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Nonlinear Optimization

Exercises 1

1. Compute the gradient $\nabla f_i(x)$ and Hessian matrix $\nabla^2 f_i(x)$, for $i = 1, 2$
   - $f_1(x) = x^T Q x + b^T x$, with $Q^T = Q \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $x \in \mathbb{R}^n$.
   - $f_2(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$, $x \in \mathbb{R}^2$ (Rosenbrock function.)
2. Let A be a symmetric matrix of order n. Show that the following statements are equivalent:
   i) A is positive semidefinite, i.e., $v^T A v \geq 0 \ \ \forall v \in \mathbb{R}^n$.
   ii) $\lambda_i \geq 0$ where λ_i is an eigenvalue of A, $i = 1, 2, \ldots, n$.
   iii) $\exists C \in \mathbb{R}^{m\times n}$ such that $A = C^T C$ (and $\operatorname{rank}(C) = \operatorname{rank}(A)$).
3. Generate various symmetric 2×2 matrices via specifying their eigenvalues (not necessarily all positive) and eigenvectors and, using Matlab, produce mesh and contour plots of the respective quadratic functions.
4. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the linear model of f at $\bar{x}$ is
   $$f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x})$$
   and the quadratic model of f at $\bar{x}$ is
   $$f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T \nabla^2 f(\bar{x})(x - \bar{x}).$$
   Write a Matlab routine that generates for an arbitrary point in $[-1, 2] \times [-\frac{1}{2}, 3]$ a mesh plot of the Rosenbrock function with its linear and quadratic model.
5. Suppose Newton's method is applied to $f(x) = \cos(x)$.
   a) Find a starting point so that (in theory) the sequence of iterates diverges.
   b) Find a starting point so that the iterates alternate between two non optimal points.
   c) If the procedure starts from a point near 0, then to what point does the method converge? Why?
   d) Determine the maximal interval containing π so that for all starting points in this interval the iterates remain in this interval and converge to π.