Einführung in die Diskrete Mathematik
Solution to exercise 5, problem 1

Prove
a) $O(f(n))O(g(n)) = O(f(n)g(n))$,
b) if $\forall n : f(n) > 0$, then $O(f(n)g(n)) = f(n)O(g(n))$, and
c) $O(f(n)) + O(g(n)) = O(|f(n)| + |g(n)|)$.

Lösung:

a)
If $h(n) \in O(f(n))O(g(n))$, then there are functions h_f and h_g such that

- $\forall n : h(n) = h_f(n)h_g(n)$
- $\exists n_f, c_f : \forall n \geq n_f : |h_f(n)| \leq c_f|f(n)|$
- $\exists n_g, c_g : \forall n \geq n_g : |h_g(n)| \leq c_g|g(n)|$

Therefore, $\forall n \geq \max\{n_f, n_g\}$ we have $|h(n)| = |h_f(n)| \times |h_g(n)| \leq c_f c_g|f(n)g(n)|$ and $h(n) \in O(f(n)g(n))$ becomes obvious (with $n_h = \max\{n_f, n_g\}$ and $c_h = c_f c_g$).

If $h(n) \in O(f(n)g(n))$, then there are positive numbers n_h and c_h such that $\forall n \geq n_h : |h(n)| \leq c_h|f(n)g(n)|$. We have to prove, that there are functions $h_f(n) \in O(f(n))$ and $h_g(n) \in O(g(n))$ such that $h(n) = h_f(n)h_g(n)$.

We define h_f in such a way, that $h_f(n) = 0 \rightarrow h(n) = 0$ (we have to be able to divide non-zero values $h(n)$ by $h_f(n)$) and $h_f(n) \in O(f(n))$:

$$h_f(n) = \begin{cases} f(n) & : \quad n \geq n_h \\ 1 & : \quad n < n_h \end{cases}$$

The latter condition is obvious (with $n_f = n_h$ and $c_f = 1$), while the former condition holds because of the following:
If $h_f(n) = 0$ we know $f(n) = 0$, $n \geq n_h$ and therefore $0 \leq |h(n)| \leq c_h|f(n)| = 0$.

We define h_g in such way, that $\forall n : h_f(n)h_g(n) = h(n)$.

$$h_g(n) = \begin{cases} \frac{h(n)}{h_f(n)} & : \quad h_f(n) > 0 \\ 0 & : \quad h_f(n) = 0 \end{cases}$$

Our condition holds, because if $h_f(n) = 0$ we know already, that $h(n) = 0 = 0^2$. It remains to prove $h_g(n) \in O(g(n))$. We choose $c_g = c_h$ and $n_g = n_h$. If $h_f(n) = 0$, we get obviously $|h_g(n)| = 0 \leq c_g|g(n)|$. If $h_f(n) > 0$, we get $\forall n \geq n_g : |h_g(n)| = \left|\frac{h(n)}{h_f(n)}\right| = \frac{|h(n)|}{|f(n)|} \leq \frac{c_h|f(n)g(n)|}{|f(n)|} = c_g|g(n)|$, and the proof is done.

b) Because $f(n) \in O(f(n))$ we get $\{f(n)\} \subseteq O(f(n))$ and therefore:

$$f(n)O(g(n)) = \{f(n)\}O(g(n)) \subseteq O(f(n))O(g(n))$$

It remains to prove $h(n) \in f(n)O(g(n))$ for all h with $h(n) \in O(f(n))O(g(n))$. If $h(n) \in O(f(n))O(g(n))$ then there is an h_f with $h_f(n) \in O(f(n))$ and an h_g with $h_g(n) \in O(g(n))$ with $\forall n : h_f(n)h_g(n) = h(n)$. But then for large enough n (in the previous notation $n \geq n_h = \max\{n_f, n_g\}$) we have $|h(n)| = |h_f(n)||h_g(n)| \leq c_f|f(n)|c_g|g(n)| = c_f c_g|f(n)g(n)|$ and with $c_h = c_f c_g$ the proof is done.

c) $\subseteq$: $\forall n \geq \max\{n_f, n_g\} : |h(n)| = |h_f(n) + h_g(n)| \leq |h_f(n)| + |h_g(n)| \leq c_f|f(n)| + c_g|g(n)| \leq \max\{c_f, c_g\}(|f(n)| + |g(n)|)$
$\supseteq$: Define $h_f(n) = h(n)$ and $h_g(n) = 0$ if $|f(n)| \geq |g(n)|$ and $h_f(n) = 0$, $h_g(n) = h(n)$ otherwise. We have $h(n) = h_f(n) + h_g(n)$ If $h_f(n) \neq 0$ we have
$2c_h|f(n)| = c_h(|f(n)| + |f(n)|) \geq c_h(|f(n)| + |g(n)|) \geq |h(n)| = |h_f(n)|$ and therefore:

$$h_f(n) \in O(f(n))$$

If $h_g(n) \neq 0$ we have
$2c_h|g(n)| = c_h(|g(n)| + |g(n)|) \geq c_h(|f(n)| + |g(n)|) \geq |h(n)| = |h_g(n)|$ and therefore:

$$h_g(n) \in O(g(n))$$

This completes the proof.