On cycles through specified vertices

Tobias Gerlach[1)], Frank Göring[1)], Jochen Harant[1)], Michal Tkáč[2)]
[1)]Department of Mathematics, Technical University of Ilmenau,
D-98684 Ilmenau, Germany
[2)]Department of Mathematics, The Faculty of Economics in Kosice,
University of Economics in Bratislava, Tajovskeho 13, 04130 Kosice, Slovakia

Abstract

For a set X of vertices of a graph fulfilling local connectedness conditions the existence of a cycle containing X is proved.
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1 Introduction and Results

We use [5] for terminology and notation not defined here and consider finite simple graphs only. Let G be a graph, $X \subseteq V(G)$, and $G[X]$ be the subgraph of G induced by X. A set $S \subset V(G)$ *splits* X if the graph $G - S$ obtained from G by removing S contains at least two components each containing a vertex of X. Let $\kappa(X)$ be infinity if $G[X]$ is complete or the minimum cardinality of a set $S \subset V(G)$ splitting X. Given $t > 0$, X is called to be $t-$*tough (in G)* if for every set $S \subset V(G)$ splitting X the number of components of $G - S$ each containing a vertex of X is at most $\frac{|S|}{t}$. We remark that the usual global concepts of connectedness and toughness are obtained with $X = V(G)$ from these local ones. We call a cycle of G containing all vertices of X an $X-$*cycle* of G.
Results on cycles through specified vertices of a graph can be found in [2, 3, 4, 6, 7, 8, 9, 10, 11, 13, 14, 15]. Theorem 1 and Theorem 3 are consequences of results in [3] and in [8, 15], respectively. Theorem 2 is proved in [8].

Theorem 1. ([3]) *Let G be a graph, $X \subseteq V(G)$ with $|X| \leq \kappa(X) \geq 2$. Then there is an $X-$cycle.*

Theorem 2. ([8]) *Let G be a graph, $X \subseteq V(G)$ with $|X| \leq \kappa(X) + 1 \geq 3$, and $e \in E(G[X])$. Then there is an X-cycle of G containing e.*

Theorem 3. ([8, 15]) *Let G be a graph, $X \subseteq V(G)$ with $|X| = \kappa(X) + 1 \geq 4$ such that X is $1-$tough. Then there is an $X-$cycle.*

In [10] and later in [11] the notion of $A-$separators was introduced as follows: Let A be a set of independent vertices of a graph G. A pair (Y, Z) is called an $A-$*separator* of G if $Y \subseteq V(G-A)$, $Z \subseteq E(G-A-Y)$, and $|A| > c(Y, Z) = |Y| + \sum_{C \in \mathcal{C}(Z)} \left\lfloor \frac{|\partial_{G-A-Y} C|}{2} \right\rfloor$.

Here $\mathcal{C}(Z)$ denotes the set of components of the minimum subgraph of G containing Z as its edge set. Furthermore, $\partial_G C$ denotes the set of vertices of C incident with edges contained in $E(G) \setminus E(C)$. It is easy to see that there is no $A-$cycle if there is an $A-$separator.

Theorem 4. ([10, 11]) *For an integer $k \geq 2$ let G be a k-connected graph and X be a set of at most $k+2$ vertices of G. Then G contains an X-cycle if and only if G has no $A-$separator for each $A \subseteq X$.*

The outlined proof of Theorem 4 in [11] only used the local connectivity of X in G instead of the global one. Therefore, even the following theorem is proved:

Theorem 5. *Let G be a graph and X be a set of at least four vertices with $|X| \leq \kappa(X)+2$. Then G contains an $X-$cycle if and only if G has no $A-$separator for each $A \subseteq X$.*

Our results are Theorem 6, Theorem 7, and Theorem 8.

Theorem 6. *Let G be a graph, $X \subseteq V(G)$ with $|X| = \kappa(X)+2 \geq 5$ such that X is $1-$tough. Then X is independent or there is an $X-$cycle.*

Theorem 7. *Let $t > 1$, G be a graph, $X \subseteq V(G)$ with $|X| = \kappa(X)+2 \geq 6$ such that X is $t-$tough, and $e \in E(G[X])$. Then there is an $X-$cycle containing e.*

Theorem 8. *Let $t > 1$, G be a graph, $X \subseteq V(G)$ with $|X| = \kappa(X)+2 \geq 6$ such that X is $t-$tough. Then there is an $X-$cycle.*

2 Remarks

Using the properties that
(π_1) *A is $t-$tough if B is $t-$tough for $A \subseteq B \subseteq V(G)$* and
(π_2) *$\kappa(A) \geq \kappa(B)$ if $A \subseteq B \subseteq V(G)$*,
global versions of the previous theorems are obtained if $\kappa(X)$, X is $1-$tough, and X is $t-$tough are replaced by $\kappa(V(G))$, $V(G)$ is $1-$tough, and $V(G)$ is $t-$tough, respectively.
Given two disjoint sets A and B of vertices. Let $K_{A,B}$ be the complete bipartite graph with $V(K_{A,B}) = A \cup B$ and $E(K_{A,B}) = \{ab|\ a \in A,\ b \in B\}$. For $|B| \geq 2$ and $b, b' \in B$ $(b \neq b')$ let $K_{A,B}(b, b')$ be the graph obtained from $K_{A,B}$ by adding the edge bb'.
The graph $K_{A,B}$ with $|A| = k+1$ and $|B| = k$ is an example showing that Theorem 1 $(X = A)$ is best possible and that Theorem 3 $(X = A)$ and Theorem 6 $(X = A \cup \{b\},\ b \in B)$ do not hold without the assumption that X is $1-$tough, respectively.

The graph $K_{A,B}(b,b^{'})$ with $|A| = |B| = k$ shows that Theorem 7 does not hold without the assumption that X is $t-$tough with $t > 1$ ($X = A \cup \{b, b^{'}\}$).
Let j, k, and l be three positive integers with $j \geq k$ and $j \geq l$. Given three disjoint sets X, B, and C of vertices such that $X = \{x_1, x_2, \ldots, x_j\}$, $B = \{b_1, b_2, \ldots, b_k\}$, and $C = \{c_1, c_2, \ldots, c_l\}$, respectively. We define the graph $G(j,k,l)$ by $V(G(j,k,l)) = X \cup B \cup C$ and $E(G(j,k,l)) = \{c_{i_1}c_{i_2}|\ 1 \leq i_1 < i_2 \leq l\} \cup \{xb|\ x \in X;\ b \in B\} \cup \{x_i c_i|\ 1 \leq i \leq l\} \cup \{bc|\ b \in B;\ c \in C\}$. Clearly, $G(j,k,l)$ is $k-$connected. The graph $G(k+2,k,3)$ is an example showing that Theorem 8 does not hold without the assumption that X is $t-$ tough with $t > 1$. The graph $G(k+3,k,5)$ is an example showing that Theorem 8 does not hold for $|X| = \kappa(X) + 3$.
Considering three edges of K_4 incident with a common vertex and subdividing each of them by a vertex shows that Theorem 6 is also not true if $\kappa(X) = 2$.
Consider the graph $K_{A,B}(b,b^{'})$ with $|A| = |B| = 3$. Let $a, a^{'} \in A$ ($a \neq a^{'}$) and $b^{''} \in B \setminus \{b, b^{'}\}$. The graph obtained by subdividing the edge $ab^{''}$ by an vertex u, subdividing the edge $a^{'}b^{''}$ by an vertex v, and adding the edge uv shows that Theorem 7 does not hold if $\kappa(X) = 3$ ($X = A \cup \{b, b^{'}\}$).
Given $k \geq 2$, let $G(k)$ be the graph consisting of a disjoint union of a clique H on $2k-1$ vertices and a $K_{X,B}$ with $|X| = 2k-1$, $|B| = k-1$, and, additionally, a matching between X and the vertices of H. $G(2)$ and $G(3)$ show that Theorem 3 and Theorem 8 do not hold with $\kappa(X) = 2$ and $\kappa(X) = 3$, respectively.

3 Proofs

For $A, B \subseteq V(G)$ an $A-B-$*path* is a path P between A and B such that $|V(P) \cap A| = |V(P) \cap B| = 1$. A common vertex of A and B is also an $A-B-$path. A set $S \subseteq V(G)$ *separates A and B* if any $A-B-$path contains a vertex in S. Let $N(v)$ be the neighbourhood of $v \in V(G)$. Without mentioning in each case, we shall use the following properties.

(π_3) *Let $A, B, B' \subseteq V(G)$ such that $B' \subseteq B$. If $S \subseteq V(G)$ separates A and B then S also separates A and B'.*
(π_4) *Let $a \in A \subseteq V(G)$ and $\kappa(A) < \infty$. Then $|N(a)| \geq \kappa(A)$ or $A \subseteq \{a\} \cup N(a)$.*
(π_5) *Let $A \subset V(G)$ and $b \in V(G) \setminus A$. If $|A| \geq \kappa(A \cup \{b\})$ then A and $N(b)$ cannot be separated by a set of at most $\kappa(A \cup \{b\}) - 1$ vertices .*

For a set $\mathcal{P}$ of paths put $V(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} V(P)$. A more detailed version of Menger's Theorem ([12]) is the following lemma ([1]).

Lemma 1. ([1]) *Let s be a non-negative integer, G be a graph, $A, B \subseteq V(G)$ such that A and B cannot be separated by a set of at most s vertices. Furthermore, let $\mathcal{Q}$ be a set of s disjoint $A-B-$paths. Then there is a set $\mathcal{R}$ of $s+1$ disjoint $A-B-$paths, such that $A \cap V(\mathcal{Q}) \subset A \cap V(\mathcal{R})$ and $B \cap V(\mathcal{Q}) \subset B \cap V(\mathcal{R})$.*

Lemma 2. *Let G be a graph, $X \subseteq V(G)$ with $|X| = \kappa(X) + 2 \geq 4$. Moreover, let $a \in X$, C be a cycle with $X \setminus \{a\} \subseteq V(C) \subseteq V(G) \setminus \{a\}$ such that there is an $\{a\} - V(C)-$path W containing a vertex $b \in X \setminus \{a\}$. Then there is an $X-$cycle or there is a set $Y \subseteq V(G) \setminus X$ with $|Y| = \kappa(X) - 1$ such that G contains a subdivision of $K_{X\setminus\{b\},Y\cup\{b\}}$.*

Proof of Lemma 2. Assume that G has no $X-$cycle and put $Z = X \setminus \{a\}$. Let ϕ be an arbitrary but fixed orientation of C. For $u, v \in V(C)$ let $[u, v]$ be the subpath of C from u to v following ϕ. Denote by (u, v) the path obtained from $[u, v]$ by deleting $\{u, v\}$. If $|V([u, v])| = 2$ then (u, v) is considered to be empty. Put $A = N(a)$, $B = V(C)$, $B' = Z$, $s = 1$ and let $\mathcal{Q}$ contain the $N(a) - \{b\}-$subpath of W. Using (π_2), (π_5), and Lemma 1 repeatedly, consider a set $\mathcal{P}$ of $\kappa(X)$ $\{a\} - V(C)-$paths having only a in common. Note that $\mathcal{P}$ contains an $\{a\} - \{b\}-$path. For $P \in \mathcal{P}$ let $T(P) \in V(P) \cap V(C)$. Put $T(\mathcal{P}) = \bigcup_{P\in\mathcal{P}}\{T(P)\}$. If $\kappa(X) = 2$, then we are done with $Y = T(\mathcal{P})\setminus\{b\}$. In the sequel let $\kappa(X) \geq 3$. Given $v \in V(C)$, $S \subseteq V(C)$, $S - \{v\} \neq \emptyset$, let $v^+_S \in S\setminus\{v\}$ and $v^-_S \in S\setminus\{v\}$ such that $V((v, v^+_S)) \cap S = \emptyset$ and $V((v^-_S, v)) \cap S = \emptyset$, respectively. Because there is no $X-$cycle we have $|V([z, z^+_Z]) \cap T(\mathcal{P})| \leq 1$ for $z \in Z$, hence,

(a) $|V((z, z^+_Z)) \cap T(\mathcal{P})| = 1$ for $z \in Z \setminus \{b^-_Z, b\}$ and $V((b^-_Z, b^+_Z)) \cap T(\mathcal{P}) = \{b\}$.

(b) Given $z, z' \in Z \setminus \{b\}$, there is no $\{z\} - \{z'\}$-path Q such that $V(Q) \cap (V(C) \cup V(\mathcal{P})) = \{z, z'\}$,

otherwise, it is easy to see that there is an $X-$cycle. Consider $z \in Z \setminus \{b\}$ and, using (a), let $p = z^-_{Z\cup T(\mathcal{P})} \in T(\mathcal{P})$, $p' = z^+_{Z\cup T(\mathcal{P})} \in T(\mathcal{P})$, $q = z^-_{V(C)} \in N(z)$, and $q' = z^+_{V(C)} \in N(z)$. Note that the cases $p = q$, $p = q = b$, $p' = q'$ or $p' = q' = b$ are included. Put $A = N(z), B = (V(C) \cup V(\mathcal{P})) \setminus V((p, p'))$, $t = 2$, $\mathcal{Q} = \{[p, q], [q', p']\}$, $B' = X \setminus \{z\}$, note (π_2), (π_3), and (π_4), and apply Lemma 1. $\mathcal{R}$ contains a $\{y\} - \{p\}-$path P, a $\{y'\} - \{p'\}-$path P', and a $\{y''\} - \{p''\}-$path P'' where $\{q, q'\} \subset \{y, y', y''\} \subseteq N(z)$ and $p'' \in B$. The cycle obtained from C by replacing $[p, p']$ by the union of P, P', $\{z\}$, and the two edges zy, zy' is again denoted by C, i.e. in the sequel the cycle C may vary permanently without changing the notation C. The path obtained by adding z and the edge zy'' to P'' is a $\{z\} - (V(C) \cup V(\mathcal{P})) \setminus \{z\}-$path with this new cycle C. Again using the assumption that there is no $X-$cycle it is easy to see that $p'' \in T(\mathcal{P})$. Hence, using Lemma 1 and possibly varying C repeatedly, we obtain (c).

(c) Given $z \in Z \setminus \{b\}$, there is a set $\mathcal{R}(z)$ of $\kappa(X) - 2$ $\{z\} - (V(C) \cup V(\mathcal{P}))-$paths having only z in common and ending all in $T(\mathcal{P}) \setminus \{z^-_{Z\cup T(\mathcal{P})}, z^+_{Z\cup T(\mathcal{P})}\}$.

By (b), a path from $\mathcal{R}(z)$ and a path from $\mathcal{R}(z')$ can intersect only in $T(\mathcal{P})$ if

$z, z' \in Z \setminus \{b\}$ and $z \neq z'$. With $Y = T(\mathcal{P}) \setminus \{b\}$, the union of C and of all paths in $\mathcal{P}$ and in $\mathcal{R}(z)$ for $z \in Z \setminus \{b\}$ is the desired subdivision of $K_{X\setminus\{b\},Y\cup\{b\}}$. $\square$

Proof of Theorem 6. Assume that there is an edge connecting $a, b \in X$ and that there is no $X-$cycle of G. Using (π_1), (π_2), $\kappa(X \setminus \{a\}) \geq \kappa(X) = |X| - 2 = |X \setminus \{a\}| - 1$, and Theorem 2, there is a cycle containing $X \setminus \{a\}$. With Lemma 2, there is a set $Y \subseteq V(G) \setminus X$ with $|Y| = \kappa(X) - 1$ such that G contains a subdivision of $K_{X\setminus\{b\},Y\cup\{b\}}$. The graph obtained from $K_{X\setminus\{b\},Y\cup\{b\}}$ by deleting the $\kappa(X)$ vertices of $Y \cup \{b\}$ has $\kappa(X) + 1$ components, each containing exactly one vertex of $X \setminus \{b\}$. Since there is no $X-$cycle of G and $\kappa(X) \geq 3$ an easy case study shows that there is no path in $G - (Y \cup \{b\})$ connecting two of these components - contradicting that X is $1-$tough. $\square$

Lemma 3. *Let G be a graph, $X \subseteq V(G)$ with $|X| = \kappa(X) + 2 \geq 4$, and e an edge connecting two vertices $a, b \in X$. Then there is an $X-$cycle containing the edge e or there are a set $Y \subseteq V(G) \setminus (X \setminus \{a, b\})$ with $|Y| = \kappa(X)$ and two vertices $y, y' \in Y$ such that G contains a subdivision of $K_{X\setminus\{a,b\},Y}(y, y')$ and the $\{y\} - \{y'\}-$ path of the subdivision contains the edge e.*

Proof of Lemma 3. We use the notation as in the proof of Lemma 2. Assume that G has no $X-$cycle containing e. Let $c \in X \setminus \{a, b\}$ and put $Z = X \setminus \{c\}$. Since $|Z| = \kappa(X) + 1$, by π_1, π_2, and Theorem 2 there exists a cycle C containing Z and the edge e. Let ϕ be choosen such that $[a, b] = e$. Using (π_2) and Lemma 1 repeatedly, there must be a set $\mathcal{P}$ of $\{c\} - V(C)-$paths having only c in common, with $|\mathcal{P}| = \kappa(X)$. If $\kappa(X) = 2$, then we are done with $Y = T(\mathcal{P})$. In the sequel let $\kappa(X) \geq 3$. Because there is no X-cycle of G containing e we have $|V([z, z_Z^+]) \cap T(\mathcal{P})| \leq 1$ for $z \in Z \setminus \{a\}$. Proceeding in a similar manner as in Lemma 2 to prove the properties (a)-(c), we obtain

(α) $|V((z, z_Z^+)) \cap T(\mathcal{P})| = 1$ *for* $z \in Z \setminus \{a_Z^-, a, b\}$ *and* $|V((a_Z^-, b_Z^+)) \cap T(\mathcal{P})| = 2$.

(β) *Given $z, z' \in Z \setminus \{a, b\}$, there is no $\{z\} - \{z'\}$-path Q such that $V(Q) \cap (V(C) \cup V(\mathcal{P})) = \{z, z'\}$.*

(γ) *Given $z \in Z \setminus \{a, b\}$, there is a set $\mathcal{R}(z)$ of $\kappa(X) - 2$ $\{z\} - (V(C) \cup V(\mathcal{P}))-$paths having only z in common and ending all in $T(\mathcal{P}) \setminus \{z^-_{Z\cup T(\mathcal{P})}, z^+_{Z\cup T(\mathcal{P})}\}$.*

Let $y, y' \in V((a_Z^-, b_Z^+)) \cap T(\mathcal{P})$ such that $V([y, y']) \cap Z = \{a, b\}$. By (β), a path from $\mathcal{R}(z)$ and a path from $\mathcal{R}(z')$ can intersect only in $T(\mathcal{P})$ if $z, z' \in Z \setminus \{a, b\}$ and $z \neq z'$. With $Y = T(\mathcal{P})$, the union of C and of all paths in $\mathcal{P}$ and in $\mathcal{R}(z)$ for $z \in Z \setminus \{a, b\}$ is the desired subdivision of $K_{X\setminus\{a,b\},Y}(y, y')$. $\square$

Proof of Theorem 7. Assume that there is an e edge connecting $a, b \in X$ and that there is no $X-$cycle of G containing e. Let $c \in X \setminus \{a,b\}$. Using (π_1), (π_2), $\kappa(X \setminus \{c\}) \geq \kappa(X) = |X|-2 = |X \setminus \{c\}|-1$, and Theorem 2, there is a cycle containing $X \setminus \{c\}$ and the edge e. With Lemma 4, there are a set $Y \subseteq V(G) \setminus (X \setminus \{a,b\})$ with $|Y| = \kappa(X)$ and two vertices $y, y^{'} \in Y$ such that G contains a subdivision of $K_{X\setminus\{a,b\},Y}(y,y^{'})$ and the $\{y\} - \{y^{'}\}-$ path of the subdivision contains the edge $e = ab$. The graph obtained from $K_{X\setminus\{a,b\},Y}(y,y^{'})$ by deleting the $\kappa(X)$ vertices of Y has $\kappa(X)$ components, each containing exactly one vertex of $X \setminus \{a,b\}$. Since there is no $X-$cycle of G containing the edge e and $\kappa(x) \geq 4$ an easy case study shows that there is no path in $G - Y$ connecting two of these components - contradicting that X is $t-$tough with $t > 1$. □

Lemma 4. *Let G be a graph and $X \subseteq V(G)$. If $2 \leq \kappa(X) \leq |X| \leq 2(\kappa(X)-1)$ and G contains an $A-$separator for an $A \subseteq X$ then the toughness of X in G is at most* $2 - \frac{\kappa(X)+2}{|X|}$.

Proof of Lemma 4. Let (Y,Z) be an $A-$separator for an $A \subseteq X$ such that $|Y|$ is maximum. Furthermore, let $\kappa(X) = k, |A| = a, |X| = x, |Y| = y, |\mathcal{C}(Z)| = z$, and $\sum\limits_{C \in \mathcal{C}(Z)} |\partial_{G-A-Y} C| = r$. Because of the maximality of $|Y|$, $|\partial_{G-A-Y} C|$ is an odd number at least three for each $C \in \mathcal{C}(Z)$. Furthermore, $c(Y,Z) = y + \frac{r-z}{2} \leq a-1$. If we delete the set $T(Y,Z)$ consisting of Y and all but one vertex of $\partial_{G-A-Y} C$ of each $C \in \mathcal{C}(Z)$ then we get at least a components. Let $t = |T(Y,Z)|$. Because (Y,Z) is an $A-$separator we get $t \leq y+2(a-y-1) = 2a-y-2$. Starting with $(y+2)x \leq (y+2)a$, subtracting this inequality from $2xa = 2xa$, and dividing the resulting inequality by the positive integer x we get $t \leq y + 2(a-y-1) \leq \left(2 - \frac{y+2}{x}\right) a$ which proves the Lemma in the case that $y \geq k$.
Therefore it suffices to disprove the assumption $y < k$: We get $r \geq (k-y)a$ by Menger's theorem ([6]) used for each vertex of A in $G - Y$. Notice that (Y,Z) is an $A-$separator and thus no vertex of $G-Y-Z$ can be connected in $G-Y-Z$ with two vertices of A. This leads to $a-1 \geq c(Y,Z) \geq y + \frac{r}{3} \geq y + \frac{a}{3}(k-y)$, hence, $k-y \leq 2$. If $k-y = 2$ then $a-1 \geq y + \frac{2}{3}a$ and $2(k-1) \geq x \geq a \geq 3(y+1) = 3(k-1)$ - contradicting $k \geq 2$. Consequently, the remaining case is $y = k-1$. In this case $r \geq a+z-1$ holds, since $G-Y$ has a component containing A. This leads to $a-1 \geq c(Y,Z) = y + \frac{r-z}{2} \geq y + \frac{a-1}{2}$ and finally we are done with $k = y+1 \leq \frac{a+1}{2} \leq \frac{x+1}{2}$ - contradicting $2k-2 \geq x$. □

If we combine Theorem 5 and Lemma 4 with $|X| = \kappa(X)+2$ we obtain Theorem 8. □

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