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Primal and Dual Interface Concentrated Iterative Substructuring Methods

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Abstract

This paper is devoted to the fast solution of interface concentrated finite element equations. The interface concentrated finite element schemes are constructed on the basis of a non-overlapping domain decomposition where a conforming boundary concentrated finite element approximation is used in every subdomain. Similar to data-sparse boundary element domain decomposition methods the total number of unknowns per subdomain behaves like $O((H/h)^{(d-1)})$, where H, h, and d denote the usual scaling parameter of the subdomains, the average discretization parameter of the subdomain boundaries, and the spatial dimension, respectively. We propose and analyze primal and dual substructuring iterative methods which asymptotically exhibit the same or at least almost the same complexity as the number of unknowns. In particular, the so-called All-Floating Finite Element Tearing and Interconnecting solvers are highly parallel and very robust with respect to large coefficient jumps.

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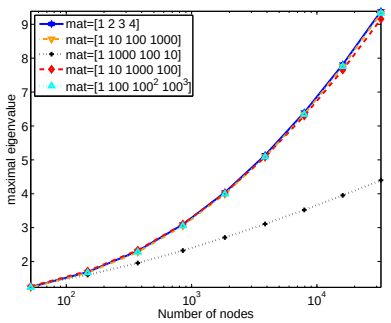

Figure 5: Maximal eigenvalue of the FETI preconditioner.

proposed primal and dual domain decomposition preconditioners resulting in asymptotically optimal or almost optimal solvers for the interface concentrated finite element equations. The first class of preconditioners are based on inexact Dirichlet-Dirichlet preconditioning techniques. The second class of preconditioners are constructed on the basis of the so-called All-Floating Finite Element Tearing and Interconnecting technology. The latter class of preconditioners is very robust with respect to jumps in the coefficients of the PDE. It is clear that the AF-FETI technology can be replaced by the FETI-DP technique. The number of unknowns involved in the IC-FEM is comparable to the number of unknowns living on the skeleton of the domain decomposition. Thus, the coupling of IC-FEM with the Boundary Element Method (BEM) seems to be very attractive for many applications. In contrast to the standard FEM the IC-FEM will not perturb the complexity of the BEM. We refer the reader to the survey paper [25] for the standard symmetric BEM-FEM coupling.

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1 Introduction

Boundary concentrated finite element methods (BC-FEM) introduced and investigated by Khoromskij and Melenk in [18] allow us to solve potential problems with smooth coefficients as accurate as the corresponding standard finite element methods (FEM) but with a significantly smaller number of unknowns. Indeed, the total number of unknowns in the BC-FEM is proportional to $O(h^{-(d-1)})$ whereas the total number of unknowns in the standard FEM behaves like $O(h^{-d})$, where h and d denote the the average discretization parameter of the boundary and the spatial dimension, respectively. Moreover, the stiffness matrices of the BC-FEM are significantly better conditioned than the usual stiffness matrices. This property was already recovered by Yserentant [37].
Interface concentrated finite element methods (IC-FEM) are a combination of nonoverlapping domain decomposition (DD) methods with the BC-FEM that is used in the subdomains Ω_k in which the initial domain Ω is decomposed. Now we can admit piecewise smooth coefficients in the partial differential equation that we are going to solve. An efficient method for the solution of symmetric and positive definite systems of linear algebraic equations arising from the finite element discretization of elliptic boundary value problems is the preconditioned conjugate gradient method with DD preconditioners. The nonoverlapping Dirichlet-Dirichlet DD preconditioning technique and the Finite-Element-Tearing and Interconnecting (FETI) method are typical representatives of primal and dual nonoverlapping DD approaches, respectively. We refer the reader to the monograph [36] for an excellent overview over the different DD methods.
The Dirichlet-Dirichlet DD-preconditioner requires preconditioners for the local Dirichlet problems on the subdomains, a Schur complement preconditioner related to the interfaces and approximate discrete harmonic extension operators, or more precisely, bounded discrete extension operators acting from the subdomain boundaries into the interior, see [27], [31], [14], [15] and [21].

Nowadays, for preconditioning the local Dirichlet problems, a lot of very efficient techniques are available from preconditioning standard h, p or hp finite element equations, see, e.g., [21]. First preconditioners for the Schur complement were proposed by Dryja for some special cases in [6] and [7]. The BPS preconditioners proposed and developed by Bramble, Pasciak and Schatz in a series of papers can be seen as a generalization to more general decompositions and discretizations [2]. During the last 20 years many alternative Schur complement preconditioners have been proposed in the literature. Let us here only mention the Schur complement BPX [35], techniques borrowed from the boundary element method [25] and $\mathcal{H}$-matrix techniques [17]. In the case of uniform refined meshes, two basic techniques for the fast evaluation of the approximate discrete harmonic extension of a finite element function on the boundary to the interior exist, namely, an averaging technique [31], [28] and hierarchical decomposition techniques [15], [12], [16].

The FETI technology introduced by Farhat and Roux in [11] has been developed to a very efficient and robust parallel solver technique for large scale finite element equations. There are a lot of modifications and new versions of the classical one-level FETI method like the FETI-DP (Dual-Primal) [10] and BDDC (Balanced Domain Decomposition by Constraints) [5], [26] methods. The main building blocks of the FETI methods are solvers or preconditioners for the local Dirichlet and the local Neumann problems in the subdomains and a component

managing the global information exchange. We again refer to the monograph [36] for more information on and references to FETI methods.

In this paper, we propose and investigate fast, robust and highly parallel DD solvers for large-scale interface concentrated finite element equations. The polynomials of high order are removed by an overlapping preconditioner with inexact subproblem solvers which was suggested by Pavarino [33]. This preconditioner requires

- a solver for the each patch corresponding to a node, and
- a coarse space solver for $p = 1$ on the computational domain.

The first problem is treated by a direct solver, whereas the second problem is treated by two different types of domain decomposition preconditioners. The first preconditioner is a nonoverlapping domain decomposition preconditioner of Dirichlet-Dirichlet type with inexact subproblem solvers. The second type of preconditioners are All-Floating FETI preconditioners. We will prove that the design of both preconditioner leads to an almost optimal solver for the corresponding system of algebraic finite element equations the complexity of which is almost proportional to the number of unknowns corresponding to the skeleton of the domain decomposition. Moreover, we show that the FETI preconditioners are robust with respect to jump of the coefficients.

The rest of the paper is organized as follows: In Section 2, we formulate the boundary value problem and describe its discretization with the help of the IC-FEM. In Section 3, we consider the primal DD-preconditioners. In Section 4, we investigate the FETI-preconditioners. Several numerical experiments are presented in Section 5.

2 Model Problem and Finite Element Discretization

2.1 Model Problem

For a bounded Lipschitz domain Ω with

$$\overline{\Omega} = \bigcup_{i=1}^{s} \overline{\Omega}_i \subset \mathbb{R}^2, \qquad \Omega_i \cap \Omega_j = \emptyset \ \text{ with } i \neq j, \tag{2.1}$$

a symmetric positive definite matrix function $\mathcal{D} \in \mathbb{R}^{2,2}$ which is piecewise constant on Ω_j and a right hand side $f \in L^2(\Omega)$ analytic on $\overline{\Omega}_j$, we consider the following Dirichlet problem, given in weak formulation:

Problem 2.1. (model problem) *Find $u \in H^1_0(\Omega)$ such that*

$$a(u,v) := \int_\Omega \nabla u(x) \cdot \mathcal{D}(x) \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx =: l(v) \qquad \forall\, v \in H^1_0(\Omega). \tag{2.2}$$

2.2 Discretization - The Geometric Mesh, the Linear Degree Vector and the Fine Element Space

We will restrict our considerations to γ-shape-regular triangulations τ of Ω consisting of affine triangles, i.e., each element $K \in \tau$ is the image $F_K(\hat{K})$ of the reference triangle $\hat{K}$, where the

the constants in comparison to $\mathcal{C}_D = \mathcal{D}_L$.

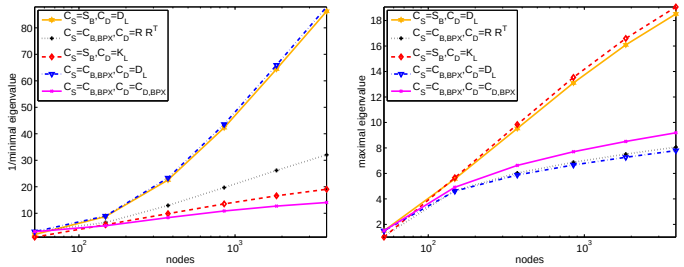

Figure 4: Eigenvalues of $\mathcal{C}^{-1}\mathcal{K}$ for strongly jumping coefficients: minimal eigenvalue left, maximal eigenvalue right

In a second experiment, we have chosen $a_i = 10^i$, $i = 1,2,3,4$. From the experiments, it can seen that the primal DD-preconditioner is not robust with respect to the jumps of the diffusion coefficient, see Figure 4.

In a last experiment, we determine the numbers of iterations of the pcg-method for the solution of $\mathcal{K}\underline{u} = \underline{b}$ using the preconditioner $\mathcal{C}$ (3.29) with $\mathcal{C}_S = \mathcal{C}_{B,BPX}$ and $\mathcal{C}_D = \mathcal{C}_{D,BPX}$ and a relative accuracy of 10^{-5}. Table 1 displays the numbers of iterations for the diffusion coefficients $a_i = i$ and $a_i = 10^i$, $= 1,2,3,4$, respectively. In both cases, the numbers of

Level	0	1	2	3	4	5	6	7	8	9
N	41	129	337	785	1713	3601	7409	15057	30385	61073
$a_i = i$	5	13	16	17	17	18	18	19	19	19
$a_i = 10^i$	5	23	35	41	53	57	57	55	56	56

Table 1: Numbers of pcg-iterations for the solution of $\mathcal{K}\underline{x} = \underline{b}$ with the preconditioner $\mathcal{C}$.

pcg-iterations are bounded.

In a last experiment, we investigate the FETI preconditioner (4.21) $\mathcal{C}_F$ for $\mathcal{F}$ (4.16). Again, the finite element mesh of Figure 2 with $p = 1$ is used. We consider five different distributions for the coefficients $\underline{mat} = [a_i]_{i=1}^4$ in the subdomains Ω_i, $i = 1,2,3,4$. The maximal eigenvalue of $\mathcal{C}_F^{-1}\mathcal{F}$ is displayed in Figure 5, whereas the minimal eigenvalue is 1 in all experiments. Both eigenvalues are robust with respect to material jumps. Moreover, a logarithmic growth of the condition number can be seen.

6 Conclusions

The solution of potential problems with piecewise smooth coefficients can be approximated by the interface concentrated finite element method with the same asymptotical accuracy as the standard finite element method but with a considerably smaller number of unknowns. We have

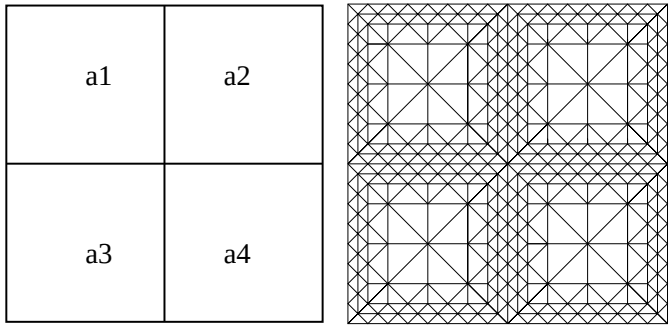

Figure 2: Coefficient matrix (left) and IC-concentrated mesh (right)

In all experiments, we use the preconditioner $\mathcal{C}$ (3.29), where $\mathcal{E}_{DB}$ is the matrix representation of the extension operator of Algorithm 3.8. and the preconditioners $\mathcal{C}_S = \{\mathcal{S}_B, \mathcal{C}_{B,BPX}\}$ and $\mathcal{C}_D = \{\mathcal{K}_L, \mathcal{D}_L, \mathcal{C}_{D,BPX}, RR^T\}$ for the Schur complement and the block of the interior functions, where

- $\mathcal{C}_{B,BPX}$ and $\mathcal{C}_{D,BPX}$ denote the BPX-preconditioner for $\mathcal{S}_B$ and $\mathcal{K}_D$, respectively,
- R denotes the Cholesky factor of the incomplete Cholesky factorization for $\mathcal{K}_D$, i.e. $\mathcal{C}_{ichol} = RR^T$, and
- $\mathcal{D}_L$ denotes the diagonal part of $\mathcal{K}_L$.

Since $p = 1$ on all elements, the block $\mathcal{K}_H$ does not exist. Experiments for the overlapping preconditioner $\mathcal{C}_D$ be defined via (3.16) have already been presented in [9], [29].

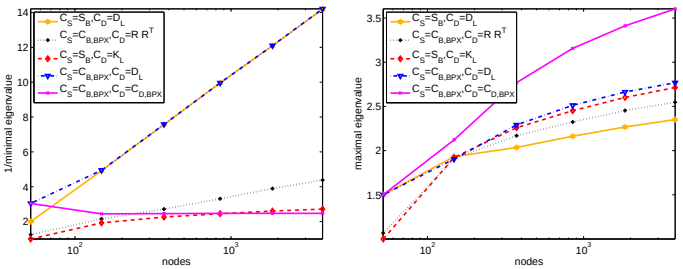

Figure 3: Eigenvalues of $\mathcal{C}^{-1}\mathcal{K}$ for moderate jumping coefficients: minimal eigenvalue left, maximal eigenvalue right

Figure 3 displays the maximal and minimal eigenvalue of $\mathcal{C}^{-1}\mathcal{K}$ for several variants of $\mathcal{C}_S$ and $\mathcal{C}_D$ in the case of moderate jumping coefficients $a_i = i$, $i = 1, 2, 3, 4$. The optimality of the condition number can be seen in all cases. The usage of $\mathcal{C}_D = C_{D,BPX}$ or $\mathcal{C}_D = RR^T$ reduces

mapping F_K satisfies the inequality

$$h_K^{-1}\|F_K'\|_{L^\infty(K)} + h_K\|\left(F_K'\right)^{-1}\|_{L^\infty(K)} \le \gamma \qquad \forall\, K \in \tau.$$

Here h_K denotes the diameter of the element K. Moreover, we assume $\tau = \cup_{j=1}^s \tau^{(j)}$ with $\tau^{(j)} := \{K \in \tau \mid \operatorname{supp}\{K\} \subset \Omega_j\}$ to be a geometric mesh with boundary mesh size h (see also Figure 2). The precise description is given in the following definition.

Definition 2.2. (geometric mesh) *There exist $c_1, c_2 > 0$ such that for all $K \in \tau^{(j)}$:*

1. *if $\overline{K} \cap \partial\Omega_j \neq \emptyset$, then $h \le h_K \le c_2 h$,*
2. *if $\overline{K} \cap \partial\Omega_j = \emptyset$, then $c_1 \inf_{\mathbf{x}\in K} \operatorname{dist}(\mathbf{x}, \partial\Omega_j) \le h_K \le c_2 \sup_{\mathbf{x}\in K} \operatorname{dist}(\mathbf{x}, \partial\Omega_j)$.*

In order to define hp-FEM spaces on the mesh τ, we associate a polynomial degree $p_K \in \mathbb{N}$ with each element $K \in \tau$ and collect these p_K in the polynomial degree vector $\mathbf{p} := (p_K)_{K\in\tau}$. Furthermore we associate a polynomial degree

$$p_e := \min\{p_K \,|\, \text{e is an edge of element } K\} \tag{2.3}$$

with each edge e of the triangulation. We denote the vector containing the polynomial distribution of the triangle $K \in \tau$ with edges $\{e_i \mid i = 1, 2, 3\}$ by

$$\mathbf{p}(K) := (p_{e1}, p_{e2}, p_{e3}, p_K). \tag{2.4}$$

In conjunction with geometric meshes a particularly useful polynomial degree distribution is the linear degree vector:

Definition 2.3 (linear degree vector)**.** *Let $\overline{\Omega} = \cup_{j=1}^s \overline{\Omega}_j$. For all $j = 1, \ldots, s$, let $\tau^{(j)}$ be a geometric mesh on Ω_j with boundary mesh size h and let $\tau = \cup_{j=1}^s \tau^{(j)}$ be a mesh on Ω. A polynomial degree vector $\mathbf{p} = (p_K)_{K\in\tau}$ is said to be a linear degree vector with slope $\alpha > 0$ if*

$$1 + \alpha c_1 \log\frac{h_K}{h} \le p_K \le 1 + \alpha c_2 \log\frac{h_K}{h} \tag{2.5}$$

for some $c_1, c_2 > 0$.

Now we are in the position to define our hp-FEM spaces:

Definition 2.4 (FEM spaces)**.** *Let $\overline{\Omega} = \cup_{j=1}^s \overline{\Omega}_j$. For all $j = 1, \ldots, s$, let $\tau_l^{(j)}$ be a geometric mesh on Ω_j with boundary mesh size h_l and let $\tau_l = \cup_{j=1}^s \tau_l^{(j)}$ be a mesh on Ω. Let $\mathbf{p}$ be a linear degree vector. Furthermore, for all edges e let p_e be given by (2.3) and for all $K \in \tau$ let $\mathbf{p}(K)$ be given by (2.4). Then we set*

$$\begin{aligned}
\mathbb{S}_{l,j}^{\mathbf{p}} = \mathbb{S}^{\mathbf{p}}(\Omega_j, \tau_l^{(j)}) &:= \{u \in H^1(\Omega_j) \mid u \circ F_K \in \mathcal{P}_{\mathbf{p}(K)}(\hat{K}) \quad \forall K \in \tau_l^{(j)}\},\\
\mathbb{S}_{l,j,0}^{\mathbf{p}} = \mathbb{S}_0^{\mathbf{p}}(\Omega_j, \tau_l^{(j)}) &:= \mathbb{S}_{l,j}^{\mathbf{p}} \cap H_0^1(\Omega_j),\\
\mathbb{V}_l^{\mathbf{p}} = \mathbb{V}^{\mathbf{p}}(\Omega, \tau_l) &:= \{u \in H^1(\Omega_j) \mid u|_{\Omega_j} \in \mathbb{S}_{l,j}^{\mathbf{p}} \quad \forall j = 1, \ldots, s\},\\
\mathbb{V}_{l,0}^{\mathbf{p}} &:= \mathbb{V}_l^{\mathbf{p}} \cap H_0^1(\Omega),
\end{aligned}$$

where

$$\mathcal{P}_{\mathbf{p}(K)}(\hat{K}) := \{u \in \mathcal{P}_{p_K}(\hat{K}) \mid u|_{e_i} \in \mathcal{P}_{p_{e_i}}, i = 1, 2, 3\}.$$

The FE-discretization of Problem 2.1 then reads:
Problem 2.5 (FE-approximation)**.** *Find* $u_h \in \mathbb{V}^{\mathbf{p}}_{l,0}$ *such that*

$$a(u_h, v_h) = l(v_h) \qquad \forall\, v_h \in \mathbb{V}^{\mathbf{p}}_{l,0}.$$

Considering the approximation properties of boundary concentrated *hp*-FEM, we have the following theorem
Theorem 2.6. *Let* $\Omega_i \subset \mathbb{R}^2$ *and* τ_i *a geometric mesh with boundary mesh size* h *on* Ω_i*. Let* $\mathbf{p}$ *be a linear degree vector with slope* α *sufficiently large. Then for* $u \in H^{1+\delta}(\Omega_i)$*,* $\delta \in (0,1)$ *we have*

$$\inf_{v_h \in \mathbb{S}^{\mathbf{p}}(\Omega_i,\tau_i)} \|u - v_h\|_{H^1(\Omega_i)} \le C h^{\delta}.$$

Proof. See [18]. □

2.3 The System of Finite Element Equations

To solve Problem 2.5 numerically, we equip the space $\mathbb{V}^{\mathbf{p}}_{l,0}$ with a basis

$$\Phi = \{\Phi_B, \Phi_L, \Phi_H\} = \{\phi_i \mid i = 1, \dots, N\} \tag{2.6}$$

in the following way:

The functions $\phi_1, \dots, \phi_{N_B}$ are the usual nodal hat functions with $\operatorname{supp}\phi_i \not\subset \overline{\Omega}_j$ for all $j = 1, \dots, s$, $\forall i = 1, \dots, N_B$, i.e. the support is not contained in one subdomain Ω_j only. The functions $\phi_{N_B+1}, \dots, \phi_{N_L}$ are the usual nodal hat functions with $\operatorname{supp}\phi_i \subset \overline{\Omega}_j$ for some j, $\forall i = N_B + 1, \dots, N_L$. The remaining functions $\phi_{N_L+1}, \dots, \phi_{N_H}$ are all polynomials of degree $p > 1$. Since $h_K = h$ for all elements with $\overline{K} \cap \partial\Omega_j \neq \emptyset$, we have $p = 1$ for all these boundary elements by (2.5). With this definition, the basis functions ϕ_i can be divided into three groups,

- the boundary functions $\Phi_B = \{\phi_i \mid i = 1, \dots, N_B\}$,
- the interior functions of low order $\Phi_L = \{\phi_i \mid i = N_B + 1, \dots, N_L\}$,
- the interior functions of high order $\Phi_H = \{\phi_i \mid i = N_L + 1, \dots, N_H\}$.

In the same way, the basis Φ_i on the subdomains Ω_i, $i = 1, \dots, s$, is introduced. Analogously, this basis can be partitioned into $\Phi_{B,i}$, $\Phi_{L,i}$ and $\Phi_{H,i}$, i.e. boundary functions (B), interior functions of low order (L) and interior functions of high order (H). More precisely, let

$$\Phi_i = \Phi\,|_{\Omega_i}\, R_i \quad \text{and} \quad \Phi_{B,i} = \Phi_B\,|_{\Omega_i}\, R_{B,i} \tag{2.7}$$

with the corresponding restriction matrices R_i and $R_{B,i}$. Now, the solution of Problem 2.5 is equivalent to solve the system of linear equations

$$\mathcal{K}\underline{u} = \underline{f}, \tag{2.8}$$

where

$$\mathcal{K} = [a(\phi_j, \phi_k]_{k,j=1}^N, \qquad \underline{f} = [f(\phi_j)]_{j=1}^N, \qquad \underline{u} = [u_k]_{k=1}^N.$$

which can be done in parallel and which can be performed by means of a sparse direct method with the matrix factorization in a preprocessing step, or with the help of the PCG method with the Dirichlet preconditioner $\mathcal{C}_{D,i}$ proposed by relation (3.16) in Subsection 3.2.

4.2.3 AF-FETI3 Solvers

The AF-FETI3 system (4.20) is again an one-fold saddle point problem which can be solved by the solvers mentioned above. In any case, we need

- an efficient preconditioners $\mathcal{C}_i$ for the regularized local Neumann matrices (4.19) and
- a FETI preconditioner $\mathcal{C}_F$,

which are available from Section 3 and (4.21)–(4.23), respectively.
Theorem 4.3. *Assume that the AF-FETI3 system (4.20) is solved by means of the Bramble Pasciak PCG method with the scaled inexact Dirichlet FETI preconditioner (4.22) and an appropriately scaled local Neumann preconditioners (3.27) composed of the optimal components* $\mathcal{C}_{D,i}$*,* $\mathcal{C}_{S,i}$ *and* $\mathcal{E}_{DB,i}$ *given in Section 3. Then not more than* $I(\varepsilon) = O((1+\log(H/h))\log\varepsilon^{-1})$ *iterations and* $ops(\varepsilon) = O((H/h)(1+\log(H/h))\log\varepsilon^{-1})$ *arithmetical operations are required in order to reduce the initial error by the factor* $\varepsilon \in (0,1)$ *in a parallel regime, where* $H/h = \max H_i/h_i$*. The number of iterations* $I(\varepsilon)$ *is robust with respect to the jumps in the coefficients. Moreover, at most* $O(H/h)$ *storage units are needed per processor.*

Proof. The estimate for the number $I(\varepsilon)$ of iterations is a direct consequence of results given in [3] (see [38] for improved estimates) and of the spectral estimates (3.14) and (4.25). The complexity estimates for the arithmetical costs and the memory demand follow from the corresponding complexity estimates for the block matrices involved in the matrix-vector multiplications and in the preconditioning. □

We emphasize that we can solve the interface concentrated finite element equations with the same complexity (even with a slightly better complexity) as data-sparse boundary element DD equations (cf. [23] and [22]). In contrary to boundary element technology we can easily include source terms $f(\cdot)$ and we can even admit non-constant, but smooth coefficients $a_i(\cdot)$ in the subdomains Ω_i.

5 Numerical Experiments

In this section, we first present some numerical experiments for the primal DD-solver proposed in Section 3. Finally, we investigate the behavior of the FETI preconditioner studied in Section 4.

We consider problem (2.2) in $\Omega = (-1,1)^2$ using a piecewise constant coefficient matrix $\mathcal{D}(x,y)$ with coefficients a_i, $i = 1,2,3,4$ and a right hand side $f(x,y) = 1$, see Figure 2. The problem is discretized by an IC-FEM with $p = 1$ in all elements. The interfaces are at the lines $x = 0$ and $y = 0$, whereas the boundaries are situated at $x = \pm 1$ and $y = \pm 1$. Here, we choose Dirichlet boundary conditions, see Figure 2.

spectral inequalities (4.25) were proved for $\widetilde{\mathcal{S}}_{B,i}^{FEM}$ (see [20], [4], or [36]), they are also valid if we replace $\widetilde{\mathcal{S}}_{B,i}^{FEM}$ in $\mathcal{F}$ and $\mathcal{C}_F$ by the matrices $\widetilde{\mathcal{S}}_{B,i}$, $\widetilde{\mathcal{S}}_{B,i}+\mathcal{T}_{B,i}$, or $\widetilde{\mathcal{W}}_{B,i}$. This completes the proof of the theorem. □

Now we can solve the SPD system (4.16) via the PCG method that can efficiently be performed on the subspace $\mathbf{\Lambda}_0$ with one of the preconditioners $\mathcal{C}_F$ proposed above. Theorem 4.2 immediately implies that we need at most $I(\varepsilon) = O((1+\log(H/h))\log\varepsilon^{-1})$ iterations in order to reduce the initial error by the factor $\varepsilon \in (0,1)$ in a parallel regime. The matrix-vector product $\mathcal{F}\underline{\lambda}$, or $\widehat{\mathcal{F}}\underline{\lambda}_0$, requires the (parallel) solution of the systems

$$\widetilde{\mathcal{S}}_{B,i}\underline{w}_{B,i} = \underline{r}_{B,i}, \tag{4.26}$$

which are equivalent to the solution of the regularized finite element Neumann problems

$$\begin{pmatrix} \widetilde{\mathcal{K}}_{B,i} & \mathcal{K}_{BD,i} \\ \mathcal{K}_{DB,i} & \mathcal{K}_{D,i} \end{pmatrix} \begin{pmatrix} \underline{w}_{B,i} \\ \underline{w}_{D,i} \end{pmatrix} = \begin{pmatrix} \underline{r}_{B,1} \\ \underline{0}_{D,1} \end{pmatrix}, \quad i = 1,\ldots,s. \tag{4.27}$$

The discrete Neumann problem (4.27) can now be solved either by means of a sparse direct method with the matrix factorization in a preprocessing step, or with the help of the PCG method with the Neumann preconditioner (3.27) proposed in Subsection 3.4. The preconditioning step $\mathcal{C}_F^{-1}\underline{\rho}$ requires

- the (parallel) solution of local discrete Dirichlet problems with the system matrices $\mathcal{K}_{D,i}$,
- the (parallel) matrix-vector multiplications $(\widetilde{\mathcal{S}}_{B,i}+\mathcal{T}_{B,i})\underline{w}_{B,i}$ costing $O((H_i/h_i))$ arithmetical operations, or
- the (parallel) multiplications $\widetilde{\mathcal{W}}_{B,i}\underline{w}_{B,i}$ costing $O((H_i/h_i)(\ln(H_i/h_i))^2)$ arithmetical operations

for the preconditioners (4.21), (4.22), or (4.23), respectively. Thus, the scaled inexact Dirichlet FETI preconditioner (4.22) is the cheapest FETI preconditioner.

4.2.2 AF-FETI2 Solvers

The saddle point problem (4.14) can be solved by the Bramble-Pasciak PCG [3] or by any other suitable Krylov subspace iterative solver [34], e.g. the preconditioned conjugate residual method [19]. In any case, we need

- efficient preconditioners $\mathcal{C}_{B,i}$ for the regularized local Schur complements $\widetilde{\mathcal{S}}_{B,i}$ and
- a FETI preconditioner $\mathcal{C}_F$

which are available from Section 3, cf. (3.27), and (4.21)–(4.23), respectively. It follows from Theorem 4.2, Theorem 3.7 and the results of [3] and [38] that the number $I(\varepsilon)$ of iterations of the Bramble-Pasciak PCG behaves like $O((1+\log(H/h))\log\varepsilon^{-1})$.

The matrix-vector multiplication $\widetilde{\mathcal{S}}_{B,i}\underline{v}_{B,i}$ requires the solution of the local Dirichlet systems

$$\mathcal{K}_{D,i}\underline{w}_{D,i} = \underline{r}_{D,i}, \tag{4.28}$$

Then, $u_h = \sum_{k=1}^N u_k\phi_k$ provides the solution of (2.5). Moreover, let $\mathcal{K}_i$ be the stiffness matrix restricted to Φ_i, $i = 1,\ldots,s$, i.e.

$$\mathcal{K}_i = [a(\phi_{j,i},\phi_{k,i}]_{k,j=1}^{N_i}. \tag{2.9}$$

Then, a consequence of (2.7)-(2.9) is the following formula

$$\mathcal{K} = \sum_{i=1}^s R_i\mathcal{K}_iR_i^\top \tag{2.10}$$

with the restriction matrix introduced in (2.7). In the following, we are interested in constructing fast solver for the systems of linear algebraic equations (2.8).

3 Primal Iterative Substructuring Methods

In this section, we will derive a primal DD preconditioner for the matrix $\mathcal{K}$ (2.8) with inexact subproblem solvers. In Subsection 3.1, we expose the ingredients of a nonoverlapping DD-preconditioner. In Subsection 3.2, we develop the preconditioner on the subdomains. Subsection 3.2 is devoted to the construction of the preconditioner on the subdomains. Using this extension, we show the equivalence of the Schur complement norm to the $H^{1/2}(\Gamma)$-seminorm. In Subsection 3.4, we prove the independence of the condition number for the preconditioned system from the discretization parameter. In Subsection 3.5, we show that the preconditioning operation $\mathcal{C}^{-1}\underline{r}$ can be performed in optimal arithmetical complexity.

3.1 Block Partitioning of the Stiffness Matrix

We partition the stiffness matrix into 3 times 3 blocks, which corresponds to the partition of the shape functions into Φ_B, Φ_L and Φ_H, see (2.6),

$$\mathcal{K} = \begin{bmatrix} \mathcal{K}_B & \mathcal{K}_{B,L} & \mathcal{K}_{B,H} \\ \mathcal{K}_{L,B} & \mathcal{K}_L & \mathcal{K}_{L,H} \\ \mathcal{K}_{H,B} & \mathcal{K}_{H,L} & \mathcal{K}_H \end{bmatrix} = \begin{bmatrix} \mathcal{K}_B & \mathcal{K}_{B,D} \\ \mathcal{K}_{D,B} & \mathcal{K}_D \end{bmatrix}$$

with $D = L \cup H$. The two times two block matrix can be factorized into

$$\mathcal{K} = \begin{bmatrix} I & \mathcal{K}_{B,D}\mathcal{K}_D^{-1} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathcal{S}_B & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_D \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathcal{K}_D^{-1}\mathcal{K}_{D,B} & I \end{bmatrix} \tag{3.1}$$

with the Schur-complement

$$\mathcal{S}_B = \mathcal{K}_B - \mathcal{K}_{B,D}\mathcal{K}_D^{-1}\mathcal{K}_{D,B}. \tag{3.2}$$

In this section, we will investigate a preconditioner of the type

$$\mathcal{C} = \begin{bmatrix} I & -\mathcal{E}_{DB}^\top \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathcal{C}_S & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_D \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ -\mathcal{E}_{DB} & I \end{bmatrix}, \tag{3.3}$$

where

- $\mathcal{C}_D$ is a preconditioner for $\mathcal{K}_D$,
- $\mathcal{C}_S$ is a preconditioner for the Schur-complement $\mathcal{S}_B$ or the inexact Schur-complement

$$\mathcal{S}_B + \mathcal{T}_B := \mathcal{S}_B + (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{K}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) \quad \text{with} \quad \mathcal{H}_{DB} = \mathcal{K}_D^{-1}\mathcal{K}_{DB}, \tag{3.4}$$

- $\mathcal{E}_{DB}$ is the matrix representation of the extension operators $\mathfrak{E}_j$ acting from $\Gamma = \overline{\cup_{i=1}^s \partial\Omega_i} \backslash \partial\Omega$ into the interior of the domains Ω_j.

The following result is the key in order to analyze the preconditioner $\mathcal{C}$, see e.g. [13], [32].

Lemma 3.1. *Let $\mathcal{C}_S$ and $\mathcal{C}_D$ be spd preconditioners for $\mathcal{S}_B$ and $\mathcal{K}_D$, i.e.*

$$c_S\,(\mathcal{C}_S\underline{v},\underline{v}) \le (\mathcal{S}_B\underline{v},\underline{v}) \quad \forall \underline{v}, \tag{3.5}$$

$$c_I\,(\mathcal{C}_D\underline{v},\underline{v}) \le (\mathcal{K}_D\underline{v},\underline{v}) \le C_I\,(\mathcal{C}_D\underline{v},\underline{v}) \quad \forall \underline{v}. \tag{3.6}$$

Moreover, let

$$\left(\mathcal{K}\begin{bmatrix} I \\ \mathcal{E}_{DB}\end{bmatrix}\underline{g}, \begin{bmatrix} I \\ \mathcal{E}_{DB}\end{bmatrix}\underline{g}\right) \le c_E^2\,(\mathcal{C}_S\underline{g},\underline{g}) \quad \forall \underline{g}. \tag{3.7}$$

Then, the inequalities

$$c\,(\mathcal{C}\underline{v},\underline{v}) \le (\mathcal{K}\underline{v},\underline{v}) \le C\,(\mathcal{C}\underline{v},\underline{v}) \quad \forall \underline{v}$$

hold with $c = \frac{c_I}{2}\min\{1, \frac{c_S}{c_I + c_E^2 - c_S}\}$ and $C = 2\max\{c_E^2, C_I\}$.

Proof. We start the proof with the following equation

$$\begin{bmatrix} I \\ \mathcal{E}_{DB}\end{bmatrix}^\top \mathcal{K}\begin{bmatrix} I \\ \mathcal{E}_{DB}\end{bmatrix} = \mathcal{S}_B + (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{K}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) \tag{3.8}$$

with the discrete harmonic extension $\mathcal{H}_{DB} = -\mathcal{K}_D^{-1}\mathcal{K}_{DB}$.

Next, we prove the upper estimate. First, we transform the inequality $\mathcal{K} \le C\mathcal{C}$ into an equivalent formulation. Using (3.1) and (3.3), we obtain

$$\begin{bmatrix} I & -\mathcal{H}_{DB}^\top \\ \mathbf{0} & I\end{bmatrix}\begin{bmatrix} \mathcal{S}_B & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_D\end{bmatrix}\begin{bmatrix} I & \mathbf{0} \\ -\mathcal{H}_{DB} & I\end{bmatrix} \le C\begin{bmatrix} I & -\mathcal{E}_{DB}^\top \\ \mathbf{0} & I\end{bmatrix}\begin{bmatrix} \mathcal{C}_S & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_D\end{bmatrix}\begin{bmatrix} I & \mathbf{0} \\ -\mathcal{E}_{DB} & I\end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} I & \mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top \\ \mathbf{0} & I\end{bmatrix}\begin{bmatrix} \mathcal{S}_B & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_D\end{bmatrix}\begin{bmatrix} I & \mathbf{0} \\ \mathcal{E}_{DB} - \mathcal{H}_{DB} & I\end{bmatrix} \le C\begin{bmatrix} \mathcal{C}_S & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_D\end{bmatrix}$$

and

$$\begin{bmatrix} \mathcal{S}_B + (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{K}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) & (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{K}_D \\ \mathcal{K}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) & \mathcal{K}_D\end{bmatrix} \le C\begin{bmatrix} \mathcal{C}_S & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_D\end{bmatrix}. \tag{3.9}$$

Both matrices are postive definite. Using the Cauchy-Schwarz inequality, (3.8), (3.7), (3.6), we can estimate

$$\begin{aligned}
&\begin{bmatrix} \mathcal{S}_B + (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{K}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) & (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{K}_D \\ \mathcal{K}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) & \mathcal{K}_D\end{bmatrix} \\
&\le 2\begin{bmatrix} \mathcal{S}_B + (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{K}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_D\end{bmatrix} \\
&= 2\left[\begin{bmatrix} I \\ \mathcal{E}_{DB}\end{bmatrix}^\top \mathcal{K}\begin{bmatrix} I \\ \mathcal{E}_{DB}\end{bmatrix} \quad \mathbf{0} \\ \mathbf{0} \quad \mathcal{K}_D\right] \le 2\max\{c_E^2, C_I\}\begin{bmatrix} \mathcal{C}_S & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_D\end{bmatrix}.
\end{aligned}$$

4.2 AF-FETI Solvers and Preconditioners

Throughout this subsection, we assume that $\mathcal{D}(x) = \alpha_i\mathcal{I}$ on Ω_i. We are interested in the robustness of our FETI preconditioners with respect to large jumps of these coefficients across the interfaces. We note that all results remain valid if the spectral condition number $\kappa(\mathcal{D}) = \lambda_{max}(\mathcal{D})/\lambda_{min}(\mathcal{D})$ of the diffusion matrix is small on Ω_i.

4.2.1 AF-FETI1 Solvers

Since the system matrix $\hat{\mathcal{F}}$ of the AF-FETI1 system (4.16) is symmetric and positive definite (SPD), it can efficiently be solved with the help of the PCG method with an appropriate FETI preconditioner $\hat{\mathcal{C}}_F$. It is clear that we only have to construct preconditioner $\mathcal{C}_F$ for $\mathcal{F}$ on the subspace $\mathbf{\Lambda}_0 = \ker\mathcal{G}^\top = (\text{range}\,\mathcal{G})^\perp$. Candidates for $\mathcal{C}_F$ are the following preconditioners:

- scaled exact Dirichlet FETI preconditioner (see, e.g., [36])

$$\mathcal{C}_F^{-1} = \mathcal{A}\mathcal{S}_B\mathcal{A}^\top, \quad \mathcal{S}_B = \text{diag}(\widetilde{\mathcal{S}}_{B,i}), \tag{4.21}$$

- scaled inexact Dirichlet FETI preconditioner

$$\mathcal{C}_F^{-1} = \mathcal{A}(\mathcal{S}_B + \mathcal{T}_B)\mathcal{A}^\top, \quad \mathcal{S}_B + \mathcal{T}_B = \text{diag}(\widetilde{\mathcal{S}}_{B,i} + \mathcal{T}_{B,i}) \tag{4.22}$$

  with the disturbed Schur complements $\mathcal{T}_B$, see (3.4),

- scaled data-sparse hypersingular BETI preconditioner

$$\mathcal{C}_F^{-1} = \mathcal{A}\mathcal{W}\mathcal{A}^\top, \quad \mathcal{W} = \text{diag}(\widetilde{\mathcal{W}}_{B,i}), \tag{4.23}$$

  with the matrices $\widetilde{\mathcal{W}}_{B,i}$ arising from an appropriate data-sparse approximation of the hypersingular operator on $\partial\Omega_i$ [24].

The matrix

$$\mathcal{A} = (\mathcal{B}\tilde{\mathcal{D}}^{-1}\mathcal{B}^\top)^{-1}\mathcal{B}\tilde{\mathcal{D}}^{-1} \tag{4.24}$$

is defined by the interconnecting matrix $\mathcal{B} = \text{diag}(\mathcal{B}_{B,i})$ and by the scaling matrix $\tilde{\mathcal{D}} = \text{diag}(\tilde{\mathcal{D}}_{B,i})$ with appropriate diagonal matrices $\tilde{\mathcal{D}}_{B,i}$. The diagonal entries depend on the coefficients α_i, see [20], [4], [36] for further details.

Theorem 4.2. *Let $\mathcal{C}_F$ be one of the FETI preconditioners defined by (4.21), (4.22), or (4.23). Then, there exist positive constants $\underline{c}_F$ and $\overline{c}_F$ such that the spectral inequalities*

$$\underline{c}_F\,(\mathcal{C}_F\underline{\lambda},\underline{\lambda}) \le (\mathcal{F}\underline{\lambda},\underline{\lambda}) \le \overline{c}_F\,(1 + \log(H/h))^2\,(\mathcal{C}_F\underline{\lambda},\underline{\lambda}) \tag{4.25}$$

hold for all $\underline{\lambda} \in \mathbf{\Lambda}_0 = ker\mathcal{G}^\top$, where the constant $\underline{c}_F$ and $\overline{c}_F$ are independent of h_i, H_i, and the α_i's (coefficient jumps), $H/h = \max H_i/h_i$.

Proof. By Theorem 3.13, the matrices $\widetilde{\mathcal{S}}_{B,i}$, $\widetilde{\mathcal{S}}_{B,i} + \mathcal{T}_{B,i}$ and $\widetilde{\mathcal{W}}_{B,i}$ are spectrally equivalent to an auxiliary Schur complement $\widetilde{\mathcal{S}}_{B,i}^{FEM}$ arising from a standard finite element discretization with linear triangular element on an auxiliary quasi-uniform triangular mesh generated from the given quasi-uniform boundary mesh on $\partial\Omega_i$ with the average mesh size h [24]. Since the

Remark 4.1. *We mention that the iteration updates for $\underline{\lambda}$ completely life in the subspace $\mathbf{\Lambda}_0$ if the initial guess is chosen as $\underline{\lambda}_e$. Therefore, a basis of $\mathbf{\Lambda}_0$, forming the columns of $\mathcal{L}_0$, is not explicitly required in the computation. We further mention that in the case of large jumps in the coefficients of the PDE the scalar product in $\mathbf{\Lambda}$ has to be changed according to the proposal made in [20], see also [4] and [36]. Of course, the change of the scalar product changes the orthoprojection $\mathcal{P}$, too.*

Eliminating the unknowns $\underline{v}_{B,1}, \ldots, \underline{v}_{B,s}$ from the AF-FETI2 system (4.14), we get the FETI Schur complement problem

$$\hat{\mathcal{F}}\underline{\lambda}_0 = \hat{\underline{d}}_0 \tag{4.16}$$

with the FETI Schur complement

$$\hat{\mathcal{F}} = \mathcal{L}_0^\top \mathcal{P}\mathcal{F}\mathcal{P}^\top \mathcal{L}_0 \quad \text{with} \quad \mathcal{F} = \sum_{i=1}^{s} \mathcal{B}_{B,i}\widetilde{\mathcal{S}}_{B,i}^{-1}\mathcal{B}_{B,i}^\top \tag{4.17}$$

and the corresponding right-hand side

$$\hat{\underline{d}}_0 = \mathcal{L}_0^\top \mathcal{P}\underline{d}_0 - \mathcal{L}_0^\top \mathcal{P}\sum_{i=1}^{s} \mathcal{B}_{B,i}\widetilde{\mathcal{S}}_{B,i}^{-1}\underline{d}_{B,i}. \tag{4.18}$$

The symmetric and positive definite system (4.16) is called AF-FETI1 system.

On the other hand, we can again unfold the AF-FETI2 system (4.14) arriving at a saddle point system that is similar to the saddle point system (4.3) but now with regular diagonal blocks

$$\widetilde{\mathcal{K}}_i = \begin{bmatrix} \widetilde{\mathcal{K}}_{B,i} & \mathcal{K}_{BD,i} \\ \mathcal{K}_{DB,i} & \mathcal{K}_{D,i} \end{bmatrix}, \tag{4.19}$$

with the regularized matrices $\widetilde{\mathcal{K}}_{B,i} = \mathcal{K}_{B,i} + \alpha_i \underline{1}_{B,i}\underline{1}_{B,i}^\top$. Moreover, let

$$\widetilde{\mathcal{B}}_i = \begin{pmatrix} \mathcal{L}_0^\top \mathcal{P}\mathcal{B}_{B,i} & 0 \end{pmatrix}, \quad \widetilde{\underline{u}}_i = \begin{pmatrix} \underline{v}_{B,i} \\ \underline{u}_{D,i} \end{pmatrix}, \quad \text{and} \quad \widetilde{\underline{f}}_i = \begin{pmatrix} \underline{f}_{B,i} - \mathcal{B}_{B,1}^\top \underline{\lambda}_e \\ \underline{f}_{D,i} \end{pmatrix}.$$

Then, the resulting system called AF-FETI3 system can be written in the form:

$$\begin{pmatrix} \widetilde{\mathcal{K}}_1 & \mathbf{0} & \ldots & \mathbf{0} & \widetilde{\mathcal{B}}_1^\top \\ \mathbf{0} & \widetilde{\mathcal{K}}_2 & & \vdots & \widetilde{\mathcal{B}}_2^\top \\ \vdots & & \ddots & \mathbf{0} & \vdots \\ \mathbf{0} & \ldots & \mathbf{0} & \widetilde{\mathcal{K}}_s & \widetilde{\mathcal{B}}_s^\top \\ \widetilde{\mathcal{B}}_1 & \ldots & & \widetilde{\mathcal{B}}_s & \mathbf{0} \end{pmatrix} \begin{pmatrix} \widetilde{\underline{u}}_1 \\ \vdots \\ \vdots \\ \widetilde{\underline{u}}_s \\ \underline{\lambda}_0 \end{pmatrix} = \begin{pmatrix} \widetilde{\underline{f}}_1 \\ \vdots \\ \vdots \\ \widetilde{\underline{f}}_s \\ \underline{0} \end{pmatrix}. \tag{4.20}$$

Let us mention that all AF-FETI systems derived in this subsection are equivalent. In the following subsection we propose and analyze fast and robust solvers for the AF-FETI1–AF-FETI3 systems (4.16), (4.14), and (4.20).

Using (3.9), we can conclude that $\mathcal{K} \le C\mathcal{C}$ with $C = 2\max\{c_E^2, C_I\}$.

Finally, we prove the lower estimate. Using similar arguments as for the proof of the upper estimate, we can show that the assertion $c\mathcal{C} \le \mathcal{K}$ is equivalent to

$$c \begin{bmatrix} \mathcal{C}_S + (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{C}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) & -(\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{C}_D \\ -\mathcal{C}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) & \mathcal{C}_D \end{bmatrix} \le \begin{bmatrix} \mathcal{S}_B & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_D \end{bmatrix},$$

see (3.9). The positive definitness of the above matrices implies

$$\begin{aligned} &\begin{bmatrix} \mathcal{C}_S + (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{C}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) & -(\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{C}_D \\ -\mathcal{C}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) & \mathcal{C}_D \end{bmatrix} \\ &\qquad \le 2 \begin{bmatrix} \mathcal{C}_S + (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{C}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_D \end{bmatrix}. \end{aligned}$$

Using (3.6), (3.8), (3.7) and (3.5), we can estimate

$$\begin{aligned} c_I\left(\mathcal{C}_S + (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{C}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}\right) &\le c_I\mathcal{C}_S + (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{K}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB}) \\ &= c_I\mathcal{C}_S + \begin{bmatrix} I \\ \mathcal{E}_{DB} \end{bmatrix}^\top \mathcal{K} \begin{bmatrix} I \\ \mathcal{E}_{DB} \end{bmatrix} - \mathcal{S}_B \\ &\le c_I\mathcal{C}_S + c_E^2\mathcal{C}_S - \mathcal{S}_B \\ &\le \frac{c_I + c_E^2 - c_S}{c_S}\mathcal{S}_B. \end{aligned}$$

Due to the postive definitness of $\mathcal{C}_S + (\mathcal{E}_{DB}^\top - \mathcal{H}_{DB}^\top)\mathcal{C}_D(\mathcal{E}_{DB} - \mathcal{H}_{DB})$, we have $c_I + c_E^2 - c_S > 0$. Together with $c_I\mathcal{C}_D \le \mathcal{K}_D$, we can conclude that $\mathcal{C} \le c\mathcal{K}$ with $c = \frac{c_I}{2}\min\left\{1, \frac{c_S}{c_I + c_E^2 - c_S}\right\}$. □

Remark 3.2. *A direct consequence of* (3.7) *is the spectrally equivalence of the Schur-complement $\mathcal{S}_B$ and the disturbed Schur-complement $\mathcal{S}_B + \mathcal{T}_B$, see [13].*

Summarizing, we have to find the preconditioners $\mathcal{C}_D$, $\mathcal{C}_S$ and the extension $\mathfrak{C} \leftrightarrow \mathcal{E}_{BD}$ in order to derive a good preconditioner $\mathcal{C}$.

For an inexact FETI preconditioner, a preconditioner $\mathcal{C}_i$ for $\mathcal{K}_i$ (cf. (2.9)), i.e. for the Neumann problem on domain Ω_i, is required. Due to (2.7), the block partitioning for $\mathcal{K}$ in (3.1), can be done for the subdomain stiffness matrices, too. A simple computation shows

$$\mathcal{K}_i = \begin{bmatrix} I & \mathcal{K}_{B,D,i}\mathcal{K}_{D,i}^{-1} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathcal{S}_{B,i} & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_{D,i} \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathcal{K}_{D,i}^{-1}\mathcal{K}_{D,B,i} & I \end{bmatrix} \tag{3.10}$$

with the local Schur-complement

$$\mathcal{S}_{B,i} = \mathcal{K}_{B,i} - \mathcal{K}_{B,D,i}\mathcal{K}_{D,i}^{-1}\mathcal{K}_{D,B,i}. \tag{3.11}$$

Then, the global Schur complement can be computed via the local Schur-complements, i.e.

$$\mathcal{S}_B = \sum_{i=1}^{s} R_{B,i}\mathcal{S}_{B,i}R_{B,i}^\top \tag{3.12}$$

with the assembling matrices $R_{B,i}$ (2.7). Using the DD-approach applied to $\mathcal{K}_i$ instead of $\mathcal{K}$ this preconditioner will be developed in Section 3.

3.2 An Overlapping Preconditioner for the Dirichlet Subproblems

In this subsection, we present the preconditioner for the matrix $\mathcal{K}_D$. Here, we use an overlapping preconditioner which has been developed in [29], see also [8], [33]. The preconditioner is based on the additive Schwarz framework (see, e.g., [36]). We decompose the *hp*-FEM space $\mathbb{S}^{\mathbf{p}}(\Omega_i, \tau_i) = \sum_{i=0}^{K} \mathbb{V}_i$ for subspaces $\mathbb{V}_j$ that will be specified below. This splitting defines an ASM-preconditioner which eliminates all high order degrees of freedom and, moreover, the properties of this preconditioner can be expressed by the two numbers C_0 and $\rho(\epsilon)$ defined below. As a well-known fact from the ASM-theory the condition number of the preconditioned system is bounded by $C_0^2(1+\rho(\epsilon))$.

Since $\mathcal{K}_D = \text{blockdiag}\,[\mathcal{K}_{D,i}]_i$ it is possible to restrict ourselves to the case of one subdomain Ω_i.

Lemma 3.3 (Clement interpolation)**.** *Let τ be a γ-shape-regular affine triangulation. For $K \in \tau$ define the patch*

$$\omega_K := \cup\{K' \in \tau \mid \overline{K} \cap \overline{K'} \neq \emptyset\}.$$

Then there exists $C > 0$ depending solely on γ and a linear operator $I^C : H^1(\Omega) \mapsto \mathbb{S}^{\mathbf{1}}(\tau)$ such that

$$\|u - I^c u\|_{L^2(K)} \leq C h_K \|\nabla u\|_{L^2(\omega_K)} \quad \text{and} \quad \|\nabla I^c u\|_{L^2(K)} \leq C \|\nabla u\|_{L^2(\omega_K)}.$$

Proof. The proof is given in [9]. □

Lemma 3.4. *Let $\hat{K}$ be the reference triangle and $\mathbf{p}(K) = (p_{e_1}, p_{e_2}, p_{e_3}, p_K)$ with $p_{e_i} \leq p_K$, $i = 1, \ldots, 3$, a polynomial degree distribution. Let i_p be the one-dimensional Gauss-Lobatto interpolation operator. Then there exists a constant $C > 0$ and a linear operator*

$$I_{\mathbf{p}(\mathbf{K})} : \mathcal{P}_{2p_K}(\hat{K}) \mapsto \mathcal{P}_{\mathbf{p}}(\hat{K}) \tag{3.13}$$

such that

- $I_{\mathbf{p}} u = u$ *for all* $u \in \mathcal{P}_{\mathbf{p}}(\hat{K})$,
- $\|I_{\mathbf{p}} u\|_{H^1(\hat{K})} \leq C \|u\|_{H^1(\hat{K})}$ *for all* $u \in \mathcal{P}_{2p_K}(\hat{K})$,
- $(I_{\mathbf{p}} u)|_{\Gamma_i} = i_{p_{e_i}, \Gamma_i} u$ *for* $i = 1, \ldots, 3$.

Proof. The proof is given in [9, Theorem 4.4]. □

Definition 3.5. *Let τ be a γ-shape-regular triangulation with polynomial degree distribution $\mathbf{p} = (p_K)_{K \in \tau}$. For each $K \in \tau$ let the operator $I_{\mathbf{p}(\mathbf{K})}$ be given by (3.13). Then we define*

$$I_{\mathbf{p}} : \mathbb{S}^{2\mathbf{p}}(\tau) \mapsto \mathbb{S}^{\mathbf{p}}(\tau) \qquad \text{by} \qquad (I_{\mathbf{p}} u)|_K = (I_{\mathbf{p}}(u|_K \circ F_K)) \circ F_K^{-1}.$$

For a finite element mesh τ, we denote the set of its vertices by $V(\tau)$ and the polynomial degree vector by $\mathbf{p} = (p_K)_{K \in \tau}$. In order to define our ASM-preconditioner we associate with each vertex v_j a patch

$$\omega_j := \cup\{K \in \tau \mid v_j \text{ is a vertex of } K\}$$

and decompose the space $\mathbb{S}^{\mathbf{p}}(\tau)$ into the following subspaces:

on $H^1(\Omega_i) \times H^1(\Omega_i)$ with a positive regularization parameter α_i, e.g. $\alpha_i = h_i^2$. Taking into account the solvability conditions for local Neumann problems

$$(\underline{g}_{B,i} - \mathcal{B}_{B,i}^\top \lambda, \underline{1}_{B,i}) = 0 \tag{4.9}$$

and the representation of the vectors $\underline{u}_{B,i}$ as

$$\underline{u}_{B,i} = \underline{v}_{B,i} + \gamma_i \underline{1}_{B,i} \quad \text{with } (\underline{v}_{B,i}, \underline{1}_{B,i}) = 0, \tag{4.10}$$

we get the two–fold saddle point problem

$$\begin{pmatrix} \widetilde{\mathcal{S}}_{B,1} & & & \mathcal{B}_{B,1}^\top & \mathbf{0} \\ & \ddots & & \vdots & \\ & & \widetilde{\mathcal{S}}_{B,s} & \mathcal{B}_{B,s}^\top & \mathbf{0} \\ \mathcal{B}_{B,1} & \ldots & \mathcal{B}_{B,s} & \mathbf{0} & \mathcal{G} \\ \mathbf{0} & \ldots & \mathbf{0} & \mathcal{G}^\top & \mathbf{0} \end{pmatrix} \begin{pmatrix} \underline{v}_{B,1} \\ \vdots \\ \underline{v}_{B,s} \\ \underline{\lambda} \\ \underline{\gamma} \end{pmatrix} = \begin{pmatrix} \underline{g}_{B,1} \\ \vdots \\ \underline{g}_{B,s} \\ \underline{0} \\ \underline{e} \end{pmatrix} \tag{4.11}$$

with the notations $\underline{\gamma} = (\gamma_i)_{i=1,\ldots,s} \in \mathbb{R}^s$, $\mathcal{G} = (\mathcal{B}_{B,1}\underline{1}_{B,1}, \ldots, \mathcal{B}_{B,s}\underline{1}_{B,s})$ and $\underline{e} = (e_i)_{i=1,\ldots,s} = (-(\underline{g}_{B,i}, \underline{1}_{B,1})_{i=1,\ldots,s} \in \mathbb{R}^s$. The solution $\underline{u}_{B,i}$ can be recovered via formula (4.10).

Applying the orthogonal projection

$$\mathcal{P} = \mathcal{I} - \mathcal{G}(\mathcal{G}^\top \mathcal{G})^{-1} \mathcal{G}^\top \tag{4.12}$$

from the space $\mathbf{\Lambda} := \mathbb{R}^L$ onto the subspace $\mathbf{\Lambda}_0 = \ker \mathcal{G}^\top = (\text{range}\,\mathcal{G})^\perp$ with respect to the scalar product $(\cdot,\cdot) = (\cdot,\cdot)_{\mathbf{\Lambda}} = (\cdot,\cdot)_{\mathbb{R}^L}$ to the last but one block equation of system (4.11), we can exclude $\underline{\gamma}$ from the first $s+1$ block equations of (4.11). Moreover, let us represent $\underline{\lambda}$ in the form

$$\underline{\lambda} = \mathcal{L}_0 \underline{\lambda}_0 + \underline{\lambda}_e \tag{4.13}$$

with known $\underline{\lambda}_e = \mathcal{G}(\mathcal{G}^\top \mathcal{G})^{-1} \underline{e} \in (\ker \mathcal{G}^\top)^\perp = \text{range}\,\mathcal{G}$, fulfilling the constraints $\mathcal{G}^\top \underline{\lambda}_e = \underline{e}$, and unknown $\mathcal{L}_0 \underline{\lambda}_0 \in \ker \mathcal{G}^\top$, i.e. $\mathcal{G}^\top \mathcal{L}_0 \underline{\lambda}_0 = \underline{0}$, where $\underline{\lambda}_0 \in \mathbb{R}^{L_0}$ and $L_0 = \dim \Lambda_0$. The columns of $\mathcal{L}_0$ span a basis of Λ_0. Now we can define $\underline{v}_{B,1}, \ldots, \underline{v}_{B,s}$ and $\underline{\lambda}_0$ from the one–fold saddle point problem

$$\begin{pmatrix} \widetilde{\mathcal{S}}_{B,1} & & & \mathcal{B}_{B,1}^\top \mathcal{P}^\top \mathcal{L}_0 \\ & \ddots & & \vdots \\ & & \widetilde{\mathcal{S}}_{B,s} & \mathcal{B}_{B,s}^\top \mathcal{P}^\top \mathcal{L}_0 \\ \mathcal{L}_0^\top \mathcal{P} \mathcal{B}_{B,1} & \ldots & \mathcal{L}_0^\top \mathcal{P} \mathcal{B}_{B,s} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \underline{v}_{B,1} \\ \vdots \\ \underline{v}_{B,s} \\ \underline{\lambda}_0 \end{pmatrix} = \begin{pmatrix} \underline{d}_{B,1} \\ \vdots \\ \underline{d}_{B,s} \\ \underline{0} \end{pmatrix}, \tag{4.14}$$

where $\underline{d}_{B,i} = \underline{g}_{B,i} - \mathcal{B}_{1,B}^\top \underline{\lambda}_e$. Once the vectors $\underline{v}_{B,1}, \ldots, \underline{v}_{B,s}$ and $\underline{\lambda}_0$ are defined from (4.14), we get $\underline{\lambda}$ from (4.13), $\underline{\gamma}$ from the last but one block equation of system (4.11), i.e.

$$\underline{\gamma} = -(\mathcal{G}^\top \mathcal{G})^{-1} \mathcal{G}^\top (\mathcal{B}_{B,1} \underline{v}_{B,1} + \cdots + \mathcal{B}_{B,s} \underline{v}_{B,s}), \tag{4.15}$$

and, finally, $\underline{u}_{B,1}, \ldots, \underline{u}_{B,s}$ from (4.10). We will call the one–fold saddle point problem (4.14) AF-FETI2 system.

interconnecting the local potential vectors across the subdomain boundaries and the nodal parameters on the boundary Γ_D with the corresponding Dirichlet data which are zero for our model problem. Each row of the matrix $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_s) = ((\mathcal{B}_{1,B}, 0), \dots, (\mathcal{B}_{s,B}, 0))$ is connected with a pair of matching nodes across the subdomain boundaries or with a Dirichlet node. The entries of the former rows are 1 and -1 for the indices corresponding to the matching nodes on the interface (coupling boundaries) $\Gamma_C = \Gamma_S \setminus \Gamma_D$ and 0 otherwise, whereas a entry corresponding to a Dirichlet node on Γ_D is 1 and again 0 otherwise. Therefore, (4.2) implies that the corresponding finite element functions $u_{i,h}$ are continuous across the interface Γ_C, i.e. $u_{i,h} = u_{j,h}$ on $\Gamma_i \cap \Gamma_j \neq \emptyset$, and are vanishing on $\Gamma_i \cap \Gamma_D$. We assume here that the number of constraints at some matching node is equal to the number of matching subdomains minus one. This method of a minimal number of constraints respectively multipliers is called non–redundant, see, e.g., [20] for the use of redundant constraints.

By introducing Lagrange multipliers $\underline{\lambda} \in \mathbb{R}^L$, the linear system (2.8) is equivalent to the following extended system

$$\begin{pmatrix} \mathcal{K}_1 & & & \mathcal{B}_1^\top \\ & \ddots & & \vdots \\ & & \mathcal{K}_s & \mathcal{B}_s^\top \\ \mathcal{B}_1 & \dots & \mathcal{B}_s & \mathbf{0} \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_s \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_s \\ \underline{0} \end{pmatrix}, \tag{4.3}$$

where now all matrices $\mathcal{K}_i$ are corresponding to local Neumann problems given by the Neumann bilinear form $\int_{\Omega_i} \nabla u \cdot \mathcal{D} \nabla v \mathrm{d}\Omega$ on $H^1(\Omega_i)$. Due to the choice of the matrices, all matrices $\mathcal{K}_i$ are singular with kernels spanned by the vector $\left[\underline{1}_{i,B}^\top, \underline{1}_{i,L}^\top, \underline{0}_{i,H}^\top\right]^\top$ as was already mentioned in Subsection 3.4.

Eliminating all interior unknowns $\underline{u}_{i,D}$, $i = 1, \dots, s$, from system (4.3), we arrive at the equivalent reduced system

$$\begin{pmatrix} \mathcal{S}_{B,1} & & & \mathcal{B}_{B,1}^\top \\ & \ddots & & \vdots \\ & & \mathcal{S}_{B,s} & \mathcal{B}_{B,s}^\top \\ \mathcal{B}_{B,1} & \dots & \mathcal{B}_{B,s} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \underline{u}_{B,1} \\ \vdots \\ \underline{u}_{B,s} \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{g}_{B,1} \\ \vdots \\ \underline{g}_{B,s} \\ \underline{0} \end{pmatrix}, \tag{4.4}$$

with the singular Schur complements

$$\mathcal{S}_{B,i} = \mathcal{K}_{B,i} - \mathcal{K}_{BD,i} \mathcal{K}_{D,i}^{-1} \mathcal{K}_{DB,i}, \quad i = 1, \dots, s, \tag{4.5}$$

and the corresponding right-hand sides

$$\underline{g}_{B,i} = \underline{f}_{B,i} - \mathcal{K}_{BD,i} \mathcal{K}_{D,i}^{-1} \underline{f}_{D,i}, \quad i = 1, \dots, s. \tag{4.6}$$

The kernels of the matrices $\mathcal{S}_{B,i}$ are spanned by the vector $\underline{1}_{B,i}$. We now replace the singular Neumann Schur complements $\mathcal{S}_{B,i}$ by the regularized matrices

$$\widetilde{\mathcal{S}}_{B,i} = \mathcal{S}_{B,i} + \alpha_i \underline{1}_{B,i} \underline{1}_{B,i}^\top, \quad i = 1, \dots, s. \tag{4.7}$$

This corresponds to the regularized Neumann bilinear form

$$\int_{\Omega_i} \nabla u(x) \cdot \mathcal{D}(x) \nabla v(x)\, dx + \left(\int_{\Gamma_i} u(x)\, ds \right) \left(\int_{\Gamma_i} v(x)\, ds \right) \tag{4.8}$$

1. The space of piecewise linear polynomials $\mathbb{V}_0 := \mathbb{S}^{\mathbf{1}}(\tau)$.
2. Local higher dimensional spaces $\mathbb{V}_j := \{u \in \mathbb{S}^{\mathbf{p}}(\tau) \mid \operatorname{supp} u \subset \overline{\omega}_j\}$.

To analyze the properties of the decomposition of $\mathbb{S}^{\mathbf{p}}(\tau)$, we introduce C_0 via:

$$\min \left\{ \sum_{j=0}^{\#V(\tau)} \|u_j\|_A^2 \;\middle|\; u = u_0 + \sum_{j=1}^{\#V(\tau)} u_j, \;\; u_j \in \mathbb{V}_j \right\} \leq C_0^2 \|u\|_A^2, \tag{3.14}$$

and define a symmetric matrix $\epsilon \in \mathbb{R}^{(\#V(\tau)) \times (\#V(\tau))}$ with entries $\epsilon_{ik} = 0$ if ω_i and ω_k are disjoint and $\epsilon_{ik} = 1$ otherwise. Thus

$$|a(u_i, u_k)| \leq \epsilon_{ik} \|u_i\|_A \|v_k\|_A \qquad \forall u_i \in \mathbb{V}_i,\ \forall u_k \in \mathbb{V}_k. \tag{3.15}$$

Theorem 3.6. *Let τ be a γ-shape regular triangulation and $\mathbf{p}$ a corresponding polynomial degree distribution. Let*

$$\mathbb{S}^{\mathbf{p}}(\tau) = \mathbb{V}_0 + \sum_{j=1}^{\#V(\tau)} \mathbb{V}_j$$

and let $a(\cdot,\cdot) : H^1 \times H^1 \mapsto \mathbb{R}$ symmetric positive definite. Then there exists a constant $C > 0$ depending solely on the bilinear form $a(\cdot,\cdot)$ and γ, such that

$$C_0 \leq C, \qquad \rho(\epsilon) \leq C$$

where C_0 and ϵ are from (3.14), (3.15), respectively.

Proof. Since there exists $M > 0$ depending only on γ, such that

$$\#\{v' \in V \mid \overline{\omega}_v \cap \overline{\omega}_{v'} \neq \emptyset\} \leq M \qquad \forall v \in V$$

we have $\rho(\epsilon) \leq \|\epsilon\|_\infty \leq M$. In order to bound C_0, we exploit the properties of the operators I^C and $I_{\mathbf{p}}$ and decompose $u \in \mathbb{S}^{\mathbf{p}}(\tau)$ as follows

$$\begin{aligned} u &= I^C u + I_{\mathbf{p}}(u - I^C u) = I^C u + I_{\mathbf{p}} \left(\sum_{j=1}^{\#V(\tau)} \phi_j (u - I^C u) \right) \\ &= u_0 + \sum_{j=1}^{\#V(\tau)} u_j \end{aligned}$$

with

$$u_0 := I^c(u) \in \mathbb{V}_0, \qquad u_j := I_{\mathbf{p}}(\phi_j (u - I^C u)) \in \mathbb{V}_j.$$

Now we want to bound $\|u_0\|_A^2 + \sum_{j=1}^{\#V(\tau)} \|u_j\|_A^2$. By Lemma 3.3 and since $\|\cdot\|_A \sim \|\cdot\|_{H^1}$ we have

$$\|u_0\|_A^2 = \|I^c(u)\|^2 \leq C \|u\|_A^2.$$

Next we consider a fixed $j \in 1, ..., \#V(\tau)$ and a fixed $K \subset \omega_j$:

$$\|u_j\|_A^2 \leq C \|u_j\|_{H^1(K)}^2 \;=\; \left\| I_{\mathbf{p}}(\phi_j (u - I^C u)) \right\|_{H^1(K)}^2$$

Denoting $\tilde{u} := (u - I^C u)$, $\hat{u} := \tilde{u}|_K \circ F_K$, $\hat{\phi} := \phi_j|_K \circ F_K$ and applying Lemmas 3.4, 3.3 we get:

$$\begin{aligned}
\|u_j\|^2_{H^1(K)} &\leq C\left(h_K^2 \|I_{\mathbf{p}}(\hat{\phi}\hat{u})\|^2_{L^2(\hat{K})} + \|\nabla I_{\mathbf{p}}(\hat{\phi}\hat{u})\|^2_{L^2(\hat{K})}\right) \\
&\leq C\|I_{\mathbf{p}}(\hat{\phi}\hat{u})\|^2_{H^1(\hat{K})} \\
&\leq C\|\hat{\phi}\hat{u}\|^2_{H^1(\hat{K})} \leq C\|\hat{u}\|^2_{H^1(\hat{K})} \\
&\leq C\left(h_K^{-2}\|u - I^C u\|^2_{L^2(K)} + \|\nabla(u - I^C u)\|^2_{L^2(K)}\right) \\
&\leq C\|\nabla u\|^2_{L^2(\omega_j)}.
\end{aligned}$$

Thus, the relations

$$\|u_j\|^2_{H^1(\omega_j)} \leq C \sum_{K \subset \omega_j} \|\nabla u\|^2_{L^2(\omega_j)}$$

and

$$\sum_{j=1}^{\#V(\tau)} \|u_j\|^2_{H^1(\omega_j)} \leq C\|\nabla u\|^2_{L^2(\Omega)} \leq C\|u\|^2_A$$

follow. □

Now we are able to define the following preconditioner for $\mathcal{K}_{D,i}$. Let

$$\mathcal{C}_D^{-1} = \sum_{j=1}^{\#V(\tau)} Q_j^\top \mathcal{K}_{P,j}^{-1} Q_j + Q_0^\top (\mathcal{D}_L)^{-1} Q_0, \tag{3.16}$$

where

- $\mathcal{K}_{P,j}$ denotes the stiffness matrix restricted to V_j, $j = 1, \ldots, \#V(\tau)$,
- Q_j, $j = 0, \ldots, \#V(\tau)$ is the corresponding finite element restriction matrix onto V_j,
- $\mathcal{D}_L$ is the diagonal part of the matrix $\mathcal{K}_L$.

Theorem 3.7. *Let $\mathcal{C}_D$ be defined via* (3.16)*. Then, we have $\mathcal{C}_D \sim \mathcal{K}_D$.*

Proof. Due to [37] and [18], the condition number of $\mathcal{K}_L$ does not depend on the discretization parameter. This gives $\mathcal{D}_L \sim \mathcal{K}_L$. Now, the result is a consequence of Theorem 3.7. □

3.3 Schur complement preconditioner and extension operator

Next, we consider the preconditioner $\mathcal{C}_S$ for $\mathcal{S}_B$ (3.2). We will prove that the norm induced by the Schur complement $\mathcal{S}_B$ (3.2) is equivalent to a suitable seminorm. Therefore, the extension operators $\mathfrak{E}$ for the nearly discrete harmonic extension of a function $g \in H^{1/2}(\partial\tilde{\Omega})$ to a function $u = \mathfrak{E}g \in H^1(\tilde{\Omega})$ has to be investigated. We follow the approach of Haase and Nepomnyaschikh [16].

In a first step, we describe the extension of Haase and Nepomnyaschikh. Then, we prove that this type of extension can be used for boundary concentrated meshes with $p = 1$ or $p > 1$ defined via Definition 2.3, too. Finally, the equivalence of the Schur complement is shown.

3.5 Computational Aspects

In the previous subsection, we have proved the optimality of the condition number of $\mathcal{C}^{-1}\mathcal{K}$. For the design of an optimal solver, the operation $\mathcal{C}^{-1}\underline{r}$ should be performed in optimal arithmetical complexity. This is the purpose of this subsection.

The preconditioner $\mathcal{C}$ (3.29) consists of three ingredients, namely the preconditioner $\mathcal{C}_D$ for $\mathcal{K}_D$, the Schur-complement preconditioner $\mathcal{C}_S$ and the extension operator $\mathcal{E}_{DB}$.

For the preconditioning operation $\mathcal{C}_D^{-1}\underline{r}_D$ (3.16), we have to solve subproblems onto the high-order subspaces V_j, $j = 1, \ldots, n$. Those problems can be treated by direct solvers since $\dim(V_j) \preceq (1 + \log \frac{h_K}{h})^3$, see (2.5). On the subspace V_0, we have to multiply with a diagonal matrix. Hence, the cost for $\mathcal{C}_D^{-1}\underline{r}_D$ is $\mathcal{O}(N)$.

Furthermore, we need a good preconditioner C_S for $\mathcal{S}_B$. Due to Theorem 3.13, the norm induced by the Schur complement is equivalent to the $H^{1/2}$-norm on the skeleton. Hence, any preconditioner which is known from the h-version of the FEM using uniform refined meshes can be used as preconditioner for the Schur complements $\mathcal{S}_{B,i}$ or the assembled Schur complement $\mathcal{S}_B$ (see, e.g. [21] or [36]). Thus, nowadays many preconditioners $\mathcal{C}_{S,i}$ for $\mathcal{S}_{B,i}$ are available such that $\kappa(\mathcal{C}_{S,i}{}^{-\frac{1}{2}}\mathcal{S}_{B,i}\mathcal{C}_{S,i}{}^{-\frac{1}{2}}) = \mathcal{O}(1)$ and the solution of a system $\mathcal{C}_{S,i}\underline{w} = \underline{r}$ requires $\mathcal{O}(N_i)$ floating point operations. Similar results are valid for the assembled Schur complement $\mathcal{S}_B$. In our numerical experiments presented in Section 5 we use the so-called Schur complement BPX preconditioner proposed in [35].

The last ingredient is the matrix representation of the extension operator $\mathcal{E}_{DB}$. Due to Algorithm 3.8, the operations $\mathcal{E}_{DB}\underline{w}_B$ and $\mathcal{E}_{DB}^T\underline{w}_D$ involve only BPX-like basis transformations at the boundary or skeleton and on the interior, which are proportionally to the numbers of unknowns at the skeleton or the interior. Therefore both operations require $\mathcal{O}(N)$ floating point operations.

Summarizing, the total cost for $\mathcal{C}^{-1}\underline{r}$ is $\mathcal{O}(N)$ flops.

4 All-Floating Interface Concentrated Finite Element Tearing and Interconnecting Methods

4.1 AF-FETI Formulations

In order to avoid assembled matrices and vectors, we tear the global potential vector $\underline{u}$ on the subdomain boundaries Γ_i by introducing the individual local unknowns

$$\underline{u}_i = R_i \underline{u}. \tag{4.1}$$

In contrast to the preceding sections the vector $\underline{u}$ of nodal parameters now contains also those nodal parameters belonging to the Dirichlet boundary Γ_D that coincides with $\Gamma = \partial\Omega$ for our model problem (2.1). The global continuity of the potentials and the Dirichlet boundary conditions are now enforced by the constraints

$$\sum_{i=1}^{s} \mathcal{B}_i \underline{u}_i = \underline{0} \tag{4.2}$$

3.4 Final condition number estimates

Now, we are in the position to summarize the results of Subsections 3.2 and 3.3. In a first step, we propose the primal DD preconditioner for $\mathcal{K}_i$ (3.10). Let

$$\mathcal{C}_i = \begin{bmatrix} I & -\mathcal{E}_{DB,i}^\top \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathcal{C}_{S,i} & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_{D,i} \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ -\mathcal{E}_{DB,i} & I \end{bmatrix}, \tag{3.27}$$

where

- $\mathcal{C}_{D,i}$ is the preconditioner (3.16) for $\mathcal{K}_{D,i}$,
- $\mathcal{C}_{S,i}$ is a preconditioner for $\mathcal{S}_{B,i}$ with $\mathcal{C}_{S,i} \sim \mathcal{S}_{B,i}$,
- $\mathcal{E}_{DB,i}$ is the matrix representation of the extension operator $\mathfrak{E}_{L,i} : \partial\Omega_i \mapsto \Omega_i$ in Algorithm 3.8.

Note that the matrix $\mathcal{C}_i$ is positive definite if $\partial\Omega \cap \partial\Omega_i \neq \emptyset$. If $\partial\Omega \cap \partial\Omega_i = \emptyset$, this matrix is positive semidefinite with $\ker \mathcal{C}_i = \ker \mathcal{K}_i = [\underline{u}_B, \underline{u}_L, \underline{u}_H]^\top = \left[\underline{1}_{i,B}^\top, \underline{1}_{i,L}^\top, \underline{0}_{i,H}^\top\right]^\top$. The choice of the preconditioners $\mathcal{C}_{S,i}$ is specified in the next subsection.

Theorem 3.14. *Let $\mathcal{C}_i$ be defined via (3.27). Then, the spectral inequalities*

$$c_8 \left(\mathcal{C}_i \underline{v}, \underline{v}\right) \leq \left(\mathcal{K}_i \underline{v}, \underline{v}\right) \leq c_9 \left(\mathcal{C}_i \underline{v}, \underline{v}\right) \tag{3.28}$$

are valid for all $\underline{v} \in \mathbb{R}^{N_i}$, where the spectral constants do not depend on N_i.

Proof. We show that the assumptions of Lemma 3.1 are satisfied. Due to Theorem 3.7, we have $\mathcal{C}_{D,i} \sim \mathcal{K}_{D,i}$. Our assumptions imply the estimate $\mathcal{C}_{S,i} \sim \mathcal{S}_{B,i}$. Using Theorem 3.13, Theorem 3.11 and $|u|^2_{H^1(\Omega_i)} \sim (\mathcal{K}_i \underline{u}, \underline{u})$ for all $u = \Phi_i \underline{u} \in \mathbb{S}^{\mathbf{1}}_{L,i}$, the relation

$$\left(\mathcal{K}_i \begin{bmatrix} I \\ \mathcal{E}_{BD,i} \end{bmatrix} \underline{g}, \begin{bmatrix} I \\ \mathcal{E}_{BD,i} \end{bmatrix} \underline{g}\right) \leq c_E^2 \left(\mathcal{C}_{S,i} \underline{g}, \underline{g}\right) \quad \forall \underline{g}$$

holds with a constant, which is independent of N_i, for the matrix representation of the extension operator in Algorithm 3.8. □

Now, we will present the solver for the global stiffness matrix $\mathcal{K}$. Let

$$\mathcal{C} = \begin{bmatrix} I & -\mathcal{E}_{DB}^\top \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathcal{C}_S & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_D \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ -\mathcal{E}_{DB} & I \end{bmatrix}, \tag{3.29}$$

where

- $\mathcal{C}_D$ is the block diagonal matrix of the preconditioner (3.16) for $\mathcal{K}_{D,i}$, $i = 1, \ldots, s$,
- $\mathcal{C}_S$ is a preconditioner for $\mathcal{S}_B$, which satisfies $\mathcal{C}_S \sim \mathcal{S}_B$,
- $\mathcal{E}_{DB}$ is the matrix representation of the extension operators $\mathfrak{E}_{L,i} : \partial\Omega_i \mapsto \Omega_i$ in Algorithm 3.8.

Theorem 3.15. *Let $\mathcal{C}$ be defined via (3.29). Then,*

$$c_{10} \left(\mathcal{C} \underline{v}, \underline{v}\right) \leq \left(\mathcal{K} \underline{v}, \underline{v}\right) \leq c_{11} \left(\mathcal{C} \underline{v}, \underline{v}\right) \quad \forall \underline{v}.$$

The constants do not depend on N.

Proof. The proof is similar to the proof of Theorem 3.14. □

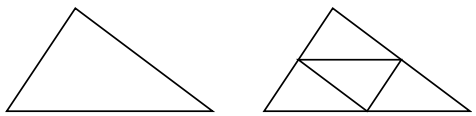

Figure 1: Refinement of each element of the mesh.

The technique of Haase and Nepomnyaschikh has originally been derived for the h-version of the finite element method with $p = 1$ using uniform refined meshes and uses the hierarchy of the nested finite element spaces. Let $\tau_0 \subset \tau_1 \subset \tau_2 \subset \ldots \subset \tau_L$ be a sequence of nested triangular ($d = 2$) or tetrahedral ($d = 3$) finite element meshes with nodes $x_i^{(l)}$, $i = 1, \ldots, N_l$, $l = 0, \ldots, L$, on a domain $\tilde{\Omega}$. Here τ_0 is the coarse mesh and τ_l is obtained from τ_{l-1} be subdividing each element of τ_{l-1} into 2^d new congruent elements, cf. Figure 1 for $d = 2$.

Due to Definition 2.4, $\mathbb{S}^{\mathbf{1}}_l$ denotes the space of all functions which are piecewise linear, linear on each element of τ_l and belong to $H^1(\tilde{\Omega})$. The corresponding trace space is denoted by $\mathbb{B}_l$, i.e. $\mathbb{B}_l = \mathbb{S}^{\mathbf{1}}_l \mid_{\partial\tilde{\Omega}}$. This space is spanned by nodal basis functions $\psi_{i,l}$, i.e.

$$\mathbb{B}_l = \operatorname{span}\{\psi_{i,l}\}_{i=1}^{N_l} = \operatorname{span} \Psi_l. \tag{3.17}$$

Moreover, let $\mathbb{W}_l^{(i)} = \operatorname{span}\{\psi_{i,l}\}$ be the one-dimensional subspace corresponding to the node $x_i^{(l)}$. Let $g \in \mathbb{B}_l \subset H^{1/2}(\partial\tilde{\Omega})$. In order to define a function $u = \mathfrak{E}g$ such that

$$\| u \|_{H^1(\tilde{\Omega})} \leq c \| g \|_{H^{1/2}(\partial\tilde{\Omega})}, \quad u \mid_{\partial\tilde{\Omega}} = g, \tag{3.18}$$

Haase/Nepomnyaschikh used a BPX-like decomposition of the function g. Let $Q_{i,l} : L_2(\partial\tilde{\Omega}) \mapsto \mathbb{W}_l^{(i)}$ be the L_2-like projection from $L_2(\partial\tilde{\Omega})$ onto $\mathbb{W}_l^{(i)}$, i.e.

$$Q_{i,l} g = d_{i,l}(g) \psi_{i,l} \quad \forall g \in \mathbb{B}_l,$$

where

$$d_{i,l} = \frac{(g, \psi_{i,l})_{L_2(\partial\tilde{\Omega})}}{(\psi_{i,l}, \mathbf{1})_{L_2(\partial\tilde{\Omega})}}, \quad \text{or,} \quad d_{i,l} = \frac{(g, \mathbf{1})_{L_2(\operatorname{supp}\psi_{i,l})}}{(\mathbf{1}, \mathbf{1})_{L_2(\operatorname{supp}\psi_{i,l})}}.$$

Moreover, let

$$Q_l = \sum_{i=1}^{N_l} Q_{i,l}, \quad l = 0, \ldots, L, \quad \text{and} \quad Q_{-1} = \mathbf{0}.$$

Now, the extension of a function $g \in \mathbb{B}_L$ into $\tilde{\Omega}$ can be defined.

Algorithm 3.8. *1. Let $\psi_l = (Q_l - Q_{l-1}) g \in \mathbb{B}_l$, $l = 0, \ldots, L$.*

2. Define the function $u \in \mathbb{S}^{\mathbf{1}}_l$ via

$$u_l(x_{i,0}) = \begin{cases} \psi_l(x_{i,0}) & \text{if } x_{i,0} \in \partial\tilde{\Omega} \\ \overline{g} & \text{if } x_{i,0} \notin \partial\tilde{\Omega} \end{cases}, \tag{3.19}$$

$$u_l(x_{i,l}) = \begin{cases} \psi_l(x_{i,l}) & \text{if } x_{i,l} \in \partial\tilde{\Omega} \\ 0 & \text{if } x_{i,l} \notin \partial\tilde{\Omega} \end{cases}. \tag{3.20}$$

3. Set $u = \mathfrak{E}_l g = u_0 + u_1 + \ldots + u_L$.

Remark 3.9. *Note that the function u_l in (3.19), (3.20) is uniquely defined by setting the function values in the nodes $x_i^{(l)}$ of the mesh τ_l. For $\overline{g}$, a coarse grid problem on the mesh τ_0 has to be solved. Alternatively, the mean value of g over $\partial\tilde{\Omega}$ can be taken.*

Now, the following result has been shown.

Theorem 3.10. *Let $\tau_0 \subset \ldots \subset \tau_L$ be a family of uniformly refined meshes and let $\mathbb{S}_l^{\mathbf{1}}$, $l = 0, \ldots, L$ be the space of the piecewise linear nodal basis functions on τ_l. Let $\mathfrak{E}_L : \mathbb{B}_L \mapsto \mathbb{S}_L^{\mathbf{1}}$ be the extension of Algorithm 3.8. Then, we have $\mathfrak{E}_L g \mid_{\partial\tilde{\Omega}} = g$. Moreover, there is a constant c_2 which is independent of L such that*

$$\| \mathfrak{E}_L g \|_{H^1(\tilde{\Omega})} \leq c_2 \| g \|_{H^{1/2}(\partial\tilde{\Omega})} \quad \forall g \in \mathbb{B}_L.$$

Proof. The proof was given in [16]. □

Next, we consider a family of boundary concentrated finite element meshes and the corresponding finite element spaces $\mathbb{S}_l$.

Theorem 3.11. *Let $\tau_0 \subset \ldots \subset \tau_L$ be a family of meshes on a domain $\tilde{\Omega}$, which are geometrically refined to the boundary. Let $\mathbb{S}_l$, $l = 0, \ldots, L$, be defined via Definition 2.4. Then, $\mathbb{S}_L \mid_{\partial\tilde{\Omega}} = \mathbb{B}_L$. Moreover, let $\mathfrak{E}_L : \mathbb{B}_L \mapsto \mathbb{S}_L$ be the extension operator of Algorithm 3.8. Then, there exists a constant c_2 which is independent of L (and N) such that*

$$\| \mathfrak{E}_L g \|_{H^1(\tilde{\Omega})} \leq c_2 \| g \|_{H^{1/2}(\partial\tilde{\Omega})} \quad \forall g \in \mathbb{B}_L. \tag{3.21}$$

Proof. Due to (2.5), the polynomial degree on the elements on $\partial\tilde{\Omega}$ is 1. Thus, $\mathbb{S}_L \mid_{\partial\tilde{\Omega}} = \mathbb{B}_L$. This proves the first assertion.

Due to (3.19) and (3.20), the image of extension operator described in Algorithm 3.8 for uniformly refined meshes belongs to the space $\mathbb{S}_L^{\mathbf{1}}$. Theorem 3.10 implies the estimate

$$\| \mathfrak{E}_L g \|_{H^1(\tilde{\Omega})} \leq c_2 \| g \|_{H^{1/2}(\partial\tilde{\Omega})} \quad \forall g \in \mathbb{B}_L$$

with the constant of Theorem 3.10. Since $\mathbb{S}_L^{\mathbf{1}} \subset \mathbb{S}_L^{\mathbf{P}}$, the second assertion follows. □

Remark 3.12. *If $\int_{\partial\tilde{\Omega}} g = 0$, or $g \mid_{\tilde{\Gamma}} = 0$ for $\tilde{\Gamma} \subset \partial\tilde{\Omega}$ with $meas(\tilde{\Gamma}) > 0$, relation (3.21) remains valid, if the seminorms are used instead of the seminorms in (3.21). This is a consequence of the Poincare-Friedrichs type inequalities, see [36, Lemma A17], or, [30].*

Hence, we have shown that the extension of Haase/Nepomnyaschikh for uniformly refined meshes can be used for geometrically refined meshes with grading factor $\frac{1}{2}$, too.

In a next theorem, we show that the norm induced by the Schur complement $\mathcal{S}_{B,i}$ (3.11) is equivalent to the $H^{1/2}(\partial\Omega_i)$ norm and that the norm induced by the Schur complement $\mathcal{S}_B$ (3.2) is equivalent to the following norm:

$$\| g \|^2_{H^{1/2}(\Gamma)} := \| g \|^2_{L^2(\Gamma)} + |g|^2_{H^{1/2}(\Gamma)}, \quad \text{where} \quad |g|^2_{H^{1/2}(\Gamma)} = \inf_{\substack{u \mid_\Gamma = g \\ u \mid_{\partial\Omega} = 0}} |u|^2_{H^1(\Omega)}.$$

Theorem 3.13. *Let $\tau_0^{(i)} \subset \ldots \subset \tau_L^{(i)}$ be a family of geometrically refined meshes on the domains Ω_i, $i = 1, \ldots, s$, and let $\mathbb{B}_{L,i}$ and $\mathbb{B}_{L,\Gamma}$ be the trace spaces of $\mathbb{S}_{L,i}$ and $\mathbb{V}_{L,0}$ onto $\partial\Omega_i$ and $\Gamma = \cup_{i=1}^s \partial\Omega_i \backslash \partial\Omega$, respectively. Let $\Phi_{B,i}$ and Φ_B be defined via (2.6) and (2.7), respectively.*

Moreover, let $\mathcal{S}_{B,i}$ be defined via (3.11). If $\partial\Omega \cap \partial\Omega_i \neq \emptyset$, we have

$$c_4 \, |g_i|^2_{H^{1/2}(\partial\Omega_i \cap \Gamma)} \leq (\mathcal{S}_{B,i}\underline{g}_i, \underline{g}_i) \leq c_5 \, |g_i|^2_{H^{1/2}(\partial\Omega_i \cap \Gamma)} \quad \forall g_i = \Phi_{B,i}\underline{g}_i \in \mathbb{B}_{L,i}, g \mid_{\partial\Omega} = 0. \tag{3.22}$$

If $\partial\Omega \cap \partial\Omega_i = \emptyset$, we have

$$c_4 \, |g_i|^2_{H^{1/2}(\partial\Omega_i \cap \Gamma)} \leq (\mathcal{S}_{B,i}\underline{g}_i, \underline{g}_i) \leq c_5 \, |g_i|^2_{H^{1/2}(\partial\Omega_i \cap \Gamma)} \quad \forall g_i = \Phi_{B,i}\underline{g}_i \in \mathbb{B}_{L,i}. \tag{3.23}$$

Moreover, the relation

$$c_6 \| g \|^2_{H^{1/2}(\Gamma)} \leq (\mathcal{S}_B \underline{g}, \underline{g}) \leq c_7 \| g \|^2_{H^{1/2}(\Gamma)} \quad \forall g = \Phi_B \underline{g} \in \mathbb{B}_{L,\Gamma} \tag{3.24}$$

holds. The constants c_4, c_5, c_6 and c_7 are independent on L and N.

Proof. The proof of (3.22) is adapted from [36], see also [1]. By the trace theorem and the Poincare Friedrichs inequality

$$\begin{aligned} c_{14} c_T^{-2} |g_i|^2_{H^{1/2}(\partial\Omega_i)} \leq c_{14} \| u \|^2_{H^1(\Omega_i)} &\leq c_{15} |u|^2_{H^1(\Omega_i)} \\ &\leq \min_{\underline{u}_D} (\mathcal{K}_i \underline{u}, \underline{u}) \\ &= (\mathcal{S}_{B,i}\underline{g}_i, \underline{g}_i), \quad \underline{u} = \begin{bmatrix} \underline{g}_i \\ \underline{u}_D \end{bmatrix}, \; u = \Phi_i \underline{u}. \end{aligned}$$

Due to Theorem 3.11 and Remark 3.12, there exists a function $u \in H^1(\Omega_i)$, $u \mid_{\partial\Omega_i} = g$ such that $|u|_{H^1(\Omega_i)} \leq c_2 |g|_{H^{1/2}(\partial\Omega_i)}$. Thus, we can conclude that

$$(\mathcal{S}_{B,i}\underline{g}_i, \underline{g}_i) \leq (\mathcal{K}_i \underline{u}, \underline{u}) \leq c_{12} |u|^2_{H^1(\Omega_i)} \leq c_{12} c_2^2 |g_i|^2_{H^{1/2}(\partial\Omega_i)}.$$

To prove (3.23), we have

$$c_4 \, |g_i|^2_{H^{1/2}(\partial\Omega_i \cap \Gamma)} \leq (\mathcal{S}_{B,i}\underline{g}_i, \underline{g}_i) \leq c_5 \, |g_i|^2_{H^{1/2}(\partial\Omega_i \cap \Gamma)} \quad \forall g_i = \Phi_{B,i}\underline{g}_i \in \mathbb{B}_{L,i}, \int_{\partial\Omega_i} g_i = 0 \tag{3.25}$$

by the same arguments.

Let $g_i \in \mathbb{B}_{L,i}$ with $\overline{g} = \int_{\partial\Omega} g_i$. Thus, one obtains $g_i = \overline{g} \cdot \mathbf{1} + g_{i,0}$ with $\int_{\partial\Omega} g_{i,0} = 0$. Since $|g_i|_{H^{1/2}(\partial\Omega_i)} = |g_{i,0}|_{H^{1/2}(\partial\Omega_i)}$, and $(\mathcal{S}_{B,i}\underline{g}_i, \underline{g}_i) = (\mathcal{S}_{B,i}\underline{g}_{0,i}, \underline{g}_{0,i})$ for $g_{i,0} = \Phi_{B,i}\underline{g}_i$, relation (3.23) is a consequence of (3.25).

To prove (3.24), we have

$$\| g \|^2_{H^{1/2}(\Gamma)} = \sum_{i=1}^s \| g \|^2_{H^{1/2}(\partial\Omega_i \cap \Gamma)} \, . \tag{3.26}$$

Since $meas(\partial\Omega) > 0$, the $H^{1/2}(\Gamma)$ seminorm is equivalent to the $H^{1/2}(\Gamma)$ norm. This is implied by the Poincare-Friedrichs inequality. Using (3.26), (3.12), (3.23) and (3.22), the assertion follows. □