds in Theorem 8, the remaining signless Laplacian eigenvalues of G are those of C and satisfy:

$$\gamma_1 \le 3 \le \gamma_2 \le 4 \le \gamma_3 \le 5 \le \gamma_4 \le 7 \le \gamma_5 \le 8 \le \gamma_6 \le 9 \le \gamma_7 \le 10 \le \gamma_8.$$

Note, the bounds hold independently of whether the integral values $p_k - b_k$ are eigenvalues of G or not. Geometrically we can illustrate the bounds in Theorem 8 by the Ferrers diagram — its rows of boxes display the sorted degree sequence — in the following way.

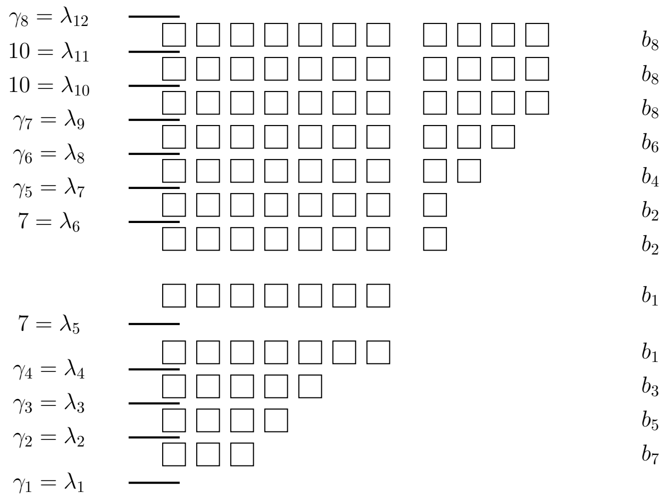


Figure 1: Illustration of the interlacing on the Ferrers diagram

The next result investigates the development of the eigenvalues upon appending a one to the binary construction sequence.

Theorem 15. *Let $b_1^{q_1} \dots b_r^{q_r}$ be an alternating binary sequence with $q_1 \ge 2$ specifying a threshold graph G on $n = \sum_{k=1}^{r} q_k$ nodes and eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_n$. The eigenvalues $0 \le \lambda'_1 \le \lambda'_2 \le \dots \le \lambda'_{n+1}$ of the signless Laplacian of the threshold graph G' specified by $b_1^{q_1} \dots b_r^{q_r} 1$ satisfy*

$$0 \le \lambda'_i \le \lambda_i + 1 \le \lambda'_{i+1} \qquad \text{for } i = 1, \dots, n. \tag{6}$$

Furthermore, $\max\{n + 1, \lambda_n + 2\} \le \lambda'_{n+1}$.

Proof. Because

$$0 \preceq Q(G') = \begin{bmatrix} Q(G) & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} + \sum_{i=1}^{n}(e_i + e_{n+1})(e_i + e_{n+1})^\top = \begin{bmatrix} Q(G) & \mathbf{1}_{n\times 1} \\ \mathbf{1}_{1\times n} & n-1 \end{bmatrix} + I_{n+1},$$

inequalities (6) follow directly from Cauchy's interlacing Theorem 3.

$\lambda'_{n+1} \geq n+1$ is a consequence of the maximum eigenvalue of $\sum_{i=1}^{n}(e_i+e_{n+1})(e_i+e_{n+1})^\top$ being $n+1$ (eigenvector $(\mathbf{1}_{1\times n}, n)^\top$). Suppose now $\lambda_n + 2 > n+1$, *i. e.*, $\lambda_n > n-1$. Then Theorem 13 ensures $\lambda'_n \leq n-1 < \lambda_n$ and $\operatorname{trace} Q(G') = \operatorname{trace} Q(G) + 2n$ yields $\sum_{i=1}^{n+1}\lambda'_i = \sum_{i=1}^{n}\lambda_i + 2n = \sum_{i=1}^{n-1}(\lambda_i+1) + \lambda_n + 1 + n = \sum_{i=1}^{n-1}(\lambda_i+1) + (n-1) + (\lambda_n+2) > \sum_{i=1}^{n}\lambda'_i + \lambda_n + 2$, implying that $\lambda'_{n+1} > \lambda_n + 2$. □

4 The signless Brouwer conjecture for threshold graphs

For an integer k with $1 \leq k \leq n$, we denote by $S_k(G)$ the sum of the k largest signless Laplacian eigenvalues of a graph G, that is $S_k(G) = \sum_{i=n+1-k}^{n}\lambda_i(Q(G))$. Ashraf et al [1] posed the following conjecture.

Conjecture 16 (Signless Brouwer Conjecture [1]). *Let G be a graph with n vertices. Then*

$$S_k(G) \leq |E| + \binom{k+1}{2},$$

where k is an integer with $1 \leq k \leq n$.

This conjecture was motivated by Brouwer's Conjecture [2], which states that the same inequality is valid for the sum of the k largest Laplacian eigenvalues of a graph G and has been studied by many researchers. For recent results on Brouwer's Conjecture see [8, 9, 11, 12, 16, 17, 13].

Ashraf el al. [1] proposed Conjecture 16 and proved it for graphs with at most 10 vertices, for all graphs when $k \in \{1, 2, n-1, n\}$ and for regular graphs. Because trees satisfy Brouwer's Conjecture (see, for example, [9]) and the spectrum of $Q(G)$ is equal to the spectrum of $L(G)$ when G is bipartite, Conjecture 16 holds for trees.

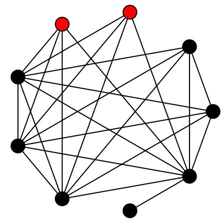
In [18], Yang and You studied and proved that Conjecture 16 is satisfied by unicyclic graphs, bicyclic graphs and tricyclic graphs with $k \neq 3$. More recently, Chen et al. [3] proved that Conjecture 16 is true for all graphs when $k = n-2$.

In this section, as an application of the main result, we prove that Conjecture 16 is true for threshold graphs. We begin by proving some technical lemmas. Lemma 17 provides an expression for $d_{n-\bar{k}}(G)$.

Lemma 17. *Let G be a threshold graph with n vertices and alternating binary sequence $b_1^{q_1}\ldots b_r^{q_r}$. We have that $d_{n-\bar{k}} = \bar{k} - q_1 b_1$.*

Proof. It follows from the definition that $\bar{k} = \sum_{h=1}^{r} b_h q_h$ and $d_{n-\bar{k}}$ represents the degree of vertices of the first block of zeros.

If $b_1 = 0$ then the first block of zeros is the first block of the binary sequence of G and its degree is $p_1 = \sum_{h=1}^{n} b_h q_h = \bar{k} - q_1 b_1$. If $b_1 = 1$ then the first block of zeros is the second block of the binary sequence of G and its degree is $p_2 = \sum_{h=3}^{r} b_h q_h = \bar{k} - q_1 b_1$. □



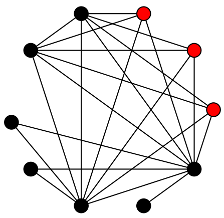


Figure 2: The red vertices have degree $d_{n-\bar{k}}$.

Figure 2 illustrates Lemma 17. On the left, we see the threshold graph given by $B_1 = 1^2 0^2 1^3 01$, in which $n = 9, b_1 = 1, q_1 = 2$ and $\bar{k} = 6$. The red vertices, which represent the first block of zeroes, have degree $p_2 = d_{n-\bar{k}} = d_3 = \bar{k} - q_1 b_1 = 6 - 2 = 4$. On the right, we have the threshold graph given by $B_2 = 0^3 1^2 0^2 101$, in which $n = 10, b_1 = 0, q_1 = 3$ and $\bar{k} = 4$. The red vertices (the first block of zeroes) have degree $p_1 = d_{n-\bar{k}} = d_6 = \bar{k} - q_1 b_1 = 6 - 2 = 4$.

In the next lemma we present a lower bound for the number of edges of the graph as a function of its sequence of degrees.

Lemma 18. *Let G be a threshold graph with n vertices and alternating binary sequence $b_1^{q_1} \dots b_r^{q_r}$. For $k \leq \bar{k}$, we have that*

$$\sum_{i=n+1-k}^{n} d_i - \binom{k}{2} + \binom{\bar{k}-k}{2} + q_1(\bar{k}-k)(1-b_1) \leq |E|.$$

Proof. The term $\sum_{i=n+1-k}^{n} d_i$ adds the degrees of the k last one-vertices in the binary sequence. In relation to all edges $|E|$ the $\binom{k}{2}$ edges between these vertices are counted twice while at least the $\binom{\bar{k}-k}{2}$ edges among the first $\bar{k} - k$ one-vertices are not counted in the sum. Furthermore, if $b_1 = 0$, there are $q_1(\bar{k} - k)$ edges between the last $\bar{k} - k$ one-vertices and the first q_1 zero-vertices that were still not counted. This completes the proof. $\square$

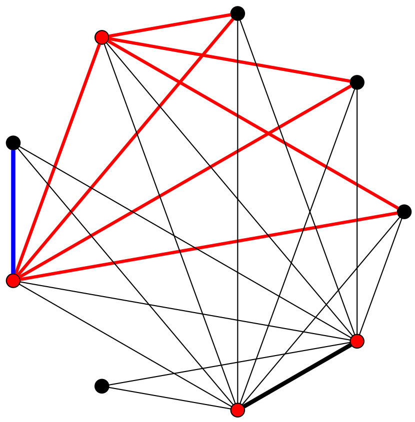


Figure 3: An illustration of the proof of Lemma 18.

Figure 3 illustrates the proof of Lemma 18. We see the threshold graph given by the binary sequence $B = 0^3 10101^2$. As $n = 9$ and $\bar{k} = 4$, consider $k = 2$. The black edges are counted in $d_n + d_{n-1}$, with the thick black edge counted twice. The red edges amount to $\binom{\bar{k}-k}{2} + q_1(\bar{k} - k)(1 - b_1)$. The blue edge is not counted.

Now, we proceed with the proof of the main result of this section.

Theorem 19. *Let G be a threshold graph with n vertices. For any integer k with $1 \le k \le n$ there holds*

$$\sum_{i=n+1-k}^{n} \lambda_i(Q(G)) \le |E| + \binom{k+1}{2}.$$

Proof. Let $b_1^{q_1} \dots b_r^{q_r}$ be the alternating binary sequence with $q_1 \ge 2$ specifying the threshold graph G, and $\bar{k}$ be the number of ones in this sequence. By Theorem 13,

$$\lambda_n \ge d_n - 1 \ge \cdots \ge \lambda_{n+1-\bar{k}} \ge d_{n+1-\bar{k}} - 1 \ge d_{n-\bar{k}} \ge \lambda_{n-\bar{k}} \ge \cdots \ge d_1 \ge \lambda_1. \quad (7)$$

Summing the $n+1-\bar{k}$ smallest signless Laplacian eigenvalues we have that

$$\sum_{i=1}^{n-\bar{k}} \lambda_i \ge \sum_{i=1}^{n-1-\bar{k}} d_i.$$

From Lemma 17, adding $d_{n-\bar{k}} = \bar{k} - q_1 b_1$ on both sides of the above inequality we obtain

$$\sum_{i=1}^{n-\bar{k}} (d_i - \lambda_i) \le \bar{k} - q_1 b_1.$$

The trace $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i = 2|E|$ now gives rise to

$$\sum_{i=n+1-\bar{k}}^{n} (\lambda_i - d_i) = \sum_{i=1}^{n-\bar{k}} (d_i - \lambda_i) \le \bar{k} - q_1 b_1.$$

Adding $\bar{k}$ on both sides, we obtain

$$\sum_{i=n+1-\bar{k}}^{n} [\lambda_i - (d_i - 1)] \le 2\bar{k} - q_1 b_1, \quad (8)$$

where $\lambda_i - (d_i - 1) \ge 0$ for $i = n+1-\bar{k}, \dots, n$ by (7).

We first consider $\sum_{i=n+1-k}^{n} \lambda_i$ for $k \le \bar{k}$. Relation (8) asserts

$$\sum_{i=n+1-k}^{n} \lambda_i \le \sum_{i=n+1-k}^{n} d_i + 2\bar{k} - q_1 b_1 - k. \quad (9)$$

With Lemma 18 this allows one to bound the sum of the largest k eigenvalues by

$$\begin{aligned} \sum_{i=n+1-k}^{n} \lambda_i \le & \\ & \le |E| + \frac{1}{2}[4\bar{k} - 2q_1 b_1 - 2k + k(k-1) - (\bar{k}-k)(\bar{k}-k-1) - 2q_1(\bar{k}-k)(1-b_1)] \\ & = |E| + \frac{1}{2}[k^2 + \bar{k} - (\bar{k}-k)^2 + (\bar{k}-k)(4 - 2q_1 + 2q_1 b_1) - 2q_1 b_1]. \end{aligned} \quad (10)$$

It remains to check that for $k \le \bar{k}$ the term added to $|E|$ is at most $\binom{k+1}{2}$, or

$$k^2 + \bar{k} - (\bar{k}-k)^2 + (\bar{k}-k)(4 - 2q_1 + 2q_1 b_1) - 2q_1 b_1 \le (k+1)k$$

which simplifies to

$$-(\bar{k}-k)^2 + (\bar{k}-k)[5 - 2q_1(1-b_1)] - 2q_1 b_1 \le 0.$$

Consider first the case $b_1 = 0$. We have $(\bar{k} - k)[(5 - 2q_1) - (\bar{k} - k)] \leq 0$. If $\bar{k} - k = 0$ the result follows. If $\bar{k} - k \geq 1$, we notice that $5 - 2q_1 \leq 1$, since $q_1 \geq 2$, and the result follows.

For $b_1 = 1$, we have $-(\bar{k} - k)^2 + 5(\bar{k} - k) - 2q_1 \leq 0$. We notice that this expression represents a parabola in $(\bar{k} - k)$, and its maximum ensures that

$$-(\bar{k} - k)^2 + 5(\bar{k} - k) - 2q_1 \leq \frac{25 - 8q_1}{4}.$$

If $q_1 \geq 3$, it follows that $-(\bar{k} - k)^2 + 5(\bar{k} - k) - 2q_1 \leq 0$ for $\bar{k} - k$ integer. If $q_1 = 2$, it follows that $-(\bar{k} - k)^2 + 5(\bar{k} - k) - 2q_1 \leq 0$ for $\bar{k} - k \leq 1$ and $\bar{k} - k \geq 4$.

It remains to consider the cases with $b_1 = 1$, $q_1 = 2$ and $k = \bar{k} - 2, \bar{k} - 3$. We have

$$\lambda_{n-\bar{k}+2}(G) \geq \bar{k} - 1$$

by Theorem 13. Thus, when $k = \bar{k} - 2$,

$$\begin{aligned}
\sum_{i=n-\bar{k}+3}^{n} \lambda_i &= \sum_{i=n-\bar{k}+2}^{n} \lambda_i - \lambda_{n-\bar{k}+2}(G) \\
&\leq |E| + \binom{\bar{k}}{2} - \lambda_{n-\bar{k}+2}(G) \\
&\leq |E| + \binom{\bar{k}}{2} - (\bar{k} - 1) \\
&= |E| + \binom{\bar{k} - 1}{2},
\end{aligned}$$

because the case $k = \bar{k} - 1$ was already proved. Similarly, when $k = \bar{k} - 3$, since $\lambda_{n-\bar{k}+3} \geq \lambda_{n-\bar{k}+2} \geq \bar{k} - 1$,

$$\begin{aligned}
\sum_{i=n-\bar{k}+4}^{n} \lambda_i &= \sum_{i=n-\bar{k}+3}^{n} \lambda_i - \lambda_{n-\bar{k}+3}(G) \\
&\leq |E| + \binom{\bar{k} - 1}{2} - \lambda_{n-\bar{k}+3}(G) \\
&\leq |E| + \binom{\bar{k} - 1}{2} - (\bar{k} - 1) \\
&\leq |E| + \binom{\bar{k} - 2}{2}.
\end{aligned}$$

For $k \geq \bar{k} + 1$ we proceed by induction on k. For the induction basis consider $k = \bar{k} + 1$. By $\lambda_{n-\bar{k}} \leq d_{n-\bar{k}} = \bar{k} - q_1 b_1 \leq \bar{k}$ there holds

$$\sum_{i=n-\bar{k}}^{n} \lambda_i = \sum_{i=n+1-\bar{k}}^{n} \lambda_i + \lambda_{n-\bar{k}} \leq |E| + \binom{\bar{k} + 1}{2} + \bar{k} \leq |E| + \binom{\bar{k} + 2}{2}.$$

For $k > \bar{k} + 1$, we have $\lambda_{n+1-k} \leq d_{n+1-k} \leq d_{n-\bar{k}} = \bar{k} - q_1 b_1 \leq \bar{k} \leq k$. Consequently,

$$\sum_{i=n+1-k}^{n} \lambda_i = \sum_{i=n+2-k}^{n} \lambda_i + \lambda_{n+1-k} \leq |E| + \binom{k}{2} + k = |E| + \binom{k + 1}{2}.$$

This completes the proof. □

Acknowledgments

The authors acknowledge the partial support of DAAD PROBRAL Grant 56267227 - Germany and MATH-AMSUD under project GSA, brazilian team financed by CAPES under project 88881.694479/2022-01. V. Trevisan also acknowledges partial support of CNPq grant 310827/2020-5, and FAPERGS grant PqG 17/2551-0001. This research is part of the doctoral studies of Guilherme Torres and financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

References

[1] F. Ashraf, G. Omidi, and B. Tayfeh-Rezaie. On the sum of signless Laplacian eigenvalues of a graph. *Linear Algebra Appl.*, 438(11):4539 – 4546, 2013.

[2] A. E. Brouwer and W. H. Haemers. *Spectra of Graphs.* Springer, 2012.

[3] X. Chen, G. Hao, D. Jin, and J. Li. Note on a conjecture for the sum of signless Laplacian eigenvalues. *Czechoslovak Mathematical Journal*, 68(3):601–610, Sep 2018.

[4] D. Cvetković, P. Rowlinson, and S. K. Simić. Eigenvalue bounds for the signless laplacian. *Publications de l'Institut Mathématique, Nouvelle Série*, 81(95):11 – 27, 2007.

[5] D. Cvetković and S. K. Simić. Towards a spectral theory of graphs based on the signless Laplacian. I. *Publ. Inst. Math. (Beograd) (N.S.)*, 85(99):19–33, 2009.

[6] D. Cvetković and S. K. Simić. Towards a spectral theory of graphs based on the signless Laplacian. II. *Linear Algebra Appl.*, 432(9):2257–2272, 2010.

[7] D. Cvetković and S. K. Simić. Towards a spectral theory of graphs based on the signless Laplacian. III. *Appl. Anal. Discrete Math.*, 4(1):156–166, 2010.

[8] Z. Du and B. Zhou. Upper bounds for the sum of Laplacian eigenvalues of graphs. *Linear Algebra Appl.*, 436(9):3672 – 3683, 2012.

[9] E. Fritscher, C. Hoppen, I. Rocha, and V. Trevisan. On the sum of the Laplacian eigenvalues of a tree. *Linear Algebra Appl.*, 435(2):371 – 399, 2011.

[10] E. Fritscher and V. Trevisan. Exploring symmetries to decompose matrices and graphs preserving the spectrum. *SIAM Journal on Matrix Analysis and Applications*, 37(1):260–289, 2016.

[11] H. A. Ganie, A. M. Alghamdi, and S. Pirzada. On the sum of the Laplacian eigenvalues of a graph and brouwer's conjecture. *Linear Algebra Appl.*, 501:376 – 389, 2016.

[12] W. Haemers, A. Mohammadian, and B. Tayfeh-Rezaie. On the sum of Laplacian eigenvalues of graphs. *Linear Algebra Appl.*, 432(9):2214 – 2221, 2010. Special Issue devoted to Selected Papers presented at the Workshop on Spectral Graph Theory with Applications on Computer Science, Combinatorial Optimization and Chemistry (Rio de Janeiro, 2008).

[13] C. Helmberg and V. Trevisan. Spectral threshold dominance, Brouwer's conjecture and maximality of Laplacian energy. *Linear Algebra Appl.*, 512:18–31, 2017.

[14] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge; New York, 2nd edition, 2013.

[15] N. V. R. Mahadev and U. N. Peled. *Threshold graphs and related topics*, volume 56 of *Annals of Discrete Mathematics.* North-Holland Publishing Co., Amsterdam, 1995.

[16] I. Rocha and V. Trevisan. Bounding the sum of the largest Laplacian eigenvalues of graphs. *Discrete Applied Mathematics*, 170:95 – 103, 2014.

[17] S. Wang, Y. Huang, and B. Liu. On a conjecture for the sum of Laplacian eigenvalues. *Mathematical and Computer Modelling*, 56(3):60 – 68, 2012.

[18] J. Yang and L. You. On a conjecture for the signless Laplacian eigenvalues. *Linear Algebra Appl.*, 446:115 – 132, 2014.