TU Chemnitz
C. Helmberg

Chemnitz, April 28, 2004
due on: May 5,2004

Nonlinear Optimization
Exercises 4

1. Complete the proof of Lemma III.4 by showing that for $B \succ 0$
$$1 - \frac{(g^T g)^2}{g^T B g g^T B^{-1} g} \geq 0.$$

2. Suppose $\{p_0, p_1, \ldots, p_{n-1}\}$ is the set of conjugate gradient directions constructed by the linear CG-Algorithm for $\frac{1}{2}x^T Ax - b^T x$ with $A \succ 0$. Show that p_k, $k = 1, 2, \ldots, n-1$, are all descent directions.

3. **Definition:** A set of vectors $\{p_0, p_1, \ldots, p_{n-1}\}$ is said to be *conjugate* with respect to a symmetric positive definite matrix A if
$$p_i^T A p_j = 0, \quad \forall i \neq j.$$
   a) Let A be an $n \times n$ symmetric positive definite matrix and $\{p_0, p_1, \ldots, p_{n-1}\}$ be a set of non-zero conjugate vectors w.r.t. A. Show that $p_k, \quad k = 1, 2, \ldots, n-1$ are linearly independent.
   b) Consider the minimization problem
$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A x - b^T x$$
   where A is an $n \times n$ symmetric positive definite matrix and $b \in \mathbb{R}^n$. Suppose that $\{p_0, p_1, \ldots, p_{n-1}\}$ is a set of non-zero conjugate vectors w.r.t A.
      i. Show that the solution x^* of the problem can be given as
$$x^* = \sum_{i=0}^{n-1} \left(\frac{p_i^T b}{p_i^T A p_i}\right) p_i.$$
      ii. Show that for any $x_0 \in \mathbb{R}^n$, the sequence $\{x_k\}$ generated according to
$$x_{k+1} = x_k + \alpha_k p_k, \quad k \geq 0$$
      with
$$\alpha_k = \frac{-r_k^T p_k}{p_k^T A p_k}, \qquad r_k = Ax_k - b$$
      converges to the solution x^* of the problem after n steps.