ADVANCED DIFFERENTIAL GEOMETRY II - SPIN GEOMETRY

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Contents

0. Motivation

Let T be a free particle in $\mathbb{R}^3$. We want to study its motion in special relativity. If we denote its (relativistic) mass, Energy and momentum by m, E and p, respectively, then we have the relation

$$E = \sqrt{c^2 p^2 + m^2 c^4}\,, \tag{0.1}$$

where c denotes the speed of light.

Now we want to additionally study T quantum mechanically which means we have to describe T by a wave function $\psi = \psi_T : \mathbb{R} \times \mathbb{R}^3 \ni (t, x) \mapsto \psi(t, x) \in \mathbb{C}$. Here, the associated function $(t, x) \mapsto |\psi(t, x)|^2 \in \mathbb{R}$ is the density of the probability law that the particle T can be found at x at time t. The energy and momentum are no longer scalars associated with T but become unbounded operators acting on appropriate Hilbert spaces of wave functions,

$$\begin{aligned} E\psi &= ih\frac{\partial \psi}{\partial t}\,, \\ p\psi &= -ih \operatorname{grad} \psi\,. \end{aligned} \tag{0.2}$$

If one wants to combine the relativistic equation (0.1) with the quantum mechanical description (0.2), one concludes that wave functions must (formally) satisfy the equation

$$ih\frac{\partial \psi}{\partial t} = \sqrt{c^2 h^2 \Delta + m^2 c^4}\psi\,,$$

where Δ denotes the Laplacian $\Delta = -\sum_{i=1}^{3} \partial^2/\partial x_i^2$. *We thus face the problem of finding the square root of a second order differential operator*. Setting all constants to 1 (as mathematicians like to do), we specifically want to find the square root $D = \sqrt{\Delta}$ of the Laplacian. There are many ways in which this can be done, e.g., via the functional calculus, but for many reasons it is desirable that D be a differential operator itself. This means of course that D must be of first order. We take the ansatz

$$D = \sum_{i=1}^{3} \gamma_i \frac{\partial}{\partial x_i}\,.$$

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The requirement $D^2 = \Delta$ holds if and only if

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1 \qquad \text{and} \qquad \gamma_i\gamma_j + \gamma_j\gamma_i = 0\,, \qquad \text{for } i \neq j\,.$$

These equations do not posses a solution in $\mathbb{C}$. They do, however, if we allow the γ_i to be elements of some algebra. The smallest algebra that contains elements satisfying these relations is the one of complex 2×2-matrices. Specifically, the matrices

$$\gamma_1 = \begin{pmatrix} i & 0 \\ o & -i \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

do satisfy above equations. Now D becomes an operator acting on $\mathbb{C}^2$-valued functions, i.e. elements of $C^1(\mathbb{R}^3, \mathbb{C}^2)$, and the equation $D^2 = \Delta$ has to be understood component-wise.

This discussion was specific to $\mathbb{R}^3$. In the following lecture, we will learn how to define the Dirac operator D on (almost) any Riemannian manifold and study its basic properties.

1. Basics

1.1. Lie groups.

Definition 1.1. *A* Lie group *is a C^∞-manifold G which is also a group with the property that*

$$G \times G \ni (a,b) \mapsto a \cdot b \in G$$
$$G \ni a \mapsto a^{-1} \in G$$

are smooth.

Example 1.2. *(i)* $(\mathbb{R}^n, +)$, $(\mathbb{C}^n, +)$, $(\mathbb{C}\setminus\{0\} = \mathbb{C}^*, \cdot)$.

(ii) $(S^1 = \{e^{it} \mid t \in \mathbb{R}\} \subseteq \mathbb{C}^*, \cdot)$.

(iii) If G, H are Lie groups, then $G \times H$ is a Lie group with the product manifold and product group structure.

(iv) $(\mathrm{GL}(n;\mathbb{C}), \cdot)$ *since* $\mathrm{GL}(n;\mathbb{C})$ *is an open subset of* $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$ *and matrix multiplication and inversion are polynomials in the entries of matrices, hence smooth. More generally, for any finite-dimensional (unital) $\mathbb{R}$-algebra $\mathscr{A}$, the set $\mathscr{A}^*$ of units of $\mathscr{A}$ is canonically a Lie group.*

(v) Any subgroup / submanifold of any Lie group G which also happens to be a submanifold / subgroup. For $G = \mathrm{GL}(n;\mathbb{H})$ the most prominent examples are: $\mathrm{GL}(n;\mathbb{R})$, $\mathrm{SL}(n;\mathbb{C})$, $\mathrm{SL}(n;\mathbb{R})$, $\mathrm{U}(n)$, $\mathrm{O}(n)$, $\mathrm{Sp}(n)$, $\mathrm{SU}(n)$, $\mathrm{SO}(n)$. *The groups* $\mathrm{O}(n)$, $\mathrm{U}(n)$ *and* $\mathrm{Sp}(n)$ *are special cases of the following more general construction: Let $\mathbb{K}$ be either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and $s : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ a bi-/sesquilinear, nondegenerate (skew-)symmetric / (skew-)hermitian form. Then* $\mathrm{O}(s) := \{A \in M(n,n;\mathbb{K}) \mid s(AX, AY) = s(X,Y) \text{ for all } X, Y \in \mathbb{K}^n\}$ *is a Lie group.*

(vi) The Heisenberg group

$$H_{2n+1} := \left\{ \gamma(x,y,z) := \begin{pmatrix} 1 & x^t & z \\ 0 & E_n & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\} \subseteq \mathrm{GL}(n;\mathbb{R})\,.$$

As a manifold, H_{2n+1} is diffeomorphic to $\mathbb{R}^{2n+1}$. Group product and inversion are given by

$$\gamma(x,y,z) \cdot \gamma(u,v,w) = \gamma(x+u, y+v, z+w+\langle x, v\rangle_{\mathrm{Eucl}})\,,$$
$$\gamma(x,y,z)^{-1} = \gamma(-x,-y,-z+\langle x,y\rangle_{\mathrm{Eucl}})\,.$$

Definition 1.3. *(i) For $a \in G$ the map $L_a : G \ni b \mapsto a \cdot b \in G$ is called* left-translation by a. *L_a is a diffeomorphism with inverse $L_a^{-1} = L_{a^{-1}}$. Analogously, $R_a : G \ni b \mapsto b \cdot a \in G$* right-translation by a.

(ii) A vector field $X \in \mathcal{V}(G)$ is called left-invariant $:\Leftrightarrow$

$$\boxed{X \circ L_a = \mathrm{d}(L_a) \circ X \quad \forall a \in G\,,}$$

i.e., $X_{a\cdot b} = \mathrm{d}(L_a)_b X_b$ for all $a, b \in G$. In other words, X is L_a-related to itself for all $a \in G$.

Remark 1.4. *The space of left-invariant vector fields on G is canonically identified with T_eG, the tangent space to G at the identity:*

$$T_eG \ni X \mapsto (\text{vector field } \widetilde{X} \text{ given by } \widetilde{X}_a := \mathrm{d}(L_a)_e X_e)$$
$$T_eG \ni Y_e \leftarrow\!\shortmid Y \in \{\text{left-invariant vector fields on } G\}$$

These two maps are vector space isomorphisms and inverses of each other.

Lemma 1.5. *If X, Y are left-invariant vector fields on G, then $[X, Y]$ is again a left-invariant vector field.*

Proof. Let $a \in G$. Then X is L_a-related to itself, and so is Y. Hence, $[X, Y]$ is L_a-related to itself. □

***Corollary and Definition* 1.6.** *(i) A* Lie algebra over $\mathbb{R}$ *is a real vector space V together with a bilinear map $[\cdot,\cdot] : V \times V \to V$ which is alternating and satisfies the Jacobi identity, i.e., $[X, Y] = -[Y, X]$ and $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in V$.*

(ii) The vector space $\mathfrak{g}$ of left-invariant vector fields on G is by Lemma 1.5 a Lie algebra over $\mathbb{R}$.

Remark 1.7. *The tangent space T_eG is canonically identified with $\mathfrak{g}$ by Remark 1.4. This means that T_eG inherits a Lie algebra structure from $\mathfrak{g}$!*

Explicitely: *If $X, Y \in T_eG$, then $[X, Y] := [$left-inv. ext. $\widetilde{X}$ of X, left-inv. ext. $\widetilde{Y}$ of $Y]_e$.*

One often encouters the notation $\mathfrak{g} = T_eG$, which should always be understood in the above sense.

Lemma 1.8. *Let X be a left-invariant vector field on G and Φ_X^t its flow. If $\Phi_X^t(e)$ is defined for all $t \in (-\varepsilon, \varepsilon)$, then so is $\Phi_X^t(a)$, and we have*

$$\boxed{\Phi_X^t(a) = a \cdot \Phi_X^t(e)\,.}$$

Proof. We need to check that $t \mapsto a \cdot \Phi_X^t(e)$ is an integral curve of X starting in a. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(a \cdot \Phi_X^t(e)\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(L_a\Phi_X^t(e)\right) = d(L_a)_{\Phi_X^t(e)}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_X^t(e)\\ &= \mathrm{d}(L_a)_{\Phi_X^t(e)}X_{\Phi_X^t(e)} = X_{a\cdot\Phi_X^t(e)}\,,\end{aligned}$$

where the second equality follows from the chain rule and the last one from X being left-invariant. □

Corollary 1.9. *Any left-invariant vector field X on G is* complete, *i.e., $\Phi_X^t(a)$ is defined* for all *$t \in R$ and all $a \in G$.*

Proof. Let $\varepsilon > 0$ be as in Lemma 1.8 and let $a \in G$. Suppose that

$$t_0 := \sup\{t \mid \Phi_X^{\cdot}(a) \text{ is defined at least until } t\} < \infty\,.$$

Let $b := \Phi_X^{t_0-\varepsilon/2}(a)$. By the previous lemma, $\Phi_X^t(b)$ is defined at least for $t \in (-\varepsilon, t_0 + \varepsilon/2)$, which is a contradiction to our assumption $t_0 < \infty$. □

Definition 1.10. *(i) A* Lie group homomorphism *$f : G \to H$ is a smooth group homomorphism between Lie groups G and H.*

(ii) A (real / quaternionic) representation *is a Lie group homomorhism $f : G \to \mathrm{GL}(V)$, where V is a complex (real / quaternionic) vector space.*

(iii) A one-parameter subgroup in *G is a Lie group homomorphism $\alpha : (\mathbb{R}, +) \to G$, i.e., α is smooth and satisfies $\alpha(s + t) = \alpha(s) \cdot \alpha(t)$ for all $s, t \in \mathbb{R}$.*

Proposition 1.11. *The map $\{$1-parameter subgroups in $G\} \ni \alpha \mapsto \dot\alpha(0) \in T_eG$ is a bijection.*

Proof. Define

$$\begin{aligned}\Lambda : T_eG \cong \mathfrak{g} \ni X &\mapsto (t \mapsto \Phi_X^t(e)) \in \{\text{1-parameter subgroups in } G\}\\ T_eG \ni \dot\alpha(0) &\leftarrow\!\shortmid \alpha \in \{\text{1-parameter subgroups in } G\} : \Psi\,.\end{aligned}$$

- $\Psi \circ \Lambda = \mathrm{id}$: $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\Phi_X^t(e) = X_e$.
- $\Lambda \circ \Psi = \mathrm{id}$: We have to show that α is indeed the integral curve of the left-invariant vector field associated with $\dot\alpha(0)$:

$$\begin{aligned}\dot\alpha(t) &= \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}\alpha(t+s) = \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}\alpha(t)\cdot\alpha(s) = \mathrm{d}(L_{\alpha(t)})_e\dot\alpha(0)\\ &= (\text{left-invariant vector field associated with } \dot\alpha(0))_{\alpha(t)}\,.\end{aligned}$$

□

Notation 1.12. *The* Lie exponential map *$\mathrm{e}^{\cdot} : \mathfrak{g} \to G$ maps $X \in T_eG \cong \mathfrak{g}$ to $\mathrm{e}^X := \Phi_X^1(e)$. Thus, $t \mapsto \mathrm{e}^{tX} = \Phi_{tX}^1(e) = \Phi_X^t(e)$ is the 1-parameter subgroup in G associated with X as in Proposition 1.11.*

Proposition 1.13. *If $X, Y \in \mathfrak{g}$, then*

$$[X, Y]_e = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}\mathrm{e}^{tX}\mathrm{e}^{sY}\mathrm{e}^{-tX}\,.$$

Note that for fixed $t = t_0$, $s \mapsto e^{tX}e^{sX}e^{-tX}$ is a curve in G starting in $e \in G$, hence $t \mapsto \frac{d}{ds}_{|s=0} e^{tX}e^{sX}e^{-tX}$ is a curve in T_eG.

Proof. Denote by $\mathcal{L}$ the Lie derivative. By its definition, we have

$$[X,Y]_e = (\mathcal{L}_X Y)_e = \frac{d}{dt}_{|t=0} d\left(\Phi_X^{-t}\right)_{\Phi_X^t(e)} Y_{\Phi_X^t(e)} = \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} \Phi_X^{-t} \circ \Phi_Y^s \circ \Phi_X^t(e)\,.$$

By Lemma 1.8 we have

$$\begin{aligned}\Phi_X^{-t}(\Phi_Y^s(\Phi_X^t(e))) = \Phi_X^{-t}(\Phi_Y^s(e^{tX})) = \Phi_X^{-t}(\Phi_Y^s(e^{tX}\cdot e)) &= \Phi_X^{-t}(e^{tX}\cdot \Phi_Y^s(e)) \\ = e^{tX}\cdot \Phi_X^{-t}(e^{sY}) = e^{tX}\cdot e^{sY}\cdot e^{-tX}\,.&\end{aligned}$$

□

Example 1.14. *Let $G = \mathscr{A}^*$ for a finite-dimensional, associative unital $\mathbb{R}$-algebra $\mathscr{A}$ (e.g., $\mathscr{A} = M(n,n;\mathbb{C})$ with $\mathscr{A}^* = \mathrm{GL}(n;\mathbb{C})$ or $\mathscr{A} = \mathsf{End}(V)$ with $\mathscr{A}^* = \mathrm{GL}(V)$), $e = 1_{\mathscr{A}}$ the multiplicative identity in $\mathscr{A}$, $C \in T_eG = \mathscr{A}$, $A \in G$. Note that for small t, $1_{\mathscr{A}} + tC \in \mathscr{A}^*$ by the Neumann series.*

$$d(L_A)_e C = \frac{d}{dt}_{|t=0} L_A(1_{\mathscr{A}} + tC) = AC\,.$$

Hence, the left-invariant vector field X^C associated with C is given by $X^C_A = A\cdot C$.

Next, we compute the Lie bracket of $C, D \in T_eG = \mathscr{A}$. We have

$$\begin{aligned}[C,D] = [X^C, X^D]_e = d(X^D)_e X^C_e - d(X^C)_e X^D_e &= \frac{d}{dt}_{|t=0} X^D_{(1_{\mathscr{A}}+tC)} - \frac{d}{dt}_{|t=0} X^C_{(1_{\mathscr{A}}+tD)} \\ = C\cdot D - D\cdot C\,,&\end{aligned}$$

where we have interpreted X^C and X^D as maps from the open set $\mathscr{A}^* \subseteq \mathscr{A}$ to $\mathscr{A} \cong \mathbb{R}^{\dim\mathscr{A}}$, hence their Lie bracket is given by the difference of their directional derivatives with respect to each other.

At last, we compute the Lie exponential map of G. For $C \in T_eG$, the algebra exponential map $t \mapsto \exp(tC) = 1_{\mathscr{A}} + tC + \frac{1}{2}(tC)^2 + \ldots$ is a 1-parameter subgroup in G ($\exp((s+t)C) = \exp(sC)\cdot\exp(tC)$) with ${}^{d}/_{dt}{}_{|t=0}\exp(tC) = C$, so it must be the one associated with C:

$$e^{tC} = \exp(tC)\,.$$

The above formulae for $[C,D]$ and e^{tC} also hold for any Lie subgroup of G!

Lemma 1.15. *Let $\Phi : G \to H$ be a Lie group homomorphism.*

(i) $\Phi(e^{tX}) = e^{t d\Phi_e X}$ *for all $t \in \mathbb{R}$, $X \in T_eG$.*
(ii) $[d\Phi_e X, d\Phi_e Y] = d\Phi_e[X,Y]$, *hence $d\Phi_e T_eG \to T_eG$ is a* Lie algebra homomorphism, *i.e., a vector space homomorphism which preserves Lie brackets.*

Proof. (i) We are done when we show that the left hand side is indeed a 1-parameter subgroup in H with the correct initial vector: $\Phi(e^{(s+t)X}) = \Phi(e^{sX}\cdot e^{tX}) = \Phi(e^{sX})\cdot\Phi(e^{tX})$ with initial vector $\frac{d}{dt}_{|t=0}\Phi(e^{tX}) = d\Phi_e(\frac{d}{dt}_{|t=0} e^{tX}) = d\Phi_e X$.

(ii)

$$\begin{aligned}[d\Phi_e X, d\Phi_e Y] &= \frac{d}{dt}_{|t=0}\frac{d}{ds}_{|s=0} e^{t d\Phi_e X} e^{s d\Phi_e X} e^{-t d\Phi_e X} \overset{(i)}{=} \frac{d}{dt}_{|t=0}\frac{d}{ds}_{|s=0}\Phi(e^{tX})\Phi(e^{sY})\Phi(e^{-tX}) \\ &= \frac{d}{dt}_{|t=0} d\Phi_e\left(\frac{d}{ds}_{|s=0} e^{tX}e^{sY}e^{-tX}\right) = d\Phi_e\left(\frac{d}{dt}_{|t=0}\frac{d}{ds}_{|s=0} e^{tX}e^{sY}e^{-tX}\right) \\ &= d\Phi_e([X,Y])\,,\end{aligned}$$

where we have used Proposition 1.13 in the first and last step.

□

Definition 1.16. *(i) For $a \in G$ let $I_a := L_a \circ R_a^{-1} : G \ni b \mapsto a\cdot b\cdot a^{-1} \in G$ be conjugation by a.*
(ii) For $a \in G$ let $\mathsf{Ad}_a := d(I_a)_e : \mathfrak{g} \cong T_eG \to T_eG \cong \mathfrak{g}$.
(iii) For $X \in \mathfrak{g}$ let $\mathsf{ad}_X := [X,\cdot] : \mathfrak{g} \ni Y \mapsto [X,Y] \in \mathfrak{g}$.

Remark 1.17. *(i) I_a is a* Lie group automorphism, *i.e., a diffeomorphism and a group automorphism. Moreover, $\mathsf{Aut}(G)$ is a Lie group and $G \ni a \mapsto I_a \in \mathsf{Aut}(G)$ is a Lie group homomorphism.*

(ii) $\mathsf{Ad}_a : \mathfrak{g} \to \mathfrak{g}$ *is by Lemma 1.15(ii) a* Lie algebra automorphism. *Moreoever,* $\mathsf{Ad} : G \ni a \mapsto \mathsf{Ad}_a \in \mathsf{Aut}(\mathfrak{g}) \subseteq \mathrm{GL}(\mathfrak{g})$ *is a Lie group homomorphism, where* $\mathrm{GL}(\mathfrak{g})$ *is the Lie group of linear transformations of the vector space(!)* $\mathfrak{g}$.

(iii) By the Jacobi-identity, we have $\mathsf{ad}_X[Y,Z] = [[X,Y],Z] + [Y,[X,Z]] = [\mathsf{ad}_X Y, Z] + [Y, \mathsf{ad}_X Z]$. *That is,* $\mathsf{ad}_X : \mathfrak{g} \to \mathfrak{g}$ *is a* Lie algebra derivation*, i.e., a vector space endomorphism* $\varphi \in \mathsf{End}(\mathfrak{g})$ *with* $\varphi[X,Y] = [\varphi X, Y] + [X, \varphi Y]$. *Moreover,* $\mathsf{ad} : \mathfrak{g} \ni X \mapsto \mathsf{ad}_X \in \mathsf{Der}(\mathfrak{g})$ *is a Lie algebra homomorphism, where the Lie bracket on* $\mathsf{Der}(\mathfrak{g})$ *is given by* $[\varphi, \psi] = \varphi \circ \psi - \psi \circ \varphi$ *and* $\mathsf{ad}_{[X,Y]} = \mathsf{ad}_X \circ \mathsf{ad}_Y - \mathsf{ad}_Y \circ \mathsf{ad}_X = [\mathsf{ad}_X, \mathsf{ad}_Y]$.

Lemma 1.18. *Let* $X, Y \in \mathfrak{g} \cong T_e G$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathsf{Ad}_{\mathrm{e}^{tX}} = \mathsf{ad}_X \,.$$

In particular,

$$\boxed{\mathrm{d}(\mathsf{Ad})_e = \mathsf{ad} \,.}$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathsf{Ad}_{\mathrm{e}^{tX}} Y = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} I_{\mathrm{e}^{tX}}(\mathrm{e}^{sY}) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \mathrm{e}^{tX} \cdot \mathrm{e}^{sY} \cdot \mathrm{e}^{-tX} \overset{1.13}{=} [X,Y] = \mathsf{ad}_X Y \,.$$

□

Corollary 1.19. *Apply Lemma 1.15(i) to* $\Phi := \mathsf{Ad} : G \to \mathsf{Aut}(\mathfrak{g}) \subseteq \mathrm{GL}(\mathfrak{g})$*:*

$$\mathsf{Ad}_{\mathrm{e}^{tX}} = \mathrm{e}^{t\mathrm{d}\,\mathsf{Ad}_e X} = \mathrm{e}^{t\,\mathsf{ad}_X} = \exp(t\,\mathsf{ad}_X) = \mathrm{id} + t\,\mathsf{ad}_x + {}^{t^2}\!/_2\, \mathsf{ad}_X^2 + \dots \,.$$

Summary.

$$\begin{array}{ccccc} \mathfrak{g} & \xrightarrow{\mathsf{ad}} & \mathsf{Der}(\mathfrak{g}) & \xrightarrow{\subseteq} & \mathsf{End}(\mathfrak{g}) \\ \downarrow {\scriptstyle \mathrm{e}^{\cdot}} & & \downarrow {\scriptstyle \mathrm{e}^{\cdot} = \exp} & & \downarrow {\scriptstyle \mathrm{e}^{\cdot} = \exp} \\ G & \xrightarrow{\mathsf{Ad}} & \mathsf{Aut}(\mathfrak{g}) & \xrightarrow{\subseteq} & \mathrm{GL}(\mathfrak{g}) \end{array}$$

1.2. **Clifford Algebras.**

Definition 1.20. *Let* K *be a field with* $\mathrm{char}\, K \neq 2$, V *a finite-dimensional* K*-vector space and* q *a quadratic form on* V. *We call* (C, ι) *a* Clifford algebra *for* (V, q) *if*

(i) C *is an associative, unital* K*-algebra.*

(ii) $\iota : V \to C$ *is a* K*-linear map with*

$$\iota(v)^2 = -q(v) \cdot \mathbb{1}_C \qquad \textit{for all } v \in V \,.$$

(iii) If $\mathcal{A}$ *is any associative, unital* K*-Algebra for which there is a map* $j : V \to \mathcal{A}$ *with*

$$j(v)^2 = -q(v) \cdot \mathbb{1}_{\mathcal{A}} \qquad \textit{for all } v \in V \,, \tag{1.1}$$

then there exists a unique K*-algebra homomorphism* $\widetilde{j} : C \to \mathcal{A}$ *such that*

$$\begin{array}{ccccc} & & C & & \\ & {\scriptstyle \iota} \nearrow & & \searrow {\scriptstyle \widetilde{j}} & \\ V & & \xrightarrow{\quad j \quad} & & \mathcal{A} \end{array}$$

is commutative.

Proposition 1.21. *For any* (V, q) *there exists a Clifford algebra* (C, ι) *unique up to canonical isomorphism. Moreover,* ι *is injective and* $\{\mathbb{1}_C\} \cup \iota(V) \subseteq C$ *generates* C.

Proof. Let us first show uniqueness of the Clifford algebra. This is a standard argument using the universal property Definition 1.20(iii). Suppose we are given two Clifford algebras (C, ι) and (C', ι'). By definition, there exist unique maps $\widetilde{\iota} : C' \to C$ with $\widetilde{\iota} \circ \iota' = \iota$ and $\widetilde{\iota'} : C \to C'$ with $\widetilde{\iota'} \circ \iota = \iota'$. The map $\widetilde{\iota} \circ \widetilde{\iota'} : C \to C$ satisfies $\widetilde{\iota} \circ \widetilde{\iota'} \circ \iota = \widetilde{\iota} \circ \iota' = \iota$. Using Definition 1.20(iii) a third time, now with $\mathcal{A} = C$ and $j = \iota$, we see that $\mathrm{id}_C \circ \iota = \iota$. By uniqueness, we have $\widetilde{\iota} \circ \widetilde{\iota'} = \mathrm{id}_C$. Analogously, $\widetilde{\iota'} \circ \widetilde{\iota} = \mathrm{id}_{C'}$. Hence, (C, ι) is unique up to canonical isomorphism.

Next, we show that (C, ι) actually exists. Let $\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ be the tensor algebra of V. Define $\mathcal{I}$ as the two-sided ideal generated by the set

$$\{v \otimes v + q(v) \mid v \in V\}$$

and $C := \mathcal{T}(V)/\mathcal{I}$. Let $\pi : \mathcal{T}(V) \to C$ be the canonical projection and define $\iota : V \hookrightarrow \mathcal{T}(V) \xrightarrow{\pi} C$, the concatenation of the injection $V \hookrightarrow \mathcal{T}(V)$ and the projection π.

Since $\mathcal{I}$ is a two-sided ideal, C inherits an associative, unital algebra structure from $\mathcal{T}(V)$. Furthermore, by the very definition of C and ι, we have $\iota(v)^2 = -q(v) \cdot \mathbb{1}_C$ for all $v \in V$.

Let now be $j : V \to \mathcal{A}$ be linear map into an associative, unital K-algebra with (1.1). By the universal property of the tensor algebra, j extends uniquely to a K-algebra homomorphism $\bar{j} : \mathcal{T}(V) \to \mathcal{A}$. Since j satisfies (1.1), we have $\mathcal{I} \subseteq \ker \bar{j}$. Hence, $\bar{j}$ descends uniquely to a map $\tilde{j} : C \to \mathcal{A}$ satisfying $\tilde{j} \circ \iota = j$.

To show that ι is injective, it suffices to prove that $V \cap \mathcal{I} = \{0\}$. This is a simple argument by induction over the degree of tensors. Finally, since $\mathcal{T}(V)$ is generated by V and $1 \in K = V^{\otimes 0}$, C is generated by $\iota(V)$ and $\mathbb{1}_C$. □

Remark 1.22. (i) *We will from now on denote the unique Clifford algebra associated with (V, q) by $(\mathcal{C}\ell(V, q), \iota)$ and view V as a subspace of $\mathcal{C}\ell(V, q)$ by virtue of ι. Moreoever, we will write $1 \in \mathcal{C}\ell(V, q)$ instead of $\mathbb{1}_{\mathcal{C}\ell(V,q)}$.*

(ii) *If $b : V \times V \ni (v, w) \mapsto 1/2(q(v + w) - q(v) - q(w)) \in K$ denotes the symmetric bilinear form associated with q, we have*

$$v \cdot w + w \cdot v = -2b(v, w) \cdot 1 \quad \text{for all } v, w \in V$$

in $\mathcal{C}\ell(V, q)$. In particular, if V has K-dimension n and $(e_1, \ldots, e_n)$ is a basis of V that diagonalizes b, then

$$e_i^2 = -q(e_i) \quad \text{for all} \quad i = 1, \ldots, n \quad \text{and} \quad e_i \cdot e_j + e_j \cdot e_i = 0 \quad \text{for all} \quad 1 \leqslant i \neq j \leqslant n\,.$$

Definition 1.20(iii) says that $\mathcal{C}\ell(V, q)$ is the smallest associative, unital algebra containing V and satisfying these relations.

(iii) *Let V, W be K-vector spaces, equipped with quadratic forms q and r, respectively. Applying Definition 1.20(iii) to $\iota_W \circ f$ for a K-linear map $f : V \to W$ which satisfies $f^*r = q$ (i.e. $r(f(v)) = q(v)$ for all $v \in V$) shows that f extends uniquely to an algebra homomorphism $\tilde{f} : \mathcal{C}\ell(V, q) \to \mathcal{C}\ell(W, r)$. The uniqueness assertion in Definition 1.20(iii) also implies that, given another linear map $g : W \to U$ into a vector space U with a quadratic form s which satisfies $g^*s = r$, we have $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$.*

Definition 1.23. *Let V be a K-vector space and $q : V \to K$ a quadratic form with associated symmetric bilinear form $b : V \times V \to K$.*

(i) *Denote by $\alpha \in \mathrm{Aut}(\mathcal{C}\ell(V, q))$ the unique continuation of $-\mathrm{id}_V \in \mathrm{O}(b)$. Explicitely, $\alpha : \mathcal{C}\ell(V, q) \to \mathcal{C}\ell(V, q)$ is the unique K-linear map which satisfies*

$$\alpha(v_1 \cdot v_2 \cdots v_k) = \alpha(v_1) \cdot \alpha(v_2) \cdots \alpha(v_k) = (-1)^k v_1 \cdot v_2 \cdots v_k \quad \text{for all} \quad k \in \mathbb{N}_0, v_1, \ldots, v_k \in V\,.$$

In particular, $\alpha^2 = \mathrm{id}$.

(ii) *For $i = 0, 1$ define $\mathcal{C}\ell(V, q)^i := \{x \in \mathcal{C}\ell(V, q) \mid \alpha(x) = (-1)^i x\}$, i.e., $\mathcal{C}\ell(V, q)^i$ is the $(-1)^i$-eigenspace of α, and*

$$\mathcal{C}\ell(V, q) = \mathcal{C}\ell(V, q)^0 \oplus \mathcal{C}\ell(V, q)^1\,.$$

Multiplication in $\mathcal{C}\ell(V, q)$ satisfies

$$\mathcal{C}\ell(V, q)^i \cdot \mathcal{C}\ell(V, q)^j \subseteq \mathcal{C}\ell(V, q)^{i+j \bmod 2}\,.$$

Remark 1.24. (i) *A K-algebra $\mathcal{A}$ with a splitting $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ such that multiplication in $\mathcal{A}$ obeys the rule $\mathcal{A}^i \cdot \mathcal{A}^j \subseteq \mathcal{A}^{i+j}$ is called a* $\mathbb{Z}_2$-graded algebra. *We call $\mathcal{A}^0$ the* even part *and $\mathcal{A}^1$ the* odd part *of $\mathcal{A}$ and we call $\deg x := i$* the degree *of $x \in \mathcal{A}^i$. Note that $\mathcal{A}^0$ is always a subalgebra of $\mathcal{A}$.*

(ii) *Given two $\mathbb{Z}_2$-graded K-algebras $\mathcal{A}$ and $\mathcal{B}$, their tensor product $\mathcal{A} \otimes \mathcal{B}$ is the K-algebra whose underlying K-vector space is the vector space tensor product $\mathcal{A} \otimes \mathcal{B}$ with multiplication $a \otimes b \cdot a' \otimes b' = a \cdot a' \otimes b \cdot b'$. Unfortunately, $\mathcal{A} \otimes \mathcal{B}$ is in general* not *a $\mathbb{Z}_2$-graded algebra. To produce a $\mathbb{Z}_2$-graded algebra, we use the* $\mathbb{Z}_2$-graded tensor product *$\mathcal{A} \hat{\otimes} \mathcal{B}$ whose underlying vector space is again the vector space tensor product $\mathcal{A} \otimes \mathcal{B}$ and whose multiplication is defined on pure tensors of pure degree by*

$$a \otimes b \cdot a' \otimes b' = (-1)^{\deg b \cdot \deg a'} a \cdot a' \otimes b \cdot b'\,. \tag{1.2}$$

The $\mathbb{Z}_2$-grading of $\mathcal{A}\hat{\otimes}\mathcal{B}$ is given by

$$(\mathcal{A}\hat{\otimes}\mathcal{B})^0 = \mathcal{A}^0 \otimes \mathcal{B}^0 + \mathcal{A}^1 \otimes \mathcal{B}^1\,,$$
$$(\mathcal{A}\hat{\otimes}\mathcal{B})^1 = \mathcal{A}^0 \otimes \mathcal{B}^1 + \mathcal{A}^1 \otimes \mathcal{B}^0\,.$$

Proposition 1.25. *Let V be a K-vector space with quadratic form q and associated symmetric bilinear form b. Assume we are given a b-orthogonal splitting $V = V_1 \oplus V_2$, i.e., $b(v_1, v_2) = 0$ for all $v_1 \in V_1, v_2 \in V_2$ (equivalently $q(v_1 + v_2) = q(v_1) + q(v_2)$). Then there is a natural isomorphism of Clifford algebras*

$$\mathcal{C}\ell(V,q) \to \mathcal{C}\ell(V_1,q_1)\hat{\otimes}\mathcal{C}\ell(V_2,q_2)\,,$$

where $q_i := q_{|V_i} : V_i \to K$ is the restriction of q to V_i.

Proof. Define $j : V = V_1 \oplus V_2 \ni v_1 + v_2 \mapsto v_1 \otimes 1 + 1 \otimes v_2 \in \mathcal{C}\ell(V_1,q_1)\hat{\otimes}\mathcal{C}\ell(V_2,q_2)$. Then, we have for all $v_1 + v_2 \in V_1 \oplus V_2$ by (1.2)

$$\begin{aligned} j(v_1+v_2)^2 &= (v_1\otimes 1 + 1\otimes v_2)^2 = v_1^2 \otimes 1 + 1 \otimes v_2 + v_1 \otimes v_2 - v_1 \otimes v_2 = -q(v_1)\cdot 1\otimes 1 - q(v_2) 1 \otimes 1 \\ &= -q(v_1+v_2)1\otimes 1\,. \end{aligned}$$

Hence, by Definition 1.20(iii), j extends uniquely to an algebra homomorphism $\tilde{j} : \mathcal{C}\ell(V,q) \to \mathcal{C}\ell(V_1,q_1)\hat{\otimes}\mathcal{C}\ell(V_2,q_2)$. To see that $\tilde{j}$ is bijective, we construct the inverse homomorphism. Let $g_i : V_i \to \mathcal{C}\ell(V,q)$, $i = 1,2$, be the concatenation of the inclusion $V_i \hookrightarrow V$ and the inclusion $V \hookrightarrow \mathcal{C}\ell(V,q)$. Then g_i extends to an algebra homomorphism $\tilde{g}_i : \mathcal{C}\ell(V_i,q_i) \to \mathcal{C}\ell(V,q)$. The map $g : \mathcal{C}\ell(V_1,q_1)\hat{\otimes}\mathcal{C}\ell(V_2,q_2) \ni x \otimes y \mapsto \tilde{g}_1(x)\cdot\tilde{g}_2(y) \in \mathcal{C}\ell(V,q)$ is the inverse of $\tilde{j}$. It suffices to check this on pure tensors of vectors from V_1 and V_2, as those generate $\mathcal{C}\ell(V_1,q_1)\hat{\otimes}\mathcal{C}\ell(V_2,q_2)$ and hence determine g uniquely. □

Definition 1.26. *Let V be a K-vector space and q a quadratic form on V. Let $t : \mathcal{T}(V) \to \mathcal{T}(V)$ be the K-linear map given on pure tensors by*

$$t(v_1 \otimes v_2 \otimes \ldots \otimes v_k) = v_k \otimes v_{k-1} \otimes \ldots \otimes v_1\,.$$

Then t preserves the ideal $\mathcal{I}$ from the proof of Proposition 1.21 and thus descends to a K-linear *map*

$$\cdot^t : \mathcal{C}\ell(V,q) \to \mathcal{C}\ell(V,q)\,,$$

the transpose. *Note that $\cdot^t$ is an* algebra antiautomorphism, *i.e., $(x\cdot y)^t = y^t \cdot x^t$ for all $x,y \in \mathcal{C}\ell(V,q)$, and an* involution, *i.e., $(x^t)^t = x$ for all $x \in \mathcal{C}\ell(V,q)$.*

With an eye on Riemannian manifolds we are interested in two particular Clifford algebras.

Notation 1.27. *Let $q_n : \mathbb{R}^n \ni x \mapsto \sum_{i=1}^n x_i^2 \in \mathbb{R}$ be the standard positive definite quadratic form on $\mathbb{R}^n$ and $q_n^{\mathbb{C}} : \mathbb{C}^n \ni z \mapsto \sum_{i=1}^n z_i^2 \in \mathbb{C}$ the standard quadratic form on $\mathbb{C}^n$. We let*

- $\mathcal{C}\ell_n = \mathcal{C}\ell(\mathbb{R}^n, q_n)$,
- $\mathbb{C}\ell_n = \mathcal{C}\ell(\mathbb{C}^n, q_n^{\mathbb{C}})$.

Remark 1.28. *It follows from Definition 1.20(iii) that the complexification $\mathcal{C}\ell_n \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathcal{C}\ell_n$, together with the complex extension of q_n, is isomorphic to $\mathbb{C}\ell_n$. In particular, from now on we will view $\mathcal{C}\ell_n$ as a subalgebra of $\mathbb{C}\ell_n$ and think of $\mathbb{C}\ell_n$ as $\mathcal{C}\ell_n$ with complex coefficients.*

Proposition 1.29. *There are algebra isomorphisms $\mathcal{C}\ell_n \cong \mathcal{C}\ell_{n+1}^0$ and $\mathbb{C}\ell_n \cong \mathbb{C}\ell_{n+1}^0$.*

Proof. Let $(e_1, \ldots, e_{n+1})$ be the standard basis of $\mathbb{R}^{n+1}$. Define $f : \mathbb{R}^n \to \mathcal{C}\ell_{n+1}^0$ by

$$f(e_i) := -e_i \cdot e_{n+1} \quad \text{for all} \quad 1 \leqslant i \leqslant n\,,$$

and linear extension. For $x = \sum_{i=1}^n x_i e_i \in \mathbb{R}^n$ we have

$$\begin{aligned} f(x)^2 &= \left(-\sum_{i=1}^n x_i e_i \cdot e_{n+1}\right)^2 = \sum_{i,j=1}^n x_i x_j e_i \cdot e_{n+1} \cdot e_j \cdot e_{n+1} = -\sum_{i,j=1}^n x_i x_j e_i \cdot e_j \cdot e_{n+1} \cdot e_{n+1} \\ &= \sum_{i,j=1}^n x_i x_j e_i \cdot e_j = \left(\sum_{i=1}^n x_i e_i\right)^2 = x \cdot x = -q_n(x)\cdot 1\,. \end{aligned}$$

By the universal property of Clifford algebras, f extends to an algebra homomorphism $\widetilde{f} : \mathcal{C}\ell_n \to \mathcal{C}\ell^0_{n+1}$. Evaluating $\widetilde{f}$ on a vector space basis of $\mathcal{C}\ell_n$ shows that it is an isomorphism (see Exercise no. 7). Finally, the isomorphism $\mathbb{C}\ell_n \cong \mathbb{C}\ell^0_{n+1}$ is the complexification of $\widetilde{f}$. □

Theorem 1.30. *For all $m \in \mathbb{N}$ there are algebra isomorphisms*

$$\Phi_{2m} : \mathbb{C}\ell_{2m} \to M(2,2;\mathbb{C}) \otimes M(2,2;\mathbb{C}) \otimes \ldots \otimes M(2,2;\mathbb{C}) \cong M(2^m,2^m;\mathbb{C}),$$

$$\Phi_{2m+1} : \mathbb{C}\ell_{2m+1} \to (M(2,2;\mathbb{C}) \otimes \ldots \otimes M(2,2;\mathbb{C})) \oplus (M(2,2;\mathbb{C}) \otimes \ldots \otimes M(2,2;\mathbb{C})) \cong M(2^m,2^m;\mathbb{C}) \oplus M(2^m,2^m;\mathbb{C}),$$

given as follows. Let $E := E_2$, $U := \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$, $V := \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $W := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$. For $1 \leqslant j \leqslant m$ define

$$\phi_{2m} : \mathbb{C}^{2m} \ni e_{2j-1} \mapsto W \otimes W \otimes \ldots \otimes W \underset{j\text{-th slot}}{\otimes U \otimes} E \otimes \ldots \otimes E,$$

$$\phi_{2m} : \mathbb{C}^{2m} \ni e_{2j} \mapsto W \otimes W \otimes \ldots \otimes W \underset{j\text{-th slot}}{\otimes V \otimes} E \otimes \ldots \otimes E$$

and extend linearly. Then, $\phi_{2m}(x)^2 = -q^{\mathbb{C}}_{2m}(x) \cdot 1$ for all $x \in \mathbb{C}^{2m}$ and by the universal property of Clifford algebras, ϕ_{2m} extends to an algebra homomorphism Φ_{2m}, which turns out to be an isomorphism. To obtain Φ_{2m+1}, we define

$$\phi_{2m+1} : \mathbb{C}^{2m+1} \ni e_j \mapsto \begin{cases} (\phi_{2m}(e_j), \phi_{2m}(e_j)), & 1 \leqslant j \leqslant 2m, \\ (\mathrm{i}W \otimes \ldots \otimes W, -\mathrm{i}W \otimes \ldots \otimes W), & j = 2m+1, \end{cases}$$

and proceed analogously.

Definition 1.31. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let $\mathcal{A}$ be a finite-dimensional, associative and unital $\mathbb{K}$-algebra.*

(i) A representation *of $\mathcal{A}$ is a $\mathbb{K}$-algebra homomorphism $\rho : \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(V)$, where V is a finite-dimensional $\mathbb{K}$-vector space. In this situation, V is also called an $\mathcal{A}$-*module. *If the representation ρ is fixed, we shall write $x \cdot v := \rho(x)(v)$.*

(ii) Given two representations $\rho : \mathcal{A} \to \mathrm{End}(V)$ and $\kappa : \mathcal{A} \to \mathrm{End}(W)$, their direct sum *is the representation $\rho \oplus \kappa : \mathcal{A} \to \mathrm{End}(V \oplus W)$, given by $\rho \oplus \kappa(x)(v + w) = \rho(x)(v) + \kappa(x)(w)$.*

(iii) A representation $\rho : \mathcal{A} \to \mathrm{End}(V)$ is called reducible *if it is a direct sum $\rho = \rho_1 \oplus \rho_2 : \mathcal{A} \to End(V_1 \oplus V_2)$ with $V_i \neq \{0\}$, $i = 1, 2$. In other words, ρ is reducible if V splits into a nontrivial direct sum $V = V_1 \oplus V_2$ such that $\rho(x)(V_j) \subseteq V_j$ for all $x \in \mathcal{A}$, $j = 1, 2$. If ρ is not reducible, we call it* irreducible.

(iv) Two representations $\rho : \mathcal{A} \to \mathrm{End}(V)$, $\kappa : \mathcal{A} \to \mathrm{End}(W)$ are called equivalent *or* isomorphic *if there exists a $\mathbb{K}$-vector space isomorphism $F : V \to W$ such that $\rho(x) = F^{-1} \circ \kappa(x) \circ F$ for all $x \in \mathcal{A}$.*

(v) We define modules, direct sums, irreducibility and equivalence analogously for representations of Lie groups.

Remark 1.32. *If $\rho : \mathcal{A} \to \mathrm{End}(V)$ is any representation of $\mathcal{A}$, then ρ can be decomposed into a direct sum $\rho = \rho_1 \oplus \ldots \oplus \rho_k$ of irreducible representations $\rho_i : \mathcal{A} \to \mathrm{End}(V_i)$. Indeed, we simply apply Definition 1.31(iii) recursively. This process must end by finite-dimensionality of V.*

For the next theorem, we need an important element in the Clifford algebras $\mathcal{C}\ell_n$ respectively $\mathbb{C}\ell_n$.

Definition 1.33. *Fix an orientation of $\mathbb{R}^n$ and let $(e_1, \ldots, e_n)$ be an oriented orthonormal basis w.r.t. $\langle \cdot, \cdot \rangle_{\mathrm{Eukl}}$. Define the* volume element *$\omega_n \in \mathcal{C}\ell_n$ by*

$$\omega_n := e_1 \cdot e_2 \cdots e_n,$$

and the complex volume element *$\omega^{\mathbb{C}}_n \in \mathbb{C}\ell_n$ by*

$$\omega^{\mathbb{C}}_n := \mathrm{i}^{\lfloor (n+1)/2 \rfloor} e_1 \cdot e_2 \cdots e_n = \mathrm{i}^{\lfloor (n+1)/2 \rfloor} \omega.$$

Here, $\lfloor x \rfloor$ denotes the largest integer which is smaller or equal to $x \in \mathbb{R}$.

Theorem 1.34. *There exists, up to equivalence, exactly one irreducible representation $\mathbb{C}\ell_{2m} \to \mathrm{End}_{\mathbb{C}}(V)$, where $\dim_{\mathbb{C}} V = 2^m$. There are, up to equivalence, exactly two irreducible representations $\rho : \mathbb{C}\ell_{2m+1} \to \mathrm{End}_{\mathbb{C}}(V)$, where $\dim V = 2^m$. These can be distinguished by the action of the complex volume element $\omega^{\mathbb{C}}_{2m+1}$, i.e., either $\rho(\omega^{\mathbb{C}}_{2m+1}) = +\mathrm{id}$ or $\rho(\omega^{\mathbb{C}}_{2m+1}) = -\mathrm{id}$.*

Proof. By Theorem 1.30, $\mathbb{C}\ell_{2m}$ is isomorphic to $M(2^m, 2^m; \mathbb{C})$. It is a classical fact that the only irreducible representation of $M(2^m, 2^m; \mathbb{C})$ is the standard one, given by matrix multiplication.

Again by Theorem 1.34, $\mathbb{C}\ell_{2m+1}$ is isomorphic to $M(2^m, 2^m; \mathbb{C}) \oplus M(2^m, 2^m; \mathbb{C})$. The two different representations are given by the standard representation of the first, respectively second, direct summand on $\mathbb{C}^{2^m}$.

For the proof of $\rho(\omega^{\mathbb{C}}_{2m+1}) = \pm\,\mathrm{id}$ and that these are inequalivalent representations, see Exercise no. 10. □

Proposition 1.35. *Let $\Phi_{2m} : \mathbb{C}\ell_{2m} \to M(2^m, 2^m; \mathbb{C}) \cong \mathsf{End}(\mathbb{C}^{2^m})$ be the irreducible representation given in Theorem 1.30 and $F : \mathbb{C}\ell_{2m-1} \to \mathbb{C}\ell^0_{2m}$ the algebra isomorphism from Proposition 1.29. Then the representation $\Phi_{2m} \circ F : \mathbb{C}\ell_{2m-1} \to M(2^m, 2^m; \mathbb{C}) \cong \mathsf{End}(\mathbb{C}^{2^m})$ is (equivalent to) the direct sum of the two irreducible representations of $\mathbb{C}\ell_{2m-1}$.*

Proof. See Exercise 11. □

1.3. **The Spin group, its Lie algebra and representations.**

Notation and Remarks 1.36. *Recall that the set of units $\mathcal{C}\ell_n^*$ of $\mathcal{C}\ell_n$ is a Lie group with Lie algebra $\mathcal{C}\ell_n$.*

Definition 1.37. *(i) The* Clifford group Γ_n of $\mathcal{C}\ell_n$ *is the closed subgroup of $\mathcal{C}\ell_n^*$ given by*

$$\Gamma_n := \left\{ x \in \mathcal{C}\ell_n^* \mid \alpha(x) \cdot v \cdot x^{-1} \in \mathbb{R}^n \text{ for all } v \in \mathbb{R}^n \right\} .$$

(ii) Define the continuous group homomorphism $\lambda_n : \Gamma_n \to \mathrm{GL}(n; \mathbb{R})$ by

$$\lambda_n(x)(v) := \alpha(x) \cdot v \cdot x^{-1} .$$

(iii) The norm of $\mathcal{C}\ell_n$ *is the map*

$$N : \mathcal{C}\ell_n \ni x \mapsto x \cdot \alpha(x^t) = x \cdot \alpha(x)^t \in \mathcal{C}\ell_n .$$

Remark 1.38. *(i) The maps $\alpha, \cdot^t : \mathcal{C}\ell_n \to \mathcal{C}\ell_n$ leave Γ_n invariant. Indeed, if $x \in \Gamma_n$, then $\alpha(x) \cdot v \cdot x^{-1} \in \mathbb{R}^n$ for all $v \in \mathbb{R}^n$ and by definition of α we have*

$$\alpha(\alpha(x)) \cdot v \cdot \alpha(x)^{-1} = -\alpha(\alpha(x)) \cdot \alpha(v) \cdot \alpha(x)^{-1} = -\alpha(\alpha(x) \cdot v \cdot x^{-1}) = \alpha(x) \cdot v \cdot x^{-1} \in \mathbb{R}^n$$

for all $v \in \mathbb{R}^n$ and analogously for $\cdot^t$.

(ii) Note that for $x \in \mathbb{R}^n$ we have $N(x) = x \cdot \alpha(x^t) = x \cdot \alpha(x) = -x \cdot x = q_n(x) = \|x\|^2$.

Lemma 1.39. *The kernel of the group homomorphism $\lambda_n : \Gamma_n \to \mathrm{GL}(n; \mathbb{R})$ is $\ker \lambda_n = \mathbb{R}^* \cdot 1$.*

Proof. Let $x \in \ker \lambda_n$. Then by definition $\alpha(x) \cdot v \cdot x^{-1} = v$ for all $v \in \mathbb{R}^n$, which is equivalent to

$$\alpha(x) \cdot v = v \cdot x \quad \text{for all} \quad v \in \mathbb{R}^n .$$

We decompose x into its even and odd part, $x = x^0 + x^1$ with $x^i \in \mathcal{C}\ell_n^i$. Then the above statement is equivalent to

$$x^0 \cdot v = v \cdot x^0 \quad \text{and} \quad -x^1 \cdot v = v \cdot x^1 \quad \text{for all} \quad v \in \mathbb{R}^n . \tag{1.3}$$

Let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$. We express x^0 as a linear combination of the basis vectors from Exercise 7 and write

$$x^0 = a + e_1 b ,$$

where $a \in \mathcal{C}\ell_n^0, b \in \mathcal{C}\ell_n^1$ and neither a nor b contain a term with a factor e_1. We apply the first relation in (1.3) to $v = e_1$ and obtain

$$(a + e_1 b)e_1 = e_1(a + e_1 b) .$$

Since a has even degree and contains no term with a factor e_1 we have $ae_1 = e_1 a$. Analogously, we have $e_1 b = -be_1$. Hence,

$$a + e_1 b = a - e_1 b ,$$

which in turn implies $e_1 b = 0$ and x^0 contains no term with a factor e_1. By applying the same argument to e_i, $i = 2, \ldots, n$, we conclude that x^0 is a linear combination of the elements from Exercise no. 7 no term of which contains a factor e_i, i.e., $x^0 \in \mathbb{R} \cdot 1$.

Proceeding analogously with the second relation in (1.3) shows $x^1 \in \mathbb{R} \cdot 1$. But since $1 \in \mathcal{C}\ell_n^0$ we must have $x^1 = 0$. Hence $x = x^0 \in \mathbb{R} \cdot 1 \cap \Gamma_n = \mathbb{R}^* \cdot 1$. □

Lemma 1.40. *If $x \in \Gamma_n$, then $N(x) \in \mathbb{R}^*$ and the restriction $N_{|\Gamma_n} : \Gamma_n \to \mathbb{R}^*$ is a group homomorphism with $N(\alpha(x)) = N(x)$ for all $x \in \Gamma_n$.*

Proof. Let $x \in \Gamma_n$. Then $\alpha(x) \cdot v \cdot x^{-1} \in \mathbb{R}^n$ for all $v \in \mathbb{R}^n$. Since the transpose acts as the identity on $\mathbb{R}^n$, we get $(x^t)^{-1} \cdot v \cdot \alpha(x)^t = \alpha(x) \cdot v \cdot x^{-1}$. Thus, $v = x^t \cdot \alpha(x) \cdot v \cdot (\alpha(x)^t \cdot x)^{-1} = \alpha(\alpha(x)^t \cdot x) \cdot v \cdot (\alpha(x)^t \cdot x)^{-1}$ which implies that $\alpha(x)^t \cdot x \in \ker \lambda_n$. By Remark 1.38(i), $y = \alpha(x)^t \in \Gamma_n$ and by what we just showed $\alpha(y)^t \cdot y = \alpha(\alpha(x)^t)^t \cdot \alpha(x)^t = x \cdot \alpha(x)^t = N(x) \in \ker \lambda_n$. By the last lemma, $N(x) \in \mathbb{R}^* \cdot 1$.

To show that N restricted to Γ_n is a homomorphism, note that $\mathbb{R} \cdot 1$ is central in $\mathcal{C}\ell_n$. Hence, for $x, y \in \Gamma_n$, we have $N(x \cdot y) = x \cdot y \cdot \alpha(x \cdot y)^t = x \cdot y \cdot \alpha(y)^t \cdot \alpha(x)^t = xN(y)\alpha(x)^t = x \cdot \alpha(x)^t N(y) = N(x)N(y)$.

At last, we have $N(\alpha(x)) = \alpha(x) \cdot \alpha(\alpha(x))^t = \alpha(x \cdot \alpha(x)^t) = \alpha(N(x)) = N(x)$. □

Proposition 1.41. *We have*

(i) $\mathbb{R}^n \backslash \{0\} \subseteq \Gamma_n$,

(ii) for $x \in \mathbb{R}^n \backslash \{0\}$, $\lambda_n(x) \in \mathrm{GL}(n;\mathbb{R})$ *is the reflection about the hyperplane* $x^\perp$ *and* $\lambda_n(\Gamma_n) \subseteq \mathrm{O}(n)$, *the orthogonal group.*

Proof. Let $x \in \mathbb{R}^n \backslash \{0\}$. By Lemma 1.39, $\lambda_n(x) = \lambda_n(\|x\| \cdot \frac{x}{\|x\|}) = \lambda_n(\frac{x}{\|x\|})$, which is why we can assume w.l.o.g. that $\|x\| = 1$. Choose an orthonormal basis $(e_1 = x, e_2, \ldots, e_n)$ of $\mathbb{R}^n$. Then, for $v = \sum_{i=1}^n a_i e_i$ we have by the Clifford relations

$$\begin{aligned}\lambda_n(x)(v) = \lambda_n(e_1)\left(\sum_{i=1}^n a_i e_i\right) &= \sum_{i=1}^n a_i \alpha(e_1) \cdot e_i \cdot e_1^{-1} = -\sum_{i=1}^n a_i e_1 \cdot e_i \cdot e_1^{-1} \\ &= -a_1 e_1 - \sum_{i=2}^n a_i e_1 \cdot e_i \cdot e_1^{-1} = -a_1 e_1 + \sum_{i=2}^n a_i e_i \in \mathbb{R}^n .\end{aligned}$$

In particular, $\lambda_n(x)$ is the reflection about $x^\perp$ and $\|\lambda_n(x)(v)\| = \|v\|$.

Now let $x \in \Gamma_n$ be arbitrary. Then

$$\|\lambda_n(x)v\|^2 = N(\lambda_n(x)v) = N(\alpha(x) \cdot v \cdot x^{-1}) = N(\alpha(x)) \cdot N(v) \cdot N(x^{-1}) = N(x) \cdot N(v) \cdot N(x) = N(v) = \|v\|^2 .$$

Hence, $\lambda_n(x) \in \mathrm{O}(n)$. □

Definition 1.42. *The* Pin group $\mathrm{Pin}(n) \subseteq \mathcal{C}\ell_n^*$ *is the kernel of* $N : \Gamma_n \to \mathbb{R}^*$. *The* Spin group $\mathrm{Spin}(n)$ *is the group* $\mathrm{Pin}(n) \cap \mathcal{C}\ell_n^0$.

Theorem 1.43. *(i) The Pin and Spin groups are Lie groups explicitely given by*

$$\begin{aligned}\mathrm{Pin}(n) &= \{v_1 \cdot v_2 \cdots v_k \mid v_i \in \mathbb{R}^n, \|v_i\| = 1, 0 \leqslant i \leqslant k, k \in \mathbb{N}_0\} , \\ \mathrm{Spin}(n) &= \{v_1 \cdot v_2 \cdots v_{2k} \mid v_i \in \mathbb{R}^n, \|v_i\| = 1, 0 \leqslant i \leqslant k, k \in \mathbb{N}_0\} .\end{aligned}$$

(ii) $\lambda_{n|\mathrm{Pin}(n)} : \mathrm{Pin}(n) \to \mathrm{O}(n)$ *is a surjective Lie group homomorphism with kernel* $\{\pm 1\}$.

(iii) $(\lambda_{n|\mathrm{Pin}(n)})^{-1}(\mathrm{SO}(n)) = \mathrm{Spin}(n)$.

(iv) $\mathrm{Spin}(n)$ *is connected for* $n \geqslant 2$.

Proof. Recall that any orthogonal map $A \in \mathrm{O}(n)$ can be written as the concatenation $A_{v_1} \circ \ldots \circ A_{v_k}$ of reflections A_{v_i} about hyperplanes $v_i^\perp$, where $v_i \in \mathbb{R}^n$ with $\|v_i\| = 1$. By Proposition 1.41 and the definition of $\mathrm{Pin}(n)$, $v_1 \cdots v_k \in \mathrm{Pin}(n)$ and $\lambda_n(v_1 \cdots v_k) = \lambda_n(v_1) \cdots \lambda_n(v_k) = A_{v_1} \circ \ldots \circ A_{v_k} = A$. Furthermore, the kernel of $\lambda_{n|\mathrm{Pin}(n)}$ is $\ker \lambda_n \cap \ker N = \{x \in \mathbb{R}^* \cdot 1 \mid N(x) = 1\} = \{\pm 1\}$, which also shows the explicit expression for $\mathrm{Pin}(n)$.

Recall also that the group $\mathrm{SO}(n) \subseteq \mathrm{O}(n)$ can be characterized as the group of maps which can be written as the concatenation of an even number of reflections. This shows (iii) and the explicit expression for $\mathrm{Spin}(n)$.

To see that $\mathrm{Pin}(n)$ is a Lie group, recall that Γ_n is a closed subgroup of the Lie group $\mathcal{C}\ell_n^*$. It is a theorem (see, e.g., [Le13]) that any algebraic subgroup of a Lie group which is topologically closed is automatically a Lie group in its own right. This makes Γ_n into a Lie group and $N : \Gamma_n \to \mathbb{R}^*$ a Lie group homomorphism. Now $\mathrm{Pin}(n)$ is the kernel of N, which makes it a topologically closed algebraic subgroup and therefore a Lie group. Similarly, $\mathrm{Spin}(n)$ is the inverse image of the Lie group $\mathrm{SO}(n)$ and therefore, again, a topologically closed algebraic subgroup, hence a Lie group. The map $\lambda_{n|\mathrm{Pin}(n)}$ is the concatenation of multiplication, inversion and the (restriction of the) linear map α, hence smooth and therefore a Lie group homomorphism.

In light of (iii), it suffices to connect -1 to 1 with an arc in $\mathrm{Spin}(n)$ to see (iv). Such an arc is

$$c : [0, \pi] \ni t \mapsto \cos(t) + \sin(t) e_1 \cdot e_2 = (\sin \tfrac{t}{2} e_1 - \cos \tfrac{t}{2} e_2)(\sin \tfrac{t}{2} e_1 + \cos \tfrac{t}{2} e_2) \in \mathrm{Spin}(n) .$$

□

Remark 1.44. *We will henceforth only be interested in the Spin group and will from now on view* λ_n *as a map*

$$\begin{aligned}\lambda := \lambda_n : \mathrm{Spin}(n) &\to SO(n) \\ g &\mapsto (v \mapsto \alpha(g) \cdot v \cdot g^{-1} = g \cdot v \cdot g^{-1}) .\end{aligned}$$

For the next proposition, recall that the Lie group $\mathcal{C}\ell_n^*$ is an open subset of $\mathcal{C}\ell_n$. Hence, $T_1 \mathcal{C}\ell_n^* = \mathcal{C}\ell_n$. Since $\mathrm{Spin}(n)$ is a submanifold of $\mathcal{C}\ell_n^*$, $T_1 \mathrm{Spin}(n)$ is a subspace of $\mathcal{C}\ell_n$.

Proposition 1.45. *The tanget space to* $\mathrm{Spin}(n)$ *at 1 is*

$$T_1 \mathrm{Spin}(n) = \mathrm{span}_{\mathbb{R}}\{e_i \cdot e_j \,|\, 1 \leqslant i < j \leqslant n\} \subseteq \mathcal{C}\ell_n\,.$$

Proof. For $1 \leqslant i < j \leqslant n$, consider the curve

$$\gamma : \mathbb{R} \ni t \mapsto \cos(t) + \sin(t) e_i \cdot e_j = (\sin \tfrac{t}{2} e_i - \cos \tfrac{t}{2} e_j)(\sin \tfrac{t}{2} e_i + \cos \tfrac{t}{2} e_j) \in \mathrm{Spin}(n) \subseteq \mathcal{C}\ell_n\,.$$

We have $\gamma(0) = 1$ and $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\gamma(t) = e_i \cdot e_j$. This shows "$\supseteq$". By Exercise 7, the stated subset of $\mathcal{C}\ell_n$ clearly has dimension $\frac{1}{2}n(n-1)$. But from Theorem 1.43, we already know that $\dim T_1 \mathrm{Spin}(n) = \dim \mathrm{Spin}(n) = \dim \mathrm{SO}(n) = \frac{1}{2}n(n-1)$, showing "$\subseteq$". ☐

Corollary 1.46. *The Lie algebra of* $\mathrm{Spin}(n)$ *is*

$$\mathfrak{spin}(n) \cong \mathrm{span}_{\mathbb{R}}\{e_i \cdot e_j \,|\, 1 \leqslant i < j \leqslant n\} \subseteq \mathcal{C}\ell_n$$

with the Lie bracket $[x, y] = x \cdot y - y \cdot x$.

Proof. Following Example 1.14, one checks that the Lie algebra of $\mathcal{C}\ell_n^*$ is $\mathcal{C}\ell_n$ equipped with the Lie bracket $[x, y] = x \cdot y - y \cdot x$. The Lie algebra of $\mathrm{Spin}(n)$ then inherits this Lie bracket. ☐

Proposition 1.47. *The differential* $\lambda_* = \mathrm{d}\lambda_e : T_1 \mathrm{Spin}(n) \cong \mathfrak{spin}(n) \to \mathfrak{so}(n) \cong T_{E_n} \mathrm{SO}(n)$ *is an isomorphism explicitely given by*

$$\lambda_*(e_i \cdot e_j) = 2X_{e_i,e_j}\,,$$

where X_{e_i,e_j} *are the maps from Exercise 6.*

Proof. Since $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a surjective Lie group homomorphism between Lie groups of equal dimension, its differential at 1 must be an isomorphism. We consider again for $1 \leqslant i < j \leqslant n$ the path $\gamma : \mathbb{R} \ni t \mapsto \cos(t) + \sin(t) e_i \cdot e_j \in \mathrm{Spin}(n)$. Note that $\gamma(t)^{-1} = \gamma(-t)$. Hence, for $v = \sum_{k=1}^n v_k e_k \in \mathbb{R}^n$ we have

$$\begin{aligned}
\lambda_*(e_i \cdot e_j)(v) &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\lambda(\gamma(t))(v) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\gamma(t) \cdot v \cdot \gamma(t)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\gamma(t) \cdot v \cdot \gamma(-t) = e_i \cdot e_j \cdot v - v \cdot e_i \cdot e_j \\
&= v_i(e_i \cdot e_j \cdot e_i - e_i \cdot e_i \cdot e_j) + v_j(e_i \cdot e_j \cdot e_j - e_j \cdot e_i \cdot e_j) + \sum_{k \neq i,j} v_k(e_i \cdot e_j \cdot e_k - e_k \cdot e_i \cdot e_j) \\
&= 2v_i e_j - 2v_j e_i = 2(v_i e_j - v_j e_i) = 2X_{e_i,e_j} v\,.
\end{aligned}$$

☐

Definition 1.48. *The* (complex) fundamental spin representation *of* $\mathrm{Spin}(n)$ *is the Lie group homomorphism*

$$\kappa_n : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n)$$

given by restricting an irreducible complex representation $\mathbb{C}\ell_n \to \mathrm{End}(\Sigma_n)$ *to* $\mathrm{Spin}(n) \subseteq \mathcal{C}\ell_n \subseteq \mathbb{C}\ell_n^0 \subseteq \mathbb{C}\ell_n$. *We call* Σ_n *the* spinor module *and an element* $s \in \Sigma_n$ *a* spinor.

Proposition 1.49. *When n is odd the definition of the complex spin representation is independent of which irreducible representation of* $\mathbb{C}\ell_n$ *was used. In particular, it is well-defined. Moreover, when n is odd,* κ_n *is irreducible.*

When n is even, there is a decomposition

$$\kappa_n = \kappa_n^+ \oplus \kappa_n^-\,, \qquad \kappa_n^\pm : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n^\pm)$$

into irreducible representations $\kappa_n^\pm$ *called the* positive respectively negative half-spin representations*. Accordingly, the modules* $\Sigma_n^\pm$ *are the* positive respectively negative half-spinor modules*.*

Proof. Let $n = 2m+1$. Recall from Theorem 1.34 that $\mathbb{C}\ell_{2m+1}$ has two irreducible representations $\rho_i : \mathbb{C}\ell_{2m+1} \to \mathrm{GL}(V)$, $i = 1, 2$, which can be distinguished by $\rho_1(\omega_{2m+1}^{\mathbb{C}}) = +\mathrm{id}$ and $\rho_2(\omega_{2m+1}^{\mathbb{C}}) = -\mathrm{id}$. Since α is an algebra automorphism of $\mathbb{C}\ell_{2m+1}$, $\rho_2 \circ \alpha$ is also a representation of $\mathbb{C}\ell_{2m+1}$ with $\rho_2 \circ \alpha(\omega_{2m+1}^{\mathbb{C}}) = \rho_2(-\omega_{2m+1}^{\mathbb{C}}) = -\rho_2(\omega_{2m+1}^{\mathbb{C}}) = +\mathrm{id}$, so ρ_1 and $\rho_2 \circ \alpha$ are equivalent. Now recall that $\mathbb{C}\ell_{2m+1}^0$ is the $(+1)$-eigenspace of α, hence ρ_1 and ρ_2 are equivalent when restricted to $\mathbb{C}\ell_{2m+1}^0$.

By Proposition 1.29 there is an algebra isomorphism $F : \mathbb{C}\ell_{2m} \to \mathbb{C}\ell_{2m+1}^0$. Since $\rho_i \circ F : \mathbb{C}\ell_{2m} \to \mathrm{GL}(V)$ is a nontrivial complex representation of $\mathbb{C}\ell_{2m}$ of dimension 2^m, it must be the unique irreducible one, hence $\rho = \rho_{i|\mathbb{C}\ell_{2m+1}^0}$ is an irreducible representation of $\mathbb{C}\ell_{2m+1}^0$.

To see that $\rho_{|\,\mathrm{Spin}(2m+1)}$ is an irreducible Lie group representation, assume that $\rho_{|\,\mathrm{Spin}(2m+1)}$ splits into the direct sum of two nontrivial representations, i.e., there exists a nontrivial splitting $V = W_1 \oplus W_2$ such that $\rho(x)(W_j) \subseteq W_j$ for all $x \in \mathrm{Spin}(2m+1)$. By Exercise 6 and Theorem 1.43(i), $\mathrm{Spin}(2m+1)$ contains an additive basis $e_{i_1} \cdot e_{i_2} \cdots e_{i_{2k}}$, $1 \leqslant i_1 < i_2 < \ldots < i_{2k} \leqslant 2m+1$ of $\mathbb{C}\ell^0_{2m+1}$. Since ρ is the restriction to $\mathrm{Spin}(2m+1)$ of an irreducible representation of $\mathbb{C}\ell^0_n$, not all of these basis elements leave W_j invariant, i.e., there exists $1 \leqslant i_1 < i_2 < \ldots < i_{2k} \leqslant 2m+1$ and $j \in \{1,2\}$ such that $\rho(e_{i_1} \cdot e_{i_2} \cdots e_{i_{2k}})(W_j) \nsubseteq W_j$. A contradiction. Hence, $\rho_{\mathrm{Spin}(2m+1)}$ is an irreducible representation of $\mathrm{Spin}(2m+1)$.

Now let $n = 2m$. There is exactly one irreducible representation $\rho : \mathbb{C}\ell_{2m} \to \mathrm{GL}(V)$ of $\mathbb{C}\ell_{2m}$. If we restrict ρ to $\mathbb{C}\ell^0_{2m}$, then Proposition 1.35 tells us that $\rho_{|\mathbb{C}\ell^0_{2m}}$ splits into the direct sum of two inequivalent irreducible representations. We argue as in the case $n = 2m+1$ that their restrictions to $\mathrm{Spin}(2m) \subseteq \mathbb{C}\ell^0_{2m}$ are irreducible Lie group representations. □

Remark 1.50. *The fundamental spin representation* is not *induced by a representation of* $\mathrm{SO}(n)$ *(through λ). Indeed,* $-1 \in \mathrm{Spin}(n)$ *and* $\kappa_n(-1) = -\mathrm{id}_{\Sigma_n}$ *while for every representation* $\rho : \mathrm{SO}(n) \to \mathrm{GL}(V)$ *we have* $\rho \circ \lambda(-1) = \rho(E_n) = \mathrm{id}_V$.

Proposition 1.51. *Let* $\rho : \mathbb{C}\ell_n \to \mathrm{GL}(V)$ *be an irreducible representation of the complex Clifford algebra* $\mathbb{C}\ell_n$. *Then there exists an inner product* $\langle\cdot,\cdot\rangle$ *on V such that*

$$\langle \rho(x)(v), \rho(x)(w)\rangle = \langle v, w\rangle \qquad \text{for all} \quad x \in \mathbb{R}^n \subseteq \mathbb{C}\ell_n \quad \text{with} \quad \|x\| = 1 \quad \text{and all} \quad v, w \in V\,. \tag{1.4}$$

In particular,

(i) multiplication by unit vectors is skew-symmetric, *i.e., for all* $x \in \mathbb{R}^n$ *with* $\|x\| = 1$ *and all* $v, w \in V$ *we have*

$$\langle \rho(x)(v), w\rangle = \langle \rho(x)^2(v), \rho(x)(w)\rangle = \langle \rho(x^2)(v), \rho(x)(w)\rangle = -\langle v, \rho(x)(w)\rangle\,,$$

(ii) there exists a $\mathrm{Spin}(n)$-invariant inner product $\langle\cdot,\cdot\rangle$ on Σ_n, *i.e.,* $\langle \kappa_n(g)(\sigma), \kappa_n(g)(\tau)\rangle = \langle \sigma, \tau\rangle$ *for all* $g \in \mathrm{Spin}(n)$ *and* $\sigma, \tau \in \Sigma_n$. *In short:* $\kappa_n : \mathrm{Spin}(n) \to \mathrm{U}(\Sigma_n)$.

Proof. Since ρ is an irreducible representation, there exists a linear isomorphism $F : V \to \mathbb{C}^{2^{\lfloor n/2\rfloor}}$ such that $\rho(\cdot) = F^{-1} \circ \Phi_n(\cdot) \circ F$ in case $n = 2m$ or $\rho(\cdot) = F^{-1} \circ \pi_i \circ \Phi_n(\cdot) \circ F$ if $n = 2m+1$, where Φ_n is the algebra isomorphism from Theorem 1.30 and π_i, $i = 1, 2$, the projection on the first respectively second factor.

We define the inner product $\langle\cdot,\cdot\rangle$ on V to be the pullback $\langle v, w\rangle := (F(v), F(w))$ of the standard hermitian inner product

$$(a, b) = \sum_{i=1}^{2^{n/2}} a_i \overline{b}_i$$

on $\mathbb{C}^{2^{\lfloor n/2\rfloor}}$. Then (1.4) follows from the matrices U, V and W from Theorem 1.30 being unitary. □

Definition 1.52. *(i) A* Clifford multiplication *is a complex linear map*

$$\mu : \mathbb{R}^n \otimes \Sigma_n \to \Sigma_n$$
$$x \otimes \sigma \mapsto x \cdot \sigma := \mu(x \otimes \sigma)$$

which satisfies

$$x \cdot (y \cdot \sigma) + y \cdot (x \cdot \sigma) = -2\langle x, y\rangle \cdot \sigma \qquad \text{for all} \quad x, y \in \mathbb{R}^n, \sigma \in \Sigma_n\,.$$

(ii) Two Clifford multiplications $\mu_1, \mu_2 : \mathbb{R}^n \otimes \Sigma_n \to \Sigma_n$ *are* equivalent *if there exists a vector space isomorphism* $F : \Sigma_n \to \Sigma_n$ *such that*

$$\mu_1(x \otimes \sigma) = F^{-1}(\mu_2(x \otimes F(\sigma))) \qquad \text{for all} \quad x \in \mathbb{R}^n, \sigma \in \Sigma_n\,.$$

Proposition 1.53. *If n is even then there exists up to equivalence exactly one Clifford multiplication. If n is odd there exist up to equivalence exactly two Clifford multiplications one of which is the negative of the other. They can be distinguished by the action of the complex volume element* $\omega_n^{\mathbb{C}}$, *i.e., they satisfy*

$$\omega_n^{\mathbb{C}} \cdot \sigma := i^{\lfloor (n+1)/2\rfloor} e_1 \cdot (e_2 \cdot (\ldots (e_n \cdot \sigma))) = \pm\sigma \qquad \text{for all} \quad \sigma \in \Sigma_n\,.$$

Proof. If $\rho : \mathbb{C}\ell_n \to \mathsf{End}(\Sigma_n)$ is an irreducible representation then $\mu(x \otimes \sigma) := \rho(x)(\sigma)$ is a Clifford multiplication. This shows existence and in case n is odd that there are two Clifford multiplications which can be distinguished by the action of the complex volume element.

To see uniqueness, let $\mu : \mathbb{R}^n \otimes \Sigma_n \to \Sigma_n$ be a Clifford multiplication. Define $\rho : \mathbb{R}^n \to \mathsf{End}(\Sigma_n)$ by $\rho(x)(\sigma) := \mu(x \otimes \sigma)$. Then $\rho(x)^2 = -\|x\|^2 \cdot \mathrm{id}_{\Sigma_n}$. Hence, ρ extends uniquely to an algebra homomorphism $\widetilde{\rho} : \mathcal{C}\ell_n \to$

$\mathsf{End}(\Sigma_n)$ and by complexification to an algebra homomorphism $\tilde{\rho} : \mathbb{C}\ell_n \to \mathsf{End}(\Sigma_n)$. Since $\dim \Sigma_n = 2^{\lfloor n/2 \rfloor}$, $\tilde{\rho}$ must be an irreducible representation. This completes the proof. □

Corollary 1.54. *Every Clifford multiplication satisfies*

(i) $\langle x \cdot \sigma, x \cdot \tau \rangle = \langle \sigma, \tau \rangle$ *and*

(ii) $\langle x \cdot \sigma, \tau \rangle = -\langle \sigma, x \cdot \tau \rangle$

for all $x \in \mathbb{R}^n$ *with* $\|x\| = 1$ *and all* $\sigma, \tau \in \Sigma_n$, *where* $\langle \cdot, \cdot \rangle$ *is the* $\mathrm{Spin}(n)$*-invariant inner product on* Σ_n.

Remark 1.55. *The group* $\mathrm{Spin}(n)$ *acts on* Σ_n *by the fundamental spin representation* $\kappa_n : \mathrm{Spin}(n) \to \mathrm{U}(\Sigma_n)$ *and on* $\mathbb{R}^n$ *by* $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$. *If we form the tensor product* $\mathbb{R}^n \otimes \Sigma_n$, *then* $\mathrm{Spin}(n)$ *acts thereon via the* tensor representation

$$\begin{aligned} \lambda \otimes \kappa_n : \mathrm{Spin}(n) &\to \mathrm{U}(\mathbb{R}^n \otimes \Sigma_n) \\ g &\mapsto (x \otimes \sigma \mapsto \lambda(g)(x) \otimes \kappa_n(g)(\sigma)) \,. \end{aligned}$$

Proposition 1.56. *Every Clifford multiplication* $\mu : \mathbb{R}^n \otimes \Sigma_n \to \Sigma_n$ *is* $\mathrm{Spin}(n)$-equivariant, *i.e., we have*

$$\mu(\lambda \otimes \kappa_n(g)(x \otimes \sigma)) = \kappa_n(g)(\mu(x \otimes \sigma)) \qquad \text{for all} \quad g \in \mathrm{Spin}(n), x \in \mathbb{R}^n, \sigma \in \Sigma_n \,.$$

Put differently, the diagram

$$\begin{array}{ccc} \mathbb{R}^n \otimes \Sigma_n & \xrightarrow{\ \mu\ } & \Sigma_n \\ \Big\downarrow{\scriptstyle \lambda \otimes \kappa_n} & & \Big\downarrow{\scriptstyle \kappa_n} \\ \mathbb{R}^n \otimes \Sigma_n & \xrightarrow{\ \mu\ } & \Sigma_n \end{array}$$

is commutative.

Remark. *The following proof actually shows: If we choose one representative* μ *from the given equivalence class of Clifford multiplications, then there exists precisely one representative* κ_n *of the equivalence class of the fundamental spin representation such that* μ *is* $\mathrm{Spin}(n)$*-equivariant w.r.t.* $\lambda \otimes \kappa_n$ *and* κ_n.

Proof. The Clifford multiplication μ satisfies $\mu(x \otimes \sigma) = \rho(x)(\sigma)$ where $\rho : \mathbb{C}\ell_n \to \mathsf{End}(\Sigma_n)$ is an irreducible representation. We also have $\kappa = \rho_{| \mathrm{Spin}(n)}$. The claim is now a straightforward calculation:

$$\begin{aligned} \mu(\lambda \otimes \kappa_n(g)(x \otimes \sigma)) &= \mu(\lambda(g)(x) \otimes \kappa_n(g)(\sigma)) = \mu(g \cdot x \cdot g^{-1} \otimes \rho(g)(\sigma)) \\ &= \rho(g \cdot x \cdot g^{-1})(\rho(g)(\sigma)) = \rho(g \cdot x \cdot g^{-1} \cdot g)(\sigma) = \rho(g \cdot x)(\sigma) = \rho(g) \circ \rho(x)(\sigma) \\ &= \kappa_n(g)(\mu(x \otimes \sigma)) \,. \end{aligned}$$

□

Remark 1.57. *Since there is no ambiguity about how* $\mathrm{Spin}(n)$ *acts on* Σ_n *(via* κ_n*) respectively* $\mathbb{R}^n$ *(via* λ*), we can abbreviate notation and simply write* $g\sigma$ *respectively* gx *for all* $g \in \mathrm{Spin}(n), x \in \mathbb{R}^n$ *and* $\sigma \in \Sigma_n$.

The equivariance of Clifford multiplication can now be stated very concisely:

$$gx \cdot g\sigma = g(x \cdot \sigma) \qquad \text{for all} \quad g \in \mathrm{Spin}(n), x \in \mathbb{R}^n, \sigma \in \Sigma_n \,.$$

In fact, using this shorthand notation, the proof of Proposition 1.56 becomes very short:

$$gx \cdot g\sigma = g \cdot x \cdot g^{-1} \cdot g\sigma = g \cdot x \cdot \sigma = g(x \cdot \sigma) \,.$$

Note, however, that it is not easy to unravel what exactly is happening here.

2. Intermezzo: Gauge theory

Definition 2.1. *Let* P *be a smooth manifold and* G *a Lie group.*

(i) *A* (right-)action of G on P *is a smooth map*

$$P \times G \ni (p, g) \mapsto p \cdot g \in P$$

such that

- $p \cdot e = p$ *for all* $p \in P$ *and*
- $(p \cdot g) \cdot h = p \cdot (g \cdot h)$ *for all* $g, h \in G$ *and* $p \in P$.

For $g \in G$ *the map* $R_g : P \ni p \mapsto p \cdot g \in P$ *is called* right-translation by g. R_g *is a diffeomorphism with inverse* $R_g^{-1} = R_{g^{-1}}$.

(ii) *A right action of* G *on* P *is*

- free *if* $p \cdot g = p$ *for* $p \in P$ *and* $g \in G$ *implies* $g = e$*, i.e., the only right translation that has fixed points is* R_e*,*
- transitive *if for all* $p, q \in P$ *there exists a* $g \in G$ *such that* $p \cdot g = q$*,*
- simply-transitive *if it is free and transitive, i.e., if for all* $p, q \in P$ *there exists precisely one* $g \in G$ *such that* $p \cdot g = q$*.*

Example 2.2. *Let* V *be a real* n*-dimensional vector space and let* $P := \{v = (v_1, \dots, v_n) \in V^n \mid v \text{ is a basis of } V\}$*. Then* P *is a smooth manifold of dimension* n^2*. The group* $G = \mathrm{GL}(n; \mathbb{R})$ *acts on* P *from the right by*

$$P \times G \ni (v, A) \mapsto v \cdot A = \left(\sum_{i=1}^{n} A_{i,1} v_i, \dots, \sum_{i=1}^{n} A_{i,n} v_i \right) \in P .$$

Indeed, we have $v \cdot E_n = v$ *for all* $v \in P$ *and if* $v \in P$*,* $A, B \in \mathrm{GL}(n; \mathbb{R})$ *then*

$$\begin{aligned}(v \cdot A) \cdot B &= \left(\sum_{i=1}^{n} A_{i,1} v_i, \dots, \sum_{i=1}^{n} A_{i,n} v_i \right) \cdot B = \left(\sum_{j=1}^{n} B_{j,1} \sum_{i=1}^{n} A_{i,j} v_i, \dots, \sum_{j=1}^{n} B_{j,n} \sum_{i=1}^{n} A_{i,j} v_i \right) \\ &= \left(\sum_{i,j=1}^{n} A_{i,j} B_{j,1} v_i, \dots, \sum_{i,j=1}^{n} A_{i,j} B_{j,n} v_i \right) = v \cdot (A \cdot B) .\end{aligned}$$

The action is smooth since it is a polynomial in the entries of its arguments. Moreover, it is easy to see that the action is simply-transitive.

Definition 2.3. *Let* G *be a Lie group and* M *a smooth manifold.*

(i) A G-principal fibre bundle over M *is a triple* $(P, \pi_P; G)$ *consisting of a manifold* P*, a smooth map* $\pi_P : P \to M$ *and a right-action of* G *on* P *such that*
  - *(a)* π_P *is surjective,*
  - *(b) the action of* G *on* P *is free,*
  - *(c)* $\pi_P(p) = \pi_P(q)$ *if and only if there exists* $g \in G$ *such that* $p \cdot g = q$*,*
  - *(d) for every* $x \in M$ *there exists an open neighborhood* $U \subseteq M$ *containing* x *and a* section of P on U*, i.e., a smooth map* $s_U : U \to P$ *such that* $\pi_p \circ s_U = \mathrm{id}_U$*.*

(ii) Let $(P, \pi_P; G)$ *and* $(Q, \pi_Q; G)$ *be* G*-principal fibre bundles over* M*. A smooth map* $\Phi : P \to Q$ *is called* G-principal fibre bundle morphism *if*
  - *(a)* $\pi_Q \circ \Phi = \pi_P$ *and*
  - *(b)* Φ *is* (G-)equivariant*, i.e., we have* $\Phi(p \cdot g) = \Phi(p) \cdot g$ *for all* $p \in P$ *and* $g \in G$*.*

(iii) The G*-principal fibre bundles* P *and* Q *are* isomorphic*, denoted* $P \cong Q$*, if there exists a* G-principal fibre bundle isomorphism*, i.e., a bijective* G*-principal fibre bundle morphism* $\Phi : P \to Q$*.*

Remark 2.4. *(i) By Definition 2.3(i)(b) and (c)* G *acts simply-transitively on every fibre* $P_x := \pi_P^{-1}(x)$ *of* P *over* M*.*

(ii) If there is no danger of confusion we will refer to the total space P *of a* G*-principal fibre bundle* $(P, \pi_P; G)$ *as the principal fibre bundle.*

Example 2.5. *Let* M *be a smooth manifold and* G *a Lie group. Define the manifold* $P := M \times G$ *with* $\pi_P : P \ni (x, p) \mapsto x \in M$ *and the* G*-action on* P *by multiplication of* G *from the right on the second factor. Then* $(P, \pi_P; G)$ *is a* G*-principal fibre bundle called the* trivial G-principal fibre bundle over M*.*

Example 2.6. *Let* M *be a smooth* n*-dimensional manifold. For* $x \in M$ *define*

$$\mathrm{GL}(M)_x := \{v_x = (v_1, \dots, v_n) \mid v_x \text{ is a basis of } T_x M\}$$

and let

$$\mathrm{GL}(M) := \bigcup_{x \in M} \mathrm{GL}(M)_x .$$

Define the projection via

$$\begin{aligned}\pi := \pi_{\mathrm{GL}(M)} : \mathrm{GL}(M) &\to M \\ v_x &\mapsto x .\end{aligned}$$

Note that if $(U, \varphi = (x^1, \dots, x^n))$ *is a coordinate chart of* M*, then for every* $x \in U$ *the associated frame* $s_U(x) := (\partial_1(x), \dots, \partial_n(x)) \in \mathrm{GL}(M)_x$*. The set* $\mathrm{GL}(M)$ *has a unique structure as a smooth manifold if one requires that all such coordinate frames are smooth. This then turns* $\pi_{\mathrm{GL}(M)} : \mathrm{GL}(M) \to M$ *into a smooth map.*

There is a $G = \mathrm{GL}(n;\mathbb{R})$*-right-action of* $\mathrm{GL}(n;\mathbb{R})$ *on* $\mathrm{GL}(M)_x$ *as defined in Example 2.2. This action induces a right-action of* $\mathrm{GL}(n;\mathbb{R})$ *on* $\mathrm{GL}(M)$*:*

$$\begin{aligned} &\mathrm{GL}(M) \times \mathrm{GL}(n;\mathbb{R}) \to \mathrm{GL}(M) \\ &(v_x = (v_1, \dots, v_n), A) \mapsto v_x \cdot A = \left(\sum_{i=1}^n A_{i,1} v_i, \dots, \sum_{i=1}^n A_{i,n} v_i \right) . \end{aligned} \tag{2.1}$$

The principal fibre bundle $(\mathrm{GL}(M), \pi_{\mathrm{GL}(M)}; \mathrm{GL}(n;\mathbb{R}))$ *is called the* frame bundle of M.

Every additional structure on the manifold M defines a subbundle of $\mathrm{GL}(M)$.

Example 2.7. *Let M again be a smooth n-dimensional manifold.*

(i) Assume that M is oriented. Let $G = \mathrm{GL}^+(n;\mathbb{R}) = \{A \in \mathrm{GL}(n;\mathbb{R}) \mid \det A > 0\}$ *and define*

$$\mathrm{GL}^+(M) := \{v_x \in \mathrm{GL}(M)_x \mid v_x \text{ is a positively oriented basis of } T_xM, x \in M\} .$$

We define a $\mathrm{GL}^+(n;\mathbb{R})$ *right-action on* $\mathrm{GL}^+(M)$ *as the restriction of the* $\mathrm{GL}(n;\mathbb{R})$*-action on* $\mathrm{GL}(M)$*. With* $\pi_{\mathrm{GL}^+(M)} = \pi_{\mathrm{GL}(M)}{}_{|\mathrm{GL}^+(M)}$*, the tuple* $(\mathrm{GL}^+(M), \pi_{\mathrm{GL}^+(M)}; \mathrm{GL}^+(n;\mathbb{R}))$ *is then a* $\mathrm{GL}^+(n;\mathbb{R})$*-principal fibre bundle called the* bundle of positively oriented frames.

(ii) Let g be a Riemannian metric on M. Define

$$\mathrm{O}(M) := \mathrm{O}(M, g) := \{v_x \in \mathrm{GL}(M)_x \mid v_x \text{ is an orthonormal basis of } (T_xM, g_x)\} .$$

Analogously to before, we let $\pi_{\mathrm{O}(M)} : \mathrm{O}(M) \ni v_x \mapsto x \in M$ *and define an* $\mathrm{O}(n)$*-right-action on* $\mathrm{O}(M)$ *by restricting the* $\mathrm{GL}(n;\mathbb{R})$*-action on* $\mathrm{GL}(M)$*. Then the* $\mathrm{O}(n)$*-principal fibre bundle* $(\mathrm{O}(M), \pi_{\mathrm{O}(M)}; \mathrm{O}(n))$ *is called the* bundle of orthonormal frames of M.

(iii) Combining the previous two examples leads us to the $\mathrm{SO}(n)$*-principal fibre bundle of* positively oriented orthonormal frames of M. *That is, assume M is oriented and let g be a Riemannian metric on M. Define*

$$\mathrm{SO}(M) := \mathrm{SO}(M, g) := \{v_x \in \mathrm{GL}(M)_x \mid v_x \text{ is a positively oriented orthonormal basis of } (T_xM, g_x)\}$$

and $\pi_{\mathrm{SO}(M)} : \mathrm{SO}(M) \ni v_x \mapsto x \in M$*. Formula* (2.1) *defines an* $\mathrm{SO}(n)$*-right-action on* $\mathrm{SO}(M)$ *turning* $(\mathrm{SO}(M), \pi_{\mathrm{SO}(M)}; \mathrm{SO}(n))$ *into a principal fibre bundle.*

A generalization of the notion of G-principal fibre bundle morphism is the following.

Definition 2.8. *Let* $(P, \pi_P; G)$ *be a G-principal fibre bundle over M and* $f : H \to G$ *a Lie group homomorphism. An f*-reduction of P *is a pair* (Q, Φ) *consisting of an H-principal fibre bundle* $(Q, \pi_Q; H)$ *over M and a smooth map* $\Phi : Q \to P$ *such that*

(i) $\pi_P \circ \Phi = \pi_Q$ *and*
(ii) $\Phi(q \cdot h) = \Phi(q) \cdot f(h)$ *for all* $q \in Q, h \in H$.

Properties (i) and (ii) can be summarized by saying that the diagram

$$\begin{array}{ccccc} Q \times H & \xrightarrow{\ \cdot\ } & Q & & \\ \downarrow{\scriptstyle \Phi \times f} & & \downarrow{\scriptstyle \Phi} & \searrow^{\pi_Q} & \\ P \times G & \xrightarrow{\ \cdot\ } & P & \xrightarrow{\pi_P} & M \end{array}$$

is commutative.

If we are in the situation that $H \subseteq G$ *is a Lie subgroup and* $f = \iota : H \hookrightarrow G$ *is the inclusion, then we also call any f-reduction* (Q, f) *an H*-reduction of P *or* a reduction of P to H.

Example 2.9. *Any of the principal fibre bundles from Example 2.7 together with the inclusion* $\iota : H \to \mathrm{GL}(n;\mathbb{R})$*,* $H = \mathrm{GL}^+(n;\mathbb{R}), \mathrm{O}(n), \mathrm{SO}(n)$*, is an H-reduction of the frame bundle* $\mathrm{GL}(M)$*.*

Definition 2.10. *Let* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ *and M a smooth manifold.*

(i) A $\mathbb{K}$-vector bundle of rank $k < \infty$ over M *is a triple* $(E, \pi_E; V)$ *consisting of a smooth manifold E, a smooth map* $\pi_E : E \to M$ *and a k-dimensional $\mathbb{K}$-vector space V such that*
  (a) π_E *is surjective,*
  (b) $E_x := \pi_E(x)^{-1}$ *is $\mathbb{K}$-linearly isomorphic to V for all $x \in M$ and*
  (c) for all $x \in M$ there exists an open neighborhood $U \subseteq M$ of x and k pointwise linearly independent local sections of E over U, *i.e., there exist k smooth maps* $s_1, \dots, s_k : U \to E$ *such that*

(1) $\pi_E \circ s_j = \mathrm{id}_U$ *for all* $j = 1, \dots, k$ *and*
(2) $(s_1(y), \dots, s_k(y))$ *is a basis of* E_y *for all* $y \in U$.
In case $\mathbb{K} = \mathbb{R}$ *we call* E *a* real vector bundle *and in case* $\mathbb{K} = \mathbb{C}$ *a* complex vector bundle.

(ii) *We denote the space of local sections of* E *over an open set* $U \subseteq M$ *by* $\Gamma(U, E)$*, i.e.,*

$$\Gamma(U, E) = \{s : U \to E \mid s \text{ is smooth and } \pi_E \circ s = \mathrm{id}_U\} .$$

In the particular case $U = M$ *we call the elements of* $\Gamma(U, E)$ *just* sections of E *or sometimes* global sections of E.

(iii) *For sections* $s_1, \dots, s_k : U \to E$ *as in (i)(c) we call the smooth map* $s = (s_1, \dots, s_k) : U \to E^k$ *a* (local) frame for E. *In case* $U = M$*, we call* s *a* global frame for E.

(iv) *Let* E, F *be two* $\mathbb{K}$*-vector bundles over* M. *A smooth map* $\Phi : E \to F$ *is a* vector bundle homomorphism *if*
(a) $\pi_F \circ \Phi = \pi_E$ *and*
(b) $\Phi_{|E_x} : E_x \to F_x$ *is* $\mathbb{K}$*-linear for all* $x \in M$.
We call Φ *a* vector bundle isomorphism *if it is invertible and then we call* E *and* F isomorphic.

Example 2.11. *Let* M *be a smooth manifold.*

(i) *Let* V *be a* k*-dimensional* $\mathbb{K}$*-vector space. Define* $E := M \times V$ *and* $\pi_E : E = M \times V \ni (x, v) \mapsto x \in M$. *If we define*

$$(x, v) + (x, w) := (x, v + w) ,$$
$$\lambda \cdot (x, v) := (x, \lambda \cdot v)$$

for all $x \in M$*,* $v, w \in V$ *and* $\lambda \in \mathbb{K}$*, then* $(E, \pi_E; V)$ *is a rank* k *vector bundle over* M. *We call* E *the* trivial vector bundle with fibre V over M*, or simply* trivial.

The sections $\Gamma(M, E)$ *are smooth maps* $s : M \to E = M \times V$ *satisfying* $\pi_E \circ s(x) = x$*, hence they are of the form* $s(x) = (x, v(x))$ *for some* $v \in C^\infty(M, V)$.

(ii) *The tanget bundle* TM *of* M *is a real vector bundles of rank* $k = \dim M$ *over* M. *The sections* $\Gamma(M, TM)$ *of* TM *are precisely the smooth vector fields* $\mathcal{V}(M)$.

Remark 2.12. *Note that the space of sections* $\Gamma(M, E)$ *of the* $\mathbb{K}$*-vector bundle* E *over* M *is a modul over the ring* $C^\infty(M; \mathbb{K})$ *of smooth* $\mathbb{K}$*-valued functions on* M. *Here, the sum of two sections and the product of a smooth function and a section of* E *are defined pointwise, i.e., for* $f \in C^\infty(M, \mathbb{K})$ *and* $s, t \in \Gamma(M, E)$ *the sections* $s + t, fs \in \Gamma(M, E)$ *are defined by*

$$(s + t)(x) := s(x) + t(x) \in E_x ,$$
$$(fs)(x) := f(x)s(x) \in E_x$$

for all $x \in M$.

In linear algebra we learn how to construct new vector spaces out of given ones, e.g., the dual vector space, the direct sum or tensor product of two vector spaces. These constructions carry directly over to vector bundles.

Definition 2.13. (i) *Let* $(E, \pi_E; V)$ *and* $(F, \pi_F; W)$ *be two* $\mathbb{K}$*-vector bundles of rank* k *and* l*, respectively, over* M. *The* Whitney-Sum of E and F *is the* $\mathbb{K}$*-vector bundle* $(E \oplus F, \pi_{E \oplus F}; V \oplus W)$*, where*

$$E \oplus F := \bigcup_{x \in M} E_x \oplus F_x$$

and

$$\pi_{E \oplus F} : E \oplus F \ni (e_x, f_x) \mapsto x \in M .$$

If $x \in M$ *and* $U, V \subseteq M$ *are neighborhoods of* x *such that there are local frames* $s = (s_1, \dots, s_k) : U \to E^k$ *and* $t = (t_1, \dots, t_l) : V \to F^l$*, then the* $k + l$ *maps*

$$s_{1|W}, \dots, s_{k|W} : W \to E \subseteq E \oplus F , \qquad t_{1|W}, \dots, t_{l|W} : W \to F \subseteq E \oplus F$$

where $W := U \cap V$*, are a pointwise linearly independent. The requirement that all these collections of maps are smooth equips* $E \oplus F$ *with a unique topology and a smooth structure, which then turns* $\pi_{E \oplus F}$ *into a smooth map.*

(ii) *As above, let* $(E, \pi_E; V)$ *and* $(F, \pi_F; W)$ *be two* $\mathbb{K}$*-vector bundles of rank* k *and* l*, respectively, over* M. *We consider the set*

$$E \otimes F := \bigcup_{x \in M} E_x \otimes_{\mathbb{K}} F_x$$

with the projection

$$\pi_{E\otimes F} : E\otimes F \ni \sum_{i,j} e_x^i \otimes f_x^j \mapsto x \in M\,.$$

For local frames s of E and t of F as above, the $k\cdot l$ maps

$$u_{i,j} : W \to E\otimes F \quad i = 1,\dots,k \text{ and } j = 1,\dots,l$$

with

$$u_{i,j}(y) = s_i(y)\otimes t_j(y) \quad \text{for all} \quad y\in W\,,$$

are pointwise linearly independent. The requirement that all such maps constructed out of local frames of E and F are smooth turns $E\otimes F$ uniquely into a smooth manifold and $\pi_{E\otimes F}$ into a smooth map. The vector bundle $(E\otimes F, \pi_{E\otimes F}; V\otimes W)$ is called the tensor product of E and F.

(iii) *Let $(E,\pi_E;V)$ be a $\mathbb{K}$-vector bundle. We consider the set*

$$E^* := \bigcup_{x\in M} E_x^*$$

and the projection

$$\pi_{E^*} : E^* \ni \alpha_x \mapsto x\in M\,.$$

If $s = (s_1,\dots,s_k) : U\to E^k$ is a local frame of E, then we define the dual frame $\varphi = (\varphi_1,\dots,\varphi_k) : U \to (E^*)^k$ by requiring that

$$(\varphi_1(x),\dots,\varphi_k(x))$$

is the basis of E_x^* dual to the basis $(s_1(x),\dots,s_k(x))$ of E_x, for all $x\in U$. That is, $\varphi_i(x)(s_j(x)) = \delta_{i,j}$ for all $x\in U$. The requirement that all such dual frames are smooth turns E^* uniquely into a smooth manifold and π_{E^*} into a smooth map. The vector bundle $(E^*,\pi_{E^*};V^*)$ is the dual vector bundle of E.

(iv) *Let $(E,\pi_E;V)$ be a complex vector bundle over M and let $\overline{V}$ be the complex conjugate vector space. That is, $\overline{V}$ is the abelian group V together with the scalar multiplication $\mathbb{C}\times V \ni (z,v)\mapsto \overline{z}\cdot v\in V$. We consider the set*

$$\overline{E} := \bigcup_{x\in M} \overline{E_x}$$

with projection

$$\pi_{\overline{E}} : \overline{E} \ni e_x \mapsto x\in M\,.$$

Any local frame $s = (s_1,\dots,s_k) : U \to E^k$ defines a a local frame $s : U \to \overline{E}^k$. Thus, $\overline{E}$ directly inherits the topology and smooth structure from E. The vector bundle $(\overline{E},\pi_{\overline{E}};\overline{V})$ is the complex conjugate vector bundle of *E.*

In case $(E,\pi_E;V)$ is a real vector bundle we define $(\overline{E},\pi_{\overline{E}};\overline{V}) := (E,\pi_E;V)$.

(v) *There exist many more constructions like $\mathsf{Hom}(E,F)$, $\Lambda^l E,\dots$*

Remark 2.14. *(i) In case of the tangent bundle TM of a smooth manifold M, the dual bundle TM^*, called* cotangent bundle, *is denoted T^*M. Note also that in case of the tangent and cotangent bundle we denote the individual fibres by T_xM and T_x^*M instead of TM_x and T^*M_x, respectively.*

(ii) *Note that the above operations $\oplus,\otimes,{}^*,\dots$ induce associated operations on the corresponding sections. For example, if $s\in\Gamma(M,E)$ and $t\in\Gamma(M,F)$, then $s\otimes t\in\Gamma(M,E\otimes F)$ is the section defined by $(s\otimes t)(x) := s(x)\otimes t(x)$.*

Example 2.15. *We consider the real vector bundle $T^*M\otimes T^*M$. An element $b\in(T^*M\otimes T^*M)_x = T_x^*M\otimes T_x^*M$ ($x\in M$) can be thought of as a bilinear form, i.e., given $v,w\in T_xM$ we have $b(v,w)\in\mathbb{R}$. As usual, we call b symmetric if $b(v,w) = b(w,v)$ for all $v,w\in T_xM$ and positive definite if $b(v,v) > 0$ for all $v\in T_xM\backslash\{0\}$. A Riemannian metric g on M is nothing but an element of $\Gamma(M,T^*M\otimes T^*M)$ that is pointwise symmetric and positive definite. In other words, g is a pointwise inner product depending smoothly on the basepoint.*

More generally than the example of a Riemannian metric, we have the notion of a bundle metric.

Definition 2.16. *Let $(E,\pi_E;V)$ be a real or complex vector bundle over M. A bundle metric on E is a section $\langle\cdot,\cdot\rangle\in\Gamma(E^*\otimes\overline{E}^*)$ which is pointwise an inner product, that is, pointwise symmetric and positive definite ($\mathbb{K}=\mathbb{R}$) respectively hermitian and positive definite ($\mathbb{K}=\mathbb{C}$).*

Remark 2.17. *Just as for Riemannian metrics, a simple argument using a partition of unity shows that any vector bundle carries a bundle metric.*

So far, we have introduced two different types of fibre bundles, namely principal fibre bundles and vector bundles. The next definition connects these two seamingly different worlds.

Definition 2.18. *Let M be a smooth manifold, $(P, \pi_P; G)$ a G-principal fibre bundle over M and $\rho : G \to \mathrm{GL}(V)$ a real or complex representation of G on V. Define the set*

$$E := P \times_\rho V := P \times_{(G,\rho)} V := G \times V/\sim$$

where the equivalence relation $\sim$ is given by

$$(p, v) \sim (p \cdot g, \rho(g^{-1})(v)) \quad \text{for all } g \in G\,,$$

the projection $\pi_E : E \ni [p, v] \mapsto \pi_P(p) \in M$, and on each fibre $E_x = P_x \times_{(G,\rho)} V$ the vector space structure

$$\mu[p, v] + \nu[p, w] := [p, \mu v + \nu w] \quad \text{for all} \quad p \in P, v, w \in V, \mu, \nu \in \mathbb{K}\,.$$

We equip E with a topology and smooth structure by requiring that if $s : U \to P$ is a local section of P and $v \in C^\infty(U, V)$, then $U \ni x \mapsto [s(x), v(x)] \in E$ is smooth. The real (V real) resp. complex (V complex) vector bundle $(E, \pi_E; V)$ is the vector bundle associated with P and ρ.

Remark 2.19. *With respect to the construction in the last definition, the operations $\oplus, \otimes, {}^*, \mathsf{Hom}, \ldots$ on vector bundles correspond exactly to the operations denoted by the same symbols on representations.*

Example 2.20. *Let M be a smooth manifold, $\mathrm{GL}(M)$ the frame bundle of M and $\rho : \mathrm{GL}(n; \mathbb{R}) \to \mathrm{GL}(\mathbb{R}^n)$ the standard representation. Then*

$$\Phi : \mathrm{GL}(M) \times_\rho \mathbb{R}^n \to TM$$
$$[(s_1, \ldots, s_n), (x_1, \ldots, x_n)^t] \mapsto \sum_{i=1}^n x_i s_i$$

is a vector bundle isomorphism. If $\rho^* : \mathrm{GL}(n; \mathbb{R}) \to \mathrm{GL}((\mathbb{R}^n)^*)$ is the representation dual to ρ, i.e., $\rho^*(g)(l)(x) = l(\rho(g^{-1})x)$ for all $l \in (\mathbb{R}^n)^*$ and $x \in \mathbb{R}^n$, then

$$\Psi : \mathrm{GL}(M) \times_{\rho^*} (\mathbb{R}^n)^* \to T^*M$$
$$[(s_1, \ldots, s_n), (y_1, \ldots, y_n)] \mapsto \sum_{i=1}^n y_i \sigma_i\,,$$

where $(\sigma_1, \ldots, \sigma_n)$ is the basis dual to $(s_1, \ldots, s_n)$, is a vector bundle isomorphism.

Proposition 2.21. *Let M be a smooth manifold, $(P, \pi_P; G)$ a G-principal fibre bundle over M and $\rho : G \to \mathrm{GL}(V)$ a representation. If there exists a G-invariant inner product $\langle \cdot, \cdot \rangle$ on V then on the vector bundle $E = P \times_\rho V$ associated with P and ρ there exists a bundle metric given by*

$$\langle e, f \rangle_{E_x} := \langle v, w \rangle\,,$$

where $e = [p, v]$ and $f = [p, w]$ for some $p \in P_x$.

Proof. We have to show that the bundle metric is well-defined, i.e., independent of the chosen representatives. Let $q \in P_x$ and let $g \in G$ be the unique element such that $q = p \cdot g$. Then we have by definition $e = [p, v] = [p \cdot g, \rho(g^{-1})(v)] = [q, \rho(g^{-1})(v)]$ and $f = [p, w] = [p \cdot g, \rho(g^{-1})(w)] = [q, \rho(g^{-1})(w)]$. Since the inner product on V is G-invariant, we have $\langle v, w \rangle = \langle \rho(g^{-1})(v), \rho(g^{-1})(w) \rangle$. Hence, the bundle metric is well-defined. $\square$

Definition 2.22. *Let $(E, \pi_E; V)$ be a $\mathbb{K}$-vector bundle over M.*

(i) *A $\mathbb{K}$-linear map*

$$\nabla : \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$$

is called covariant derivative / connection on E *if*

$$\nabla(fs) = \mathrm{d}f \otimes s + f \cdot \nabla s \quad \text{for all} \quad f \in C^\infty(M, \mathbb{K}), s \in \Gamma(M, E)\,.$$

If $s \in \Gamma(M, E)$ and $X \in \mathcal{V}(M)$, then the section $\nabla_X s := \nabla s(X) \in \Gamma(E)$ is called covariant derivative of s in direction X.

(ii) *If E comes with a bundle metric, a covariant derivative ∇ in E is called* metric *if*

$$X\langle s, t \rangle = \langle \nabla_X s, t \rangle + \langle s, \nabla_X t \rangle$$

for all $X \in \mathcal{V}(M), s, t \in \Gamma(M, E)$. Here, $\langle s, t \rangle \in C^\infty(M)$ is the function $\langle s, t \rangle(x) := \langle s(x), t(x) \rangle_{E_x}$.

Example 2.23. *The Levi-Civita connection of a Riemannian manifold $(M, \langle\cdot,\cdot\rangle = g)$ is the unique covariant derivative ∇^{LC} on $E = TM$ given by the Koszul formula*

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\left(X\langle Y, Z\rangle + Y\langle Z, Y\rangle - Z\langle Y, X\rangle + \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle - \langle [Y,Z], X\rangle\right) .$$

The Levi-Civita connection is metric and, moreover, torsionfree, i.e., $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \equiv 0$.

Note that the torsion tensor T can in general only be defined on the tangent bundle and not on an arbitrary vector bundle E.

3. Spin Geometry

Definition 3.1. *Let (M,g) be an oriented Riemannian manifold.*

(i) A spin-structure *on M is a pair (P,π) consisting of a $\mathrm{Spin}(n)$-principal fibre bundle $(P,\pi_P;\mathrm{Spin}(n))$ over M and a smooth 2-sheeted covering map $\pi : P \to \mathrm{SO}(M,g)$ such that*
  (a) $\pi_{\mathrm{SO}(M,g)} \circ \pi = \pi_P$ and
  (b) $\pi(p\cdot g) = \pi(p)\cdot\lambda(g)$ for all $p \in P$ and $g \in \mathrm{Spin}(n)$ with $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ the Lie group homomorphism from Section 1.3.

  In other words, a spin-structure on M is a λ-reduction of the bundle $\mathrm{SO}(M,g)$ of oriented orthonormal frames of M. We can summarize properties (a) and (b) by saying that the diagram

$$\begin{array}{ccccc} P \times \mathrm{Spin}(n) & \xrightarrow{\;\cdot\;} & P & & \\ \big\downarrow{\scriptstyle \pi\times\lambda} & & \big\downarrow{\scriptstyle \pi} & \searrow^{\pi_P} & \\ \mathrm{SO}(M,g)\times\mathrm{SO}(n) & \xrightarrow{\;\cdot\;} & \mathrm{SO}(M) & \xrightarrow{\pi_{\mathrm{SO}(n)}} & M \end{array}$$

  is commutative.

(ii) Two spin-structures (P_1,π_1) and (P_2,π_2) on M are called equivalent *if there exists a $\mathrm{Spin}(n)$-principal fibre bundle isomorphism $\Phi : P_1 \to P_2$ such that $\pi_1 = \pi_2 \circ \Phi$.*

(iii) If there exists a spin-structure on a Riemannian manifold (M,g), we call M spin.

Remark 3.2. *Note that two equivalent spin-structures (P_1,π_1) and (P_2,π_2) on M provide isomorphic $\mathrm{Spin}(n)$-principal fibre bundles P_1 and P_2. However, the converse is not true. There do exist oriented Riemannian manifolds (M,g) having two inequivalent spin-structures (P_1,π_1) and (P_2,π_2) such that P_1 and P_2 are isomorphic as abstract $\mathrm{Spin}(n)$-principal fibre bundles over M.*

Example 3.3. *Let $M = \mathbb{R}^n$. By identifying $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$, we can equip $\mathbb{R}^n$ with the Riemannian metric g given by the Euclidean inner product,*

$$g_x(v,w) := \langle v,w\rangle \quad \text{for all} \quad x \in \mathbb{R}^n, v,w \in T_x\mathbb{R}^n = \mathbb{R}^n ,$$

and its standard orientation given by requiring that the canonical basis $(e_1,\dots,e_n)$ of $T_x\mathbb{R}^n = \mathbb{R}^n$ is positively oriented.

The bundle $\mathrm{SO}(\mathbb{R}^n,g)$ of oriented orthonormal frames is trivial, i.e., is given by

$$\mathrm{SO}(\mathbb{R}^n,g) = \mathbb{R}^n \times \mathrm{SO}(n) ,$$

where we have identified an OONB $(v_1,\dots,v_n)$ of $\mathbb{R}^n$ with the matrix $A \in \mathrm{SO}(n)$ whose i-th column is v_i. A spin-structure for $(\mathbb{R}^n,g)$ is now given by (P,π) with

$$P = \mathbb{R}^n \times \mathrm{Spin}(n)$$

and

$$\begin{aligned} \pi : P = \mathbb{R}^n \times \mathrm{Spin}(n) &\to \mathbb{R}^n \times \mathrm{SO}(n) = \mathrm{SO}(\mathbb{R}^n,g) \\ (x,g) &\mapsto (x,\lambda(g)) . \end{aligned}$$

Example 3.4. *We consider the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ with its round standard metric g, i.e.,*

$$g_x(v,w) := \langle v,w\rangle \quad \text{for all} \quad x \in S^n, v,w \in T_xS^n \subseteq T_x\mathbb{R}^{n+1} = \mathbb{R}^{n+1} ,$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product. By our identification $T_x\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$, $x \in \mathbb{R}^{n+1}$, we have

$$T_xS^n = x^\perp = \{v \in \mathbb{R}^{n+1} \mid \langle v,x\rangle = 0\} .$$

The orientation we endow S^n with is defined by requiring any basis $(v_1, \dots, v_n)$ of T_xS^n to be oriented if and only if $(v_1, \dots, v_n, x)$ is an oriented basis of $\mathbb{R}^{n+1}$. It follows that for any positively oriented orthonormal basis $(v_1, \dots, v_n)$ of T_xS^n, $(v_1, \dots, v_n, x)$ is an oriented orthonormal basis of $\mathbb{R}^{n+1}$. Thus, the bundle $\mathrm{SO}(S^n)$ is given by

$$\mathrm{SO}(S^n) = \mathrm{SO}(n+1)\,,$$

where we have identified the OONB $(v_1, \dots, v_n, x)$ of $\mathbb{R}^{n+1}$ with the matrix A in $\mathrm{SO}(n+1)$ having $v_1, \dots, v_n, x$ as columns, with projection

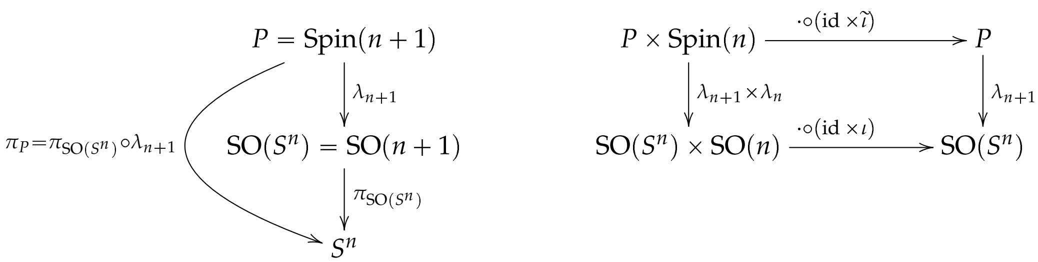
$$\begin{aligned} \pi_{\mathrm{SO}(S^n)} : \mathrm{SO}(S^n) = \mathrm{SO}(n+1) &\to S^n \\ (v_1, \dots, v_n, x) = A &\mapsto x = A \cdot e_{n+1}\,. \end{aligned}$$

The right-action of $\mathrm{SO}(n)$ on $\mathrm{SO}(S^n) = \mathrm{SO}(n+1)$ is given by the right-multiplication of $\mathrm{SO}(n+1)$ on itself precomposed with the inclusion

$$\begin{aligned} \iota : \mathrm{SO}(n) &\to \mathrm{SO}(n+1) \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Associated with the inclusion ι is an inclusion $\tilde{\iota} : \mathrm{Spin}(n) \to \mathrm{Spin}(n+1)$, which can be constructed as follows. The inclusion $\mathbb{R}^n \cong \mathbb{R}^n \times \{0\} \hookrightarrow \mathbb{R}^{n+1}$ induces an inclusion $\mathcal{C}\ell_n \hookrightarrow \mathcal{C}\ell_{n+1}$ (the image of which is the algebra generated by $e_1, \dots, e_n$), which, by restriction, induces an inclusion $\tilde{\iota} : \mathrm{Spin}(n) \to \mathrm{Spin}(n+1)$. It follows from the construction of the map λ from Section 1.3 that $\lambda_{n+1}(\tilde{\iota}(g)) = \iota(\lambda_n(g))$ for all $g \in \mathrm{Spin}(n)$.

To construct our spin-structure for S^n we set $P := \mathrm{Spin}(n+1)$. The right-action of $\mathrm{Spin}(n)$ on P is given by right-multiplication of $\mathrm{Spin}(n+1)$ on itself precomposed with the inclusion $\tilde{\iota}$. We set $\pi := \lambda_{n+1} : P = \mathrm{Spin}(n+1) \to \mathrm{SO}(n+1) = \mathrm{SO}(S^n)$ and define the projection $\pi_P : P \to S^n$ which makes P into a principal fibre bundle over S^n by $\pi_P := \pi_{\mathrm{SO}(S^n)} \circ \lambda_{n+1}$. Now (P, π) is a spin-structure for S^n. We summarize the situation in two commutative diagrams:



Example 3.5. *Let $M = S^1 \cong [0, 2\pi]/\{0, 2\pi\}$ with the metric it inherits from its embedding into $\mathbb{C} \cong \mathbb{R}^2$ and the counterclockwise orientation. Since in dimension 1 there is only one positively oriented unit-vector in each tangent space, we see that $\mathrm{SO}(S^1) \cong S^1$. Note that $\mathrm{SO}(1) = \{1\}$ and $\mathrm{Spin}(1) = \{\pm 1\} = \mathbb{Z}_2$. The first spin-structure we define is $P_1 := S^1 \times \mathbb{Z}_2$ with the obvious projections and right-action of $\mathbb{Z}_2$. We call P_1 the* trivial spin-structure on S^1. *There is a second spin-structure on S^1. Define $P_2 := [0, 2\pi] \times \mathbb{Z}_2/\sim$ where $(0, \pm 1) \sim (2\pi, \mp 1)$ with projection onto S^1 $\pi_{P_2}([x, g]) = x$. We call P_2 the* nontrivial spin-structure on S^1. *The two spin-structures are inequivalent.*

Proposition 3.6. *Let (M, g) be an oriented Riemannian manifold. Then the spin-structures on M are in natural 1:1-correspondence with the 2-sheeted coverings of $\mathrm{SO}(M, g)$ which, in case $n \geqslant 2$, are nontrivial on the fibres of $\pi_{\mathrm{SO}(M,g)}$.*

Proof. By definition, every spin-structure $\pi : P \to \mathrm{SO}(M, g)$ is a two-sheeted covering of $\mathrm{SO}(M, g)$ which, for $n \geqslant 2$, is nontrivial on the fibres of $\pi_{\mathrm{SO}(M,g)}$ since $\pi(p \cdot g) = \pi(p) \cdot \lambda(g)$.

Assume that $n \geqslant 2$ and let $\pi : P \to \mathrm{SO}(M, g)$ be an arbitrary two-sheeted covering which is nontrivial on the fibres of $\pi_{\mathrm{SO}(M,g)}$. W.l.o.g. we assume that M and thus $\mathrm{SO}(M, g)$ and P are connected. Define $\pi_P : P \to M$ by $\pi_P := \pi_{\mathrm{SO}(M,g)} \circ \pi$. Let $R : \mathrm{SO}(M, g) \times \mathrm{SO}(n) \to \mathrm{SO}(M, g)$ be the right-action and define $\overline{R} : P \times \mathrm{Spin}(n) \to \mathrm{SO}(M, g)$ by $\overline{R}(p, g) := R(\pi(p), \lambda(g))$. Then, with $\overline{R}_* : \pi_1(P \times \mathrm{Spin}(n)) \to \pi_1(\mathrm{SO}(M, g))$ the induced map on homotopy groups, we have

$$\begin{aligned} \overline{R}_*(\pi_1(P \times \mathrm{Spin}(n))) &= \overline{R}_*(\pi_1(P) \times \pi_1(\mathrm{Spin}(n))) = R_*(\pi_*(\pi_1(P)), \lambda_*(\pi_1(\mathrm{Spin}(n))))) \\ &= \pi_*(\pi_1(P)) \subseteq \pi_1(\mathrm{SO}(M, g))\,, \end{aligned}$$

since $\pi_1(\mathrm{Spin}(n))$ is trivial. Hence, there exists a unique lift $\tilde{R} : P \times \mathrm{Spin}(n) \to P$ of $\overline{R}$. One easily checks that this is indeed a group action. We have thus turned (the fibre bundle) (P, π_P) into a $\mathrm{Spin}(n)$-principal fibre bundle over M which, by assumption, is a spin-structure. ☐

Remark 3.7. *Not every Riemannian manifold allows a spin-structure. Examples are the even-dimensional real projective spaces $\mathbb{RP}^{2m}$, which are not orientable and so, in particular, not spin. Orientable examples, which are not spin, are the even-dimensional complex projective spaces $\mathbb{CP}^{2m}$. The easiest way to see this is the following theorem.*

For every smooth manifold there exist certain characteristic classes $w_i \in H^i(M;\mathbb{Z}_2)$, called **Stiefel-Whitney classes**. These are obstruction classes to the existence of everywhere linearly independent sections of the tangent bundle: if $w_i(M) \neq 0$, then there do not exist $n-i+1$ everywhere linearly independent continuous vector fields on M. In spin geometry, one is interested in the first and second Stiefel-Whitney class.

Theorem 3.8. *Let M be a smooth manifold.*

(i) M is orientable if and only if $w_1(M) = 0$.
(ii) M is spin if and only if $w_1(M) = 0$ and $w_2(M) = 0$ in the sense that for any choice of orientation and metric, there exists a spin-structure. Moreoever, if M is spin then there is a (nonunique) 1:1-correspondence between inequivalent spin-structures on M and elements of $H^1(M;\mathbb{Z}_2)$.

For the next definition recall the associated vector bundle construction from Definition 2.18.

Definition 3.9. *(i) Let (M,g) be an oriented n-dimensional Riemannian manifold with spin-structure (P,π). Let $\kappa_n : \mathrm{Spin}(n) \to \mathrm{U}(\Sigma_n)$ be the fundamental spin-representation. The complex vector bundle*

$$\Sigma M := P \times_{\kappa_n} \Sigma_n$$

is called the spinor bundle of (M,g) and the spin-structure (P,π).

(ii) A section $s \in \Gamma(M,\Sigma M)$ is called a spinor field *or, sloppily, a* spinor.

Remark 3.10. *(i) The spinor bundle ΣM has rank $\dim \Sigma_n = 2^{\lfloor \frac{n}{2} \rfloor}$. Moreover, since κ_n is a unitary representation it comes equipped with a canonical bundle metric as described in Proposition 2.21.*

(ii) Recall that in case $n = 2m$ the fundamental spin representation splits into the direct sum $\kappa_{2m} = \kappa_{2m}^+ \oplus \kappa_{2m}^-$ of the positive respectively negative half-spin representations $\kappa_{2m}^{\pm} : \mathrm{Spin}(2m) \to \mathrm{U}(\Sigma_{2m}^{\pm})$. To this splitting corresponds a splitting of the spinor bundle (see Remark 2.19)

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M\,,$$

where the vector bundles

$$\Sigma^{\pm} M := P \times_{\kappa^{\pm}} \Sigma_n^{\pm}$$

are called the bundles of positive respectively negative half-spinors. *The sections $s \in \Gamma(M,\Sigma^{\pm}M)$ are called* positive respectively negative half-spinors.

Remark 3.11. *Recall that $\mathbb{R}^n \otimes_{\mathbb{R}} \Sigma_n$ carries a canonical structure as a complex vector space where scalar multiplication with complex numbers is given by multiplication on the second factor.*

Analogously, the real tensor product $TM \otimes \Sigma M$ carries a canonical $\mathbb{C}$-vector bundle structure.

Definition 3.12. *Let (M,g) be an oriented Riemannian manifold with a spin-structure (P,π) and let ΣM be the associated spinor bundle. A* Clifford multiplication *is a vector bundle homormorphism of complex vector bundles*

$$\mu : TM \otimes \Sigma M \to \Sigma M$$
$$v \otimes \sigma \mapsto v \cdot \sigma$$

satisfying

$$v \cdot (w \cdot \sigma) + w \cdot (v \cdot \sigma) = -2g(v,w) \cdot \sigma \quad \text{for all} \quad x \in M, v, w \in T_xM, \sigma \in \Sigma M_x\,.$$

Proposition 3.13. *Let (M,g) be a Riemannian spin manifold with spin-structure (P,π) and let ΣM be the associated spinor bundle.*

(i) If n is even there exists exactly one Clifford multiplication. If n is odd there exist exactly two Clifford multiplications which are the negative of each other. They can be distinguished by the action of the complex volume element, i.e., we have either

$$\omega_n^{\mathbb{C}} \cdot \sigma := i^{\lfloor (n+1)/2 \rfloor} e_1 \cdot (e_2 \cdot (\ldots (e_n \cdot \sigma))) = \sigma \qquad \text{for all} \quad x \in M, \sigma \in \Sigma M_x\,,$$

or

$$\omega_n^{\mathbb{C}} \cdot \sigma = -\sigma \qquad \text{for all} \quad x \in M, \sigma \in \Sigma M_x\,,$$

where $(e_1,\ldots,e_n)$ is an OONB of T_xM.

(ii) Any Clifford multiplication satisfies

$$\langle v \cdot \sigma, \tau \rangle = -\langle \sigma, v \cdot \tau \rangle \quad \text{for all} \quad x \in M, v \in T_xM, \sigma, \tau \in \Sigma M_x\,.$$

Proof. To proof (i), we first note that the tangent bundle TM is associated to the spin-structure (P,π) via the representation $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$. More precisely, the vector bundle homomorphism

$$P \times_\lambda \mathbb{R}^n \to TM$$
$$[p,(x_1,\dots,x_n)^t] \mapsto \sum_{i=1}^n x_i \pi(p)_i$$

is an isomorphism. Here, for $p \in P_x$ we have $\pi(p) = (\pi(p)_1,\dots,\pi(p)_n) \in \mathrm{SO}(M,g)_x$. Alluding to Remark 2.19 again, it follows that the vector bundle $TM \otimes \Sigma M$ is associated to P and the representation $\lambda \otimes \kappa_n : \mathrm{Spin}(n) \to \mathrm{GL}(\mathbb{R}^n \otimes \Sigma_n)$ through the isomorphism

$$P \times_{\lambda\otimes\kappa_n} (\mathbb{R}^n \otimes \Sigma_n) \to TM \otimes \Sigma M$$
$$[p, x\otimes\sigma] \mapsto \sum_{i=1}^n x_i \pi(p)_i \otimes [p,\sigma]\,.$$

If $\widetilde{\mu} : \mathbb{R}^n \otimes \Sigma_n \to \Sigma_n$ is any Clifford multiplication as in Definition 1.52, we define the Clifford multiplication

$$\mu : TM \otimes \Sigma M \cong P \times_{\lambda\otimes\kappa_n} (\mathbb{R}^n \otimes \Sigma_n) \to P \times_{\kappa_n} \Sigma_n \cong \Sigma M$$
$$[p, x\otimes\sigma] \mapsto [p, \widetilde{\mu}(x\otimes\sigma)] = [p, x\cdot\sigma]\,.$$

We have to check that μ is well-defined, i.e., is independent of the chosen representative. For this, let $p, q \in P_x$ and let $g \in \mathrm{Spin}(n)$ be the unique element such that $q = p\cdot g$. Then we have

$$[p, x\otimes\sigma] = [p\cdot g, (\lambda\otimes\kappa_n)(g^{-1})(x\otimes\sigma)] = [q, \lambda(g^{-1})(x)\otimes\kappa_n(g^{-1})(\sigma)]$$

and

$$[p, \widetilde{\mu}(x\otimes\sigma)] = [p\cdot g, \kappa_n(g^{-1})(\widetilde{\mu}(x\otimes\sigma))] = [q, \kappa_n(g^{-1})(\widetilde{\mu}(x\otimes\sigma))]\,.$$

From Proposition 1.56 we know that

$$\kappa_n(g^{-1})(\widetilde{\mu}(x\otimes\sigma)) = \widetilde{\mu}((\lambda\otimes\kappa_n)(g^{-1})(x\otimes\sigma)) = \widetilde{\mu}(\lambda(g^{-1})(x)\otimes\kappa_n(g^{-1})(\sigma))$$

so that

$$[p, \widetilde{\mu}(x\otimes\sigma)] = [q, \kappa_n(g^{-1})(\widetilde{\mu}(x\otimes\sigma))] = [q, \widetilde{\mu}(\lambda(g^{-1})(x)\otimes\kappa_n(g^{-1})(\sigma))]$$

as required.

All statements now follow from Proposition 1.53 and Corollary 1.54. □

Remark 3.14. *(i) In case the dimension n of M is odd, we will always fix the Clifford multiplication for which the complex volume element acts by $+\mathrm{id}_{\Sigma M}$.*

(ii) We extend the Clifford multiplication to vector and spinor fields, that is, for $X \in \mathcal{V}(M)$ and $\varphi \in \Gamma(M,\Sigma M)$ we let $X\cdot\varphi$ be the spinor field defined by

$$(X\cdot\varphi)_x := X_x \cdot \varphi(x) \quad \textit{for all} \quad x \in M\,.$$

All relations holding pointwise then also hold as field equations, e.g., we have

$$X\cdot(Y\cdot\varphi) + Y\cdot(X\cdot\varphi) = -2g(X,Y)\cdot\varphi \quad \textit{for all} \quad X,Y \in \mathcal{V}(M), \varphi \in \Gamma(M,\Sigma M)\,.$$

Theorem 3.15. *There exists a metric connection $\nabla = \nabla^\Sigma : \Gamma(M,\Sigma M) \to \Gamma(M, T^*M \otimes \Sigma M)$ on ΣM satisfying*

$$\nabla^\Sigma_X(Y\cdot\varphi) = \nabla_X Y\cdot\varphi + Y\cdot\nabla^\Sigma_X\varphi \quad \textit{for all} \quad X,Y \in \mathcal{V}(M), \varphi \in \Gamma(M,\Sigma M)\,. \tag{3.1}$$

The connection ∇^Σ is called spinor connection *or* Levi-Civita connection.

Remark. *In fact, ∇^Σ is the unique metric connection satisfying* (3.1)*. Unfortunately, we will have to content ourselves with the existence of ∇.*

Proof. Let (P,π) be our spin-structure with which ΣM is associated.

Step 1: For any local section $s : M \subseteq U \to P$ let $(\mathbf{e}_1,\dots,\mathbf{e}_n) := \pi\circ s : U \to \mathrm{SO}(M,g)$ be the projected local OONB. For any $\varphi \in \Gamma(U,\Sigma M)$, given by $\varphi = [s,v]$ for some $v \in C^\infty(U,\Sigma_n)$, define

$$\nabla^s_X\varphi = [s, X(v)] + \frac{1}{4}\sum_{i=1}^n \mathbf{e}_i\cdot\nabla^{\mathrm{LC}}_X\mathbf{e}_i\cdot\varphi \tag{3.2}$$

for any $X \in \mathcal{V}(M)$. Obviously, ∇^s is $\mathbb{C}$-linear with respect to φ, $C^\infty(U,\mathbb{C})$-linear w.r.t. X and satisfies the Leibniz rule.

Step 2: We show that (3.2) is independent of the section s. Let s, t be local sections of P, which are, without loss of generality, defined on the same open set $W \subseteq M$. We let $\sigma : W \to \mathrm{Spin}(n)$ be the unique smooth map which satisfies

$$t = s \cdot \sigma$$

and $v, w \in \mathrm{C}^\infty(W, \Sigma_n)$ such that $\varphi = [s, v] = [t, w]$. Then

$$[s, v] = [s \cdot \sigma, \kappa_n(\sigma^{-1})(v)] = [t, w] .$$

We consider the first term on the right-hand side of (3.2). We have

$$X(w) = X(\kappa_n(\sigma^{-1})(v)) = (\mathrm{d}(\kappa_n \circ \sigma^{-1})X)(v) + \kappa_n(\sigma^{-1})(X(v)) .$$

Since $\mathrm{Spin}(n) \subseteq \mathcal{C}\ell_n^*$, we have $\mathrm{d}(L_g)X = g \cdot X$ respectively $\mathrm{d}(R_g)X = X \cdot g$ (cf. Example **??**) and using Exercise 16 we see that

$$\begin{aligned}(\mathrm{d}(\kappa_n \circ \sigma^{-1})X)(v) &= (\mathrm{d}\kappa_n \circ \mathrm{d}\,\mathrm{inv} \circ \mathrm{d}\sigma X)(v) = -\kappa_n(\mathrm{d}(L_{\sigma^{-1}}) \circ \mathrm{d}(R_{g^{-1}})\mathrm{d}\sigma X)(v) \\ &= -\kappa_n(\sigma^{-1}(\mathrm{d}\sigma X)\sigma^{-1})(v) = -\kappa_n(\sigma^{-1})\kappa_n(\mathrm{d}\sigma X \cdot \sigma^{-1})(v) ,\end{aligned}$$

so that

$$\begin{aligned}[t, X(w)] &= [s \cdot \sigma, -\kappa_n(\sigma^{-1})\kappa_n(\mathrm{d}\sigma X \cdot \sigma^{-1})(v) + \kappa_n(\sigma^{-1})(X(v))] \\ &= [s, X(v)] - [s, \kappa_n(\mathrm{d}\sigma X \cdot \sigma^{-1})(v)] .\end{aligned} \tag{3.3}$$

In order to obtain an expression for $\mathrm{d}\sigma X \cdot \sigma^{-1}$ we will first calculate $\lambda_*(\mathrm{d}\sigma X \cdot \sigma^{-1})$. Denote $A = (A_{ij}) = \lambda \circ \sigma : W \to \mathrm{SO}(n)$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a curve with $\gamma(0) = x \in W$ and $\gamma'(0) = X \in T_xM$. Then

$$\begin{aligned}\lambda_*(\mathrm{d}\sigma X \cdot \sigma^{-1}) &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\lambda(\sigma \circ \gamma(t) \cdot \sigma(x)^{-1}) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(\lambda \circ \sigma \circ \gamma)(t) \cdot \lambda(\sigma(x)^{-1}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} A \circ \gamma(t) \cdot A^t = \mathrm{d}AX \cdot A^t \\ &= \sum_{k=1}^n (X(A_{ik})A_{jk}) \\ &= \frac{1}{2}\sum_{i,j,k=1}^n X(A_{ik})A_{jk}X_{e_i,e_j} ,\end{aligned}$$

where the X_{e_i,e_j} are the matrices from Exercise 5. By Proposition 1.47 we now have

$$\mathrm{d}\sigma X \cdot \sigma^{-1} = \frac{1}{4}\sum_{i,j,k=1}^n X(A_{ik})A_{jk}e_i \cdot e_j . \tag{3.4}$$

Next, we consider the second term on the right-hand side of (3.2). Recall that the tangent bundle is (isomorphic to) the vector bundle $P \times_\lambda \mathbb{R}^n$ associated with the principal fibre bundle P of our spin-structure and the representation λ. With $(\mathbf{e}_1, \ldots, \mathbf{e}_n) = \pi \circ s$ and $(\mathbf{f}_1, \ldots, \mathbf{f}_n) = \pi \circ t$ the projected local OONBs, we have for each $i = 1, \ldots, n$,

$$\begin{aligned}\mathbf{f}_i &= [t, e_i] = [s \cdot \sigma, e_i] = [s, \lambda(\sigma)e_i] = [s, Ae_i] = \left[s, \sum_{j=1}^n A_{ji}e_j\right] = \sum_{k=1}^n A_{ji}[s, e_j] \\ &= \sum_{k=1}^n A_{ji}\mathbf{e}_j ,\end{aligned}$$

which implies

$$\nabla^{\mathrm{LC}}_X \mathbf{f}_i = \sum_{j=1}^n \nabla^{\mathrm{LC}}_X(A_{ji}\mathbf{e}_j) = \sum_{j=1}^n X(A_{ji})\mathbf{e}_j + \sum_{j=1}^n A_{ji}\nabla^{\mathrm{LC}}_X \mathbf{e}_j .$$

Hence,

$$\begin{aligned}\sum_{i=1}^{n} \mathbf{f}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{f}_i \cdot \varphi &= \sum_{i,j,k=1} A_{ji}\mathbf{e}_j \cdot \Big(X(A_{ki})\mathbf{e}_k + A_{ki}\nabla^{\mathrm{LC}}_X \mathbf{e}_k\Big) \cdot \varphi \\ &= \sum_{i,j,k=1} X(A_{ki})A_{ji}\mathbf{e}_j \cdot \mathbf{e}_k \cdot \varphi + \sum_{i,j,k=1} A_{ji}A_{ki}\mathbf{e}_j \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_k \cdot \varphi \\ &= \left[s, \kappa_n \left(\sum_{i,j,k=1} X(A_{ki})A_{ji} e_j \cdot e_k \right)(v) \right] + \sum_{i,j,k=1} A_{ji}A_{ki}\mathbf{e}_j \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_k \cdot \varphi \,.\end{aligned}$$

Since $A^{-1} = A^t$ we have $\sum_i A_{ji}A_{ki} = \delta_{kl}$ and using (3.4) we obtain

$$\sum_{i=1}^{n} \mathbf{f}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{f}_i \cdot \varphi = 4[s, \kappa_n(\mathrm{d}\sigma X \cdot \sigma^{-1})] + \sum_{i=1} \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi \,,$$

which in turn, using (3.3), implies

$$[s, X(v)] + \frac{1}{4}\sum_{i=1}^{n} \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi = [t, X(w)] + \frac{1}{4}\sum_{i=1}^{n} \mathbf{f}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{f}_i \cdot \varphi \,.$$

Step 3: We have to show that our connection is metric and satisfies (3.1). To see that ∇ is metric, let $s : U \to P$ be a local section with $(\mathbf{e}_1, \ldots, \mathbf{e}_n) = \pi \circ s : U \to \mathrm{SO}(M, g)$ the accompanying OONB, $\varphi = [s, v], \psi = [s, w] \in \Gamma(U, \Sigma M)$ with $v, w \in C^\infty(U, \Sigma_n)$ and $X \in T_x M$. Then, by definition of the bundle metric, see Proposition 2.21, we have

$$X\langle \varphi, \psi \rangle = X\langle v, w \rangle = \langle X(v), w \rangle + \langle v, X(w) \rangle = \langle [s, X(v)], \psi \rangle + \langle \varphi, [s, X(w)] \rangle \,.$$

Using the skew-symmetry of Clifford multiplication, that the Levi-Civita connection is metric and the Clifford relations, we see that

$$\begin{aligned}\langle \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi, \psi \rangle + \langle \varphi, \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \psi \rangle &= \langle \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi + \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \mathbf{e}_i \cdot \varphi, \psi \rangle \\ &= -2g(\mathbf{e}_i, \nabla^{\mathrm{LC}}_X \mathbf{e}_i)\langle \varphi, \psi \rangle \,,\end{aligned}$$

which vanishes since

$$0 = Xg(\mathbf{e}_i, \mathbf{e}_i) = 2g(\mathbf{e}_i, \nabla^{\mathrm{LC}}_X \mathbf{e}_i) \,.$$

Hence,

$$X\langle \varphi, \psi \rangle = \langle [s, X(v)], \psi \rangle + \langle \varphi, [s, X(w)] \rangle = \langle \nabla_X \varphi, \psi \rangle + \langle \varphi, \nabla_X \psi \rangle \,.$$

To see that ∇ satisies (3.1) we let $Y = [s, y] \in \Gamma(U, TM)$ with $y \in C^\infty(U, \mathbb{R}^n)$. Observe that

$$Y = [s, y] = \left[s, \sum_{i=1}^{n} y_i e_i \right] = \sum_{i=1}^{n} y_i [s, e_i] = \sum_{i=1}^{n} g(Y, \mathbf{e}_i)\mathbf{e}_i$$

and

$$Y \cdot \varphi = [s, y] \cdot [s, v] = [s, \kappa_n(y)(v)] \,.$$

Thus

$$\nabla_X(Y \cdot \varphi) = [s, X(\kappa_n(y)(v))] + \frac{1}{4}\sum_{i=1}^{n} \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot Y \cdot \varphi \,. \tag{3.5}$$

The first term on the right-hand side is

$$\begin{aligned}X(\kappa_n(y)(v)) &= X(\kappa_n(y))(v) + \kappa_n(y)(X(v)) = \kappa_n(X(y))(v) + \kappa_n(y)(X(v)) \\ &= \sum_{i=1}^{n} X(y_i)\kappa_n(e_i)(v) + \kappa_n(y)(X(v)) = \sum_{i=1}^{n} X(g(\mathbf{e}_i, Y))\kappa_n(e_i)(v) + \kappa_n(y)(X(v)) \,,\end{aligned}$$

so that

$$[s, X(\kappa_n(y)(v))] = \sum_{i=1}^{n} X(g(\mathbf{e}_i, Y))\mathbf{e}_i \cdot \varphi + Y \cdot [s, X(v)] \,. \tag{3.6}$$

Using the Clifford relations, we see that the second term on the right-hand side of (3.5) is

$$\begin{aligned}\sum_{i=1}^n \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot Y \cdot \varphi &= -\sum_{i=1}^n \mathbf{e}_i \cdot Y \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi - 2\sum_{i=1}^n g(\nabla^{\mathrm{LC}}_X \mathbf{e}_i, Y)\mathbf{e}_i, Y) \cdot \varphi \\ &= \sum_{i=1}^n Y \cdot \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi + 2\sum_{i=1}^n g(\mathbf{e}_i, Y)\nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi - 2\sum_{i=1}^n g(\nabla^{\mathrm{LC}}_X \mathbf{e}_i, Y)\mathbf{e}_i \cdot \varphi\,.\end{aligned}$$

Since the Levi-Civita connection is metric, for each $j = 1, \ldots, n$ we have

$$\begin{aligned}\sum_{i=1}^n g(g(\mathbf{e}_i, Y)\nabla^{\mathrm{LC}}_X \mathbf{e}_i, \mathbf{e}_j) &= \sum_{i=1}^n g(\mathbf{e}_i, Y) g(\nabla^{\mathrm{LC}}_X \mathbf{e}_i, \mathbf{e}_j) = -\sum_{i=1}^n g(\mathbf{e}_i, Y) g(\mathbf{e}_i, \nabla^{\mathrm{LC}}_X \mathbf{e}_j) = -g(Y, \nabla^{\mathrm{LC}}_X \mathbf{e}_j) \\ &= -\sum_{i=1}^n g(Y, \nabla^{\mathrm{LC}}_X \mathbf{e}_i) g(\mathbf{e}_i, \mathbf{e}_j) = -\sum_{i=1}^n g(g(Y, \nabla^{\mathrm{LC}}_X \mathbf{e}_i)\mathbf{e}_i, \mathbf{e}_j)\,,\end{aligned}$$

which implies

$$\frac{1}{4}\sum_{i=1}^n \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot Y \cdot \varphi = \frac{1}{4} Y \cdot \sum_{i=1}^n \mathbf{e}_i \cdot \nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi + \sum_{i=1}^n g(\mathbf{e}_i, Y)\nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi\,.$$

From this, (3.5) and (3.6) we obtain

$$\begin{aligned}\nabla_X(Y \cdot \varphi) &= \sum_{i=1}^n X(g(\mathbf{e}_i, Y))\mathbf{e}_i \cdot \varphi + \sum_{i=1}^n g(\mathbf{e}_i, Y)\nabla^{\mathrm{LC}}_X \mathbf{e}_i \cdot \varphi + Y \cdot \nabla_X \varphi \\ &= \nabla^{\mathrm{LC}}_X \left(\sum_{i=1}^n g(\mathbf{e}_i, Y)\mathbf{e}_i\right) \cdot \varphi + Y \cdot \nabla_X \varphi \\ &= \nabla^{\mathrm{LC}}_X Y \cdot \varphi + Y \cdot \nabla_X \varphi\,.\end{aligned}$$

□

Remark 3.16. *On any Riemannian manifold (M, g) there are vector bundle isomorphisms*

$$TM \underset{\sharp}{\overset{\flat}{\rightleftarrows}} T^*M$$

called musical isomorphisms *which are given by the metric, i.e., for any $x \in M$ and $X \in T_xM$ we have*

$$T_xM \ni X \mapsto X^\flat \in T_x^*M$$

with

$$X^\flat(Y) := g_x(X, Y)$$

and

$$\sharp = \flat^{-1}\,.$$

Definition 3.17. *Let (M, g) be a Riemannian spin manifold with spin-structure (P, π), associated spinor bundle ΣM and Clifford multiplication $\mu : TM \otimes \Sigma M \to \Sigma M$. The* Dirac operator *$D$ is the 1st order linear differential operator*

$$\boxed{D : \Gamma(M, \Sigma M) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes \Sigma M) \xrightarrow{\sharp\otimes \mathrm{id}} \Gamma(M, TM \otimes \Sigma M) \xrightarrow{\mu} \Gamma(M, \Sigma M)\,.}$$

Proposition 3.18. *Let $(e_1, \ldots, e_n)$ be a local ONB. Then the Dirac operator is given by*

$$\boxed{D\varphi = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \varphi}$$

for all $\varphi \in \Gamma(M, \Sigma M)$. Moreover, we have

$$D(f\varphi) = \operatorname{grad} f \cdot \varphi + f D\varphi$$

for all $f \in C^\infty(M, \mathbb{C})$ and $\varphi \in \Gamma(M, \Sigma M)$, where grad $f := (\mathrm{d}f)^\sharp$.

Proof. Let $\varepsilon_i = e_i^\flat$ for all $i = 1, \dots, n$. Then

$$\nabla\varphi = \sum_{i=1}^n \varepsilon_i \otimes \nabla_{e_i}\varphi \,,$$

so that

$$D\varphi = \mu \circ (\sharp \otimes \mathrm{id}) \left(\sum_{i=1}^n \varepsilon_i \otimes \nabla_{e_i}\varphi \right) = \mu \left(\sum_{i=1}^n e_i \otimes \nabla_{e_i}\varphi \right) = \sum_{i=1}^n e_i \cdot \nabla_{e_i}\varphi \,.$$

Using the formula we just proved, we see that

$$D(f\varphi) = \sum_{i=1}^n e_i \cdot \nabla_{e_i}(f\varphi) = \sum_{i=1}^n e_i \cdot (e_i(f)\varphi + f\nabla_{e_i}\varphi) = \sum_{i=1}^n e_i(f) e_i \cdot \varphi + \sum_{i=1}^n e_i \cdot \nabla_{e_i}\varphi = \operatorname{grad} f \cdot \varphi + fD\varphi \,.$$

□

Definition 3.19. *Let (M, g) be a Riemannian manifold.*

(i) Denote by $\mathcal{B}(M)$ the Borel σ-algebra *of M, i.e., the smallest σ-algebra containing all open sets of M. We define the* Riemannian measure / volume *$\mu := \mu_g$ on M to be the measure which in every chart (U, x) is given by*

$$\mathrm{d}\mu := \sqrt{\det(g_{ij})}\mathrm{d}\lambda \,,$$

where λ is the (pulled back) Lebesgue-measure in (U, x) and

$$g_{ij} := g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \quad \text{for} \quad i, j = 1, \dots, n \,,$$

are the components of the matrix of g associated with the coordinates $(x^1, \dots, x^n)$.

(ii) Let $(E, \pi_E; V)$ be any $\mathbb{K}$-vector bundle over M and $\varphi \in \Gamma_{C^\infty}(M, E)$. The support of φ *is the set*

$$\operatorname{supp}\varphi := \overline{\{x \in M \mid \varphi(x) \neq 0\}} \,.$$

We say that φ is compactly supported *if $\operatorname{supp}\varphi$ is compact and denote the space of all compactly supported sections by*

$$\Gamma_{C_c^\infty}(M; E) := \{\varphi \in \Gamma_{C^\infty}(M, E) \mid \operatorname{supp}\varphi \text{ is compact}\} \,.$$

In the case of $E = TM$ we additionally introduce the notation

$$\mathcal{V}_c(M) := \Gamma_{C_c^\infty}(M, TM) \,.$$

(iii) Suppose that $(E, \pi_E; V)$ comes equipped with a bundle metric $\langle\cdot,\cdot\rangle$. We define the L^2-inner product *$(\cdot,\cdot) := (\cdot,\cdot)_{L^2}$ on $\Gamma_{C_c^\infty}(M, E)$ by*

$$(\varphi, \psi)_{L^2} := \int_M \langle\varphi, \psi\rangle \mathrm{d}\mu_g$$

and the associated L^2-norm *$|\cdot| := |\cdot|_{L^2}$ by*

$$|\varphi|_{L^2} := \sqrt{(\varphi, \varphi)} \,.$$

Remark 3.20. *Note that $\Gamma_{C_c^\infty}(M; E)$ is in general not* complete *w.r.t. $|\cdot|_{L^2}$, i.e., the pair $(\Gamma_{C_c^\infty}(M; E), (\cdot,\cdot))$ is only a pre-Hilbert space.*

Definition 3.21. *Let (M, g) be a Riemannian manifold and $X \in \mathcal{V}(M)$ a vector field. The* divergence of X *is the function* $\operatorname{div} X \in C^\infty(M)$ *given locally by*

$$\operatorname{div} X = \sum_{i=1}^n g(e_i, \nabla_{e_i} X) = \operatorname{tr}_g(\nabla X) \,,$$

where $(e_1, \dots, e_n)$ is a local ONB.

The familiar Divergence Theorem from vector calculus generalizes to Riemannian manifolds and we state it here without proof.

Theorem 3.22. *Let (M, g) be a Riemannian manifold and $X \in \mathcal{V}_c(M)$. Then*

$$\int_M \operatorname{div} X \mathrm{d}\mu_g = 0 \,.$$

Notation and Remarks 3.23. *We denote by* $TM^{\mathbb{C}}$ *the* complexification of the tangent bundle*. Formally, this is the complex vector bundle over* M *given by*

$$TM^{\mathbb{C}} = \bigcup_{x\in M} (T_xM)^{\mathbb{C}}$$

where $(T_xM)^{\mathbb{C}} = T_xM \otimes_{\mathbb{R}} \mathbb{C}$ *is the complexification of* T_xM*. Each element* $z \in (T_xM)^{\mathbb{C}}$ *can be written as*

$$z = v + iw \quad \text{with} \quad v, w \in T_xM\,.$$

We denote $\mathcal{V}^{\mathbb{C}}_{(c)}(M) := \Gamma_{C^\infty_{(c)}}(M, TM^{\mathbb{C}})$ *and call its elements* complex (compactly supported) vector fields*. Each element* $Z \in \mathcal{V}^{\mathbb{C}}(M)$ *can be written in the form*

$$Z = V + iW \quad \text{for unique} \quad V, W \in \mathcal{V}(M)\,.$$

We extend the Levi-Civita connection ∇ *complex linearly to a connection of* $TM^{\mathbb{C}}$*, denoted by the same symbol, and we do the same with the divergence. The Divergence Theorem is then of course also true for all complex compactly supported vector fields.*

Proposition 3.24. *Let* (M, g) *be an oriented Riemannian spin manifold with a fixed spin-structure. Then the Dirac operator is formally selfadjoint, i.e., we have*

$$(D\varphi, \psi) = (\varphi, D\psi) \quad \text{for all} \quad \varphi, \psi \in \Gamma_{C^\infty_c}(M, \Sigma M)\,.$$

Proof. Let $p \in M$ and $(e_1, \dots, e_n)$ be an ONB defined in a neighborhood of p with $(\nabla e_i)_p = 0$. Then at p we have

$$\begin{aligned}
\langle D\varphi, \psi\rangle_p &= \sum_{i=1}^n \langle e_i \cdot \nabla_{e_i}\varphi, \psi\rangle_p = -\sum_{i=1}^n \langle \nabla_{e_i}\varphi, e_i\cdot\psi\rangle \\
&= -\sum_{i=1}^n \big((e_i)_p\langle\varphi, e_i\cdot\psi\rangle - \langle\varphi, \nabla_{e_i}e_i\cdot\psi\rangle_p - \langle\varphi, e_i\cdot\nabla_{e_i}\psi\rangle_p\big) \\
&= -\sum_{i=1}^n \big((e_i)_p\langle\varphi, e_i\cdot\psi\rangle - \langle\varphi, e_i\cdot\nabla_{e_i}\psi\rangle_p\big) \\
&= -\sum_{i=1}^n (e_i)_p\langle\varphi, e_i\cdot\psi\rangle + \langle\varphi, D\psi\rangle_p\,.
\end{aligned}$$

Denote with $g^{\mathbb{C}}$ the complex bilinear extension of g to $TM^{\mathbb{C}}$ and define a complex compactly supported vector field $X \in \mathcal{V}^{\mathbb{C}}_c(M)$ by the condition

$$g^{\mathbb{C}}_x(X_x, W) = -\langle\varphi(x), W\cdot\psi(x)\rangle_x \quad \text{for all} \quad W \in T_xM, x \in M\,.$$

Then

$$\begin{aligned}
\operatorname{div} X(p) &= \sum_{i=1}^n g^{\mathbb{C}}(\nabla_{e_i}X, e_i)_p = \sum_{i=1}^n \Big((e_i)_p g^{\mathbb{C}}(X, e_i) - g^{\mathbb{C}}(X, \nabla_{e_i}e_i)_p\Big) \\
&= \sum_{i=1}^n (e_i)_p g^{\mathbb{C}}(X, e_i) = -\sum_{i=1}^n (e_i)_p\langle\varphi, e_i\cdot\psi\rangle\,,
\end{aligned}$$

from which we deduce

$$\langle D\varphi, \psi\rangle = \operatorname{div} X + \langle\varphi, D\psi\rangle\,.$$

The statement of the theorem now follows from the Divergence Theorem. □

Corollary 3.25. *Let* (M, g) *be a compact Riemannian spin manifold with a fixed spin-structure. Then*

$$\ker D = \ker D^2\,.$$

Remark 3.26. *We call any spinor* $\varphi \in \Gamma(M, \Sigma M)$ *with* $D^2\varphi = 0$ harmonic *and in case* M *is compact, this is equivalent to* $D\varphi = 0$*.*

proof of Corollary 3.25. We only need to show $\ker D^2 \subseteq \ker D$. Let $\varphi \in \ker D^2$, i.e., $D^2\varphi = 0$. Then we also have $(D^2\varphi, \varphi) = 0$. Hence,

$$0 = (D^2\varphi, \varphi) = (D\varphi, D\varphi) = \int_M \langle D\varphi, D\varphi\rangle \mathrm{d}\mu_g\,.$$

The integrand is a nonnegative, continuous function. We claim that it must be zero. Assume it is not, i.e., there is a point $p \in M$ such that $\langle D\varphi, D\varphi\rangle_p > 0$. By continuity, there is an open neighborhood of p on which this function is positive. Since the Riemannian measure is of full support (every open set has positive measure), the integral would be positive. A contradiction. Hence, $\langle D\varphi, D\varphi\rangle \equiv 0$ which implies $D\varphi = 0$. □

For the next proposition recall that in even dimensions the spinor bundle splits as $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$. Hence, any section $\varphi \in \Gamma(M, \Sigma M)$ splits uniquely as $\varphi = \varphi^+ + \varphi^-$ with $\varphi^\pm \in \Gamma(M, \Sigma^\pm M)$.

Proposition 3.27. *Let (M^{2m}, g) be an even dimensional Riemannian spin manifold. Then the Dirac operator $D : \Gamma(M, \Sigma^+ M) \oplus \Gamma(M, \Sigma^- M) \to \Gamma(M, \Sigma^+ M) \oplus \Gamma(M, \Sigma^- M)$ splits as*

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} .$$

Proof. By the Clifford relations and skew-symmetry of Clifford multiplication, any unit vector $e \in T_x M$ induces an isometric isomorphism

$$e\cdot : (\Sigma^\pm M)_x \to (\Sigma^\mp M)_x .$$

It follows from the local formula (3.2) that the spinor connection preserves $\Gamma(M, \Sigma^\pm M)$. The local formula from Proposition 3.18 shows that the Dirac operator maps $\Gamma(M, \Sigma^\pm M)$ to $\Gamma(M, \Sigma^\mp M)$. □

Corollary 3.28. *The operators $D^\pm : \Gamma(M, \Sigma^\pm M) \to \Gamma(M, \Sigma^\mp M)$ are formal adjoints of each other w.r.t. the corresponding L^2-products.*

Definition 3.29. *Let (M, g) be a Riemannian spin manifold with a fixed spin-structure and ΣM the associated spinor bundle. Suppose we are given a complex vector bundle E over M with a bundle metric and a metric connection. We consider the bundle $\Sigma M \otimes E$ with its tensor product bundle metric, tensor product connection and the induced Clifford multiplication $\mu : TM \otimes \Sigma M \otimes E \to \Sigma M \otimes E$. Then the operator*

$$D_E : \Gamma(M, \Sigma M \otimes E) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes \Sigma M \otimes E) \xrightarrow{\sharp \otimes \mathrm{id} \otimes \mathrm{id}} \Gamma(M, TM \otimes \Sigma M \otimes E) \xrightarrow{\mu} \Gamma(M, \Sigma M \otimes E)$$

is called twisted Dirac operator with coefficients in E.

Remark 3.30. *If (M, g) is an even dimensional Riemannian spin manifold, then there is a natual splitting $\Sigma M \otimes E = (\Sigma^+ M \otimes E) \oplus (\Sigma^- M \otimes E)$ and a correspoding splitting of the twisted Dirac operator*

$$D_E = \begin{pmatrix} 0 & D_E^- \\ D_E^+ & 0 \end{pmatrix} ,$$

with $D_E^\pm : \Gamma(M, \Sigma^\pm M \otimes E) \to \Gamma(M, \Sigma^\mp M \otimes E)$.

3.1. **The Lichnerowicz formula.** The goal of this section is to come back to the very first lecture and see that, in a suitable sense, the square of the Dirac operator is a Laplacian. The corresponding formula is called the Lichnenrowicz formula (see Theorem 3.37) and it shows that there is an interesting interplay between the geometry of a manifold and the existence of harmonic spinors, i.e., solutions to the equation $D^2\varphi = 0$.

Let (M, g) be a Riemannian manifold. Recall the definition of the Riemannian curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z ,$$

the Ricci curvature tensor

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^n g(R(e_i, X)Y, e_i) = \mathrm{tr}(U \mapsto R(U, X)Y) ,$$

and the scalar curvature

$$\mathrm{scal} = \sum_{i=1}^n \mathrm{Ric}(e_i, e_i) = \mathrm{tr}_g((U, V) \mapsto \mathrm{Ric}(U, V)) = \sum_{i,j=1}^n g(R(e_i, e_j)e_j, e_i) .$$

The Riemannian curvature tensor has the following symmetry properties,

$$\begin{aligned} R(X, Y)Z &= -R(Y, X)Z , \\ g(R(X, Y)Z, W) &= -g(R(X, Y)W, Z) , \\ g(R(X, Y)Z, W) &= g(R(Z, W)X, Y) , \\ R(X, Y)Z + R(Y, Z)X + R(Z, X)Y &= 0 . \end{aligned}$$

The last equation is 1st Bianchi-identity.

It follows from the symmetry properties of the Riemannian curvature tensor, that the Ricci tensor is symmetric, i.e., $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$. It thus defines, by duality, a (pointwise) selfadjoint endomorphism field ric,

$$g(\mathrm{ric}(X),Y) = \mathrm{Ric}(X,Y)\,.$$

Definition 3.31. *Let M be a manifold and $(E,\pi_E;V)$ a $\mathbb{K}$-vector bundle over M equipped with a connection $\nabla^E : \Gamma(M,E) \to \Gamma(M,T^*M\otimes E)$. We define the* curvature tensor R^E of (E,∇^E) *by*

$$R^E(X,Y)\varphi = \nabla^E_X\nabla^E_Y\varphi - \nabla^E_Y\nabla^E_X\varphi - \nabla^E_{[X,Y]}\varphi \quad \textit{for all} \quad X,Y\in\mathcal{V}(M),\varphi\in\Gamma(M,E)\,.$$

Remark 3.32. *A calculation completely analogous to the one for the Riemannian curvature tensor shows that R^E is indeed C^∞-linear in all three arguments so that it is indeed a tensor, i.e., a section $R^E\in\Gamma_{C^\infty}(M,T^*M\otimes T^*M\otimes\mathsf{End}(E))$, and that it is antisymmetric in the first two arguments, i.e., $R^E(X,Y)\sigma = -R^E(Y,X)\sigma$ for all $X,Y\in T_xM$, $\sigma\in E_x$, $x\in M$. Therefore, it can also be viewed as an endomorphism-valued two-form, $R^E\in\Omega^2_{C^\infty}(M,\mathsf{End}(E))$.*

Proposition 3.33. *Let (M,g) be a Riemannian spin manifold with a fixed spin-structure (P,π). Then*

$$R^{\Sigma M}(X,Y)\sigma = \frac{1}{4}\sum_{i=1}^n e_i\cdot R(X,Y)e_i\cdot\sigma\,,$$

where $(e_1,\dots,e_n)$ is an ONB of the corresponding tangent space.

Proof. Let $p\in M$ and let $(e_1,\dots,e_n)$ be a local OONB defined on a neighborhood U of p with $(\nabla e_i)_p = 0$ for all $i=1,\dots,n$. Choose a section $s: U\to P$ such that $\pi\circ s = (e_1,\dots,e_n)$. Let $X,Y\in\mathcal{V}(M)$, $v\in C^\infty(U,\Sigma_n)$ and let $\varphi = [s,v]\in\Gamma(U;\Sigma M)$. Then we have (cmp. the proof of Theorem 3.15, Step 1)

$$\begin{aligned}
\nabla^\Sigma_X\nabla^\Sigma_Y\varphi &= \nabla^\Sigma_X\left([s,Y(v)] + \tfrac14\sum_{i=1} e_i\cdot\nabla_Y e_i\cdot\varphi\right)\\
&= [s,X(Y(v))] + \tfrac14\sum_{i=1}^n e_i\cdot\nabla_X e_i\cdot[s,Y(v)] + \tfrac14\sum_{i=1}^n \nabla^\Sigma_X\left(e_i\cdot\nabla_Y e_i\cdot\varphi\right)\\
&= [s,X(Y(v))] + \tfrac14\sum_{i=1}^n e_i\cdot\nabla_X e_i\cdot[s,Y(v)] + \tfrac14\sum_{i=1}^n\left(\nabla_X e_i\cdot\nabla_Y e_i\cdot\varphi + e_i\cdot\nabla_X\nabla_Y e_i\cdot\varphi + e_i\cdot\nabla_Y e_i\cdot\nabla^\Sigma_X\varphi\right)\,.
\end{aligned}$$

Analogously, we have

$$\nabla^\Sigma_Y\nabla^\Sigma_X\varphi = [s,Y(X(v))] + \tfrac14\sum_{i=1}^n e_i\cdot\nabla_Y e_i\cdot[s,X(v)] + \tfrac14\sum_{i=1}^n\left(\nabla_Y e_i\cdot\nabla_X e_i\cdot\varphi + e_i\cdot\nabla_Y\nabla_X e_i\cdot\varphi + e_i\cdot\nabla_X e_i\cdot\nabla^\Sigma_Y\varphi\right)\,,$$

and also

$$\nabla^\Sigma_{[X,Y]}\varphi = [s,[X,Y](v)] + \tfrac14\sum_{i=1}^n e_i\cdot\nabla_{[X,Y]}e_i\cdot\varphi\,,$$

so that, at the point p, we have

$$R^{\Sigma M}(X_p,Y_p)(\varphi(p)) = \tfrac14\sum_{i=1}^n (e_i)_p\cdot R(X_p,Y_p)(e_i)_p\cdot\varphi(p)\,,$$

as claimed. □

Definition 3.34. *Let (M,g) be a Riemannian manifold and $(E,\pi_E;V)$ a $\mathbb{K}$-vector bundle over M, equipped with a connection ∇^E. The associated* Bochner Laplacian, *also called the* connection Laplacian, *is the linear second order differential operator*

$$\begin{aligned}
\Delta^E : \Gamma_{C^\infty}(M,E) &\to \Gamma_{C^\infty}(M,E)\\
\varphi &\mapsto -\sum_{i=1}^n\left(\nabla^E_{e_i}\nabla^E_{e_i}\varphi - \nabla^E_{\nabla_{e_i}e_i}\varphi\right)\,,
\end{aligned}$$

where $(e_1,\dots,e_n)$ is a local ONB. In case M is a spin manifold and $E=\Sigma M$ is the spinor bundle associated with a spin-structure, we call $\Delta^\Sigma := \Delta^{\Sigma M}$ the spinor Laplacian.

Proposition 3.35. *Let (M, g) be a Riemannian manifold and $(E, \pi_E; V)$ a $\mathbb{K}$-vector bundle with a bundle metric $\langle\cdot,\cdot\rangle$ and a metric connection ∇^E. Then the associated Bochner Laplacian satisfies*

$$(\Delta^E\varphi, \psi) = (\nabla^E\varphi, \nabla^E\psi) \quad \textit{for all} \quad \varphi, \psi \in \Gamma_{C_c^\infty}(M; E).$$

In particular, Δ^E is nonnegative and formally self-adjoint, i.e.,

$$(\Delta^E\varphi, \varphi) \geqslant 0 \quad \textit{and} \quad (\Delta^E\varphi, \psi) = (\varphi, \Delta^E\psi) \quad \textit{for all} \quad \varphi, \psi \in \Gamma_{C_c^\infty}(M; E).$$

Remark. *The expression $|\nabla\varphi|^2$ has to be read as follows. The Riemannian metric g induces a bundle metric g^* on T^*M by*

$$g_x^*(\alpha, \beta) = g_x(\alpha^\sharp, \beta^\sharp) \quad \textit{for all} \quad \alpha, \beta \in T_x^*M, x \in M.$$

The bundle metric g^* is sometimes called the cometric. *Now we can use the tensor product metric $\langle\cdot,\cdot\rangle_\otimes$ on $T^*M \otimes E$ which is given on pure tensors by*

$$\langle \alpha \otimes \sigma, \beta \otimes \tau\rangle_{\otimes x} := g_x^*(\alpha, \beta)\langle\sigma, \tau\rangle_x \quad \textit{for all} \quad \alpha, \beta \in T_x^*M, \sigma, \tau \in E_x, x \in M.$$

Then $|\nabla\varphi|^2$ is the square of the corresponding L^2-norm of $\nabla\varphi$.

Proof. As before, we fix a point $p \in M$ and choose a local ONB $(e_1, \ldots, e_n)$ defined on a neighborhood of p with $(\nabla e_i)_p = 0$ for all $i = 1, \ldots, n$. Then, at p, we have

$$\begin{aligned}
\langle \Delta^E\varphi, \psi\rangle_p &= -\sum_{i=1}^n \langle \nabla_{e_i}\nabla_{e_i}\varphi, \psi\rangle_p = -\sum_{i=1}^n \Big(e_i\langle\nabla^E_{e_i}\varphi, \psi\rangle - \langle\nabla^E_{e_i}\varphi, \nabla^E_{e_i}\psi\rangle\Big)_p \\
&= -\sum_{i=1}^n (e_i)_p\langle\nabla^E_{e_i}\varphi, \psi\rangle + \sum_{i,j=1}^n g(e_i, e_j)_p\langle\nabla^E_{e_i}\varphi, \nabla^E_{e_j}\psi\rangle_p \\
&= -\sum_{i=1}^n (e_i)_p\langle\nabla^E_{e_i}\varphi, \psi\rangle + \sum_{i,j=1}^n g^*(\varepsilon_i, \varepsilon_j)_p\langle\nabla^E_{e_i}\varphi, \nabla^E_{e_j}\psi\rangle_p \\
&= -\sum_{i=1}^n (e_i)_p\langle\nabla^E_{e_i}\varphi, \psi\rangle + \sum_{i,j=1}^n \langle\varepsilon_i \otimes \nabla^E_{e_i}\varphi, \varepsilon_j \otimes \nabla^E_{e_j}\psi\rangle_p \\
&= -\sum_{i=1}^n (e_i)_p\langle\nabla^E_{e_i}\varphi, \psi\rangle + \langle\nabla^E\varphi, \nabla^E\psi\rangle_{\otimes p}.
\end{aligned}$$

In case E is a real vector bundle, we define a compactly supported vector field $X \in \mathcal{V}_c(M)$ by

$$g_x(X_x, W) = -\langle\nabla^E_W\varphi(x), \psi(x)\rangle_x \quad \text{for all} \quad W \in T_xM, x \in M,$$

and in case E is complex we substitute $g^{\mathbb{C}}$ for g to define X as a complex compactly supported vector field. In both cases, a calculation analogous to the one in the proof of Proposition 3.24 shows that

$$\operatorname{div} X(p) = -\sum_{i=1}^n (e_i)_p\langle\nabla^E_{e_i}\varphi, \psi\rangle.$$

Hence, it follows from the Divergence Theorem that

$$(\Delta^E\varphi, \psi) = (\nabla^E\varphi, \nabla^E\psi).$$

Nonnegativity now follows by setting $\psi = \varphi$ and formal selfadjointness of Δ^E follows straightforwardly,

$$(\Delta^E\varphi, \psi) = (\nabla^E\varphi, \nabla^E\psi) = \overline{(\nabla^E\psi, \nabla^E\varphi)} = \overline{(\Delta^E\psi, \varphi)} = (\varphi, \Delta^E\psi).$$

□

Corollary 3.36. *In the situation of Proposition 3.35, every $\varphi \in \Gamma_{C_c^\infty}(M, E)$ which is Δ^E-*harmonic, *i.e., satisfies $\Delta^E\varphi = 0$, is* parallel, *i.e., satisfies $\nabla^E\varphi \equiv 0$.*

Proof. Let $\varphi \in \Gamma_{C_c^\infty}(M, E)$ be harmonic. Since $\Delta^E\varphi = 0$, we also have $(\Delta^E\varphi, \varphi) = 0$. By the last proposition,

$$0 = (\Delta^E\varphi, \varphi) = (\nabla^E\varphi, \nabla^E\varphi) = \int_M \langle\nabla^E\varphi, \nabla^E\varphi\rangle\, d\mu_g.$$

The same argument as in the proof of Corollary 3.25 shows that $\nabla^E\varphi = 0$. □

Theorem 3.37 (Lichnerowicz formula). *Let (M,g) be a Riemannian spin manifold with a fixed spin-structure. Then we have*

$$\boxed{D^2\varphi = \Delta^\Sigma\varphi + \frac{1}{4}\operatorname{scal}\cdot\varphi \quad \textit{for all} \quad \varphi\in\Gamma(M,\Sigma M)\,.}$$

Proof. Let $p\in M$ and choose a local ONB $(e_1,\dots,e_n)$ with $(\nabla e_i)_p = 0$ for all $i=1,\dots,n$. Then, at p, we have

$$\begin{aligned} D^2\varphi &= \sum_{i,j=1}^n e_i\cdot\nabla_{e_i}\left(e_j\cdot\nabla_{e_j}\varphi\right) = \sum_{i,j=1}^n e_i\cdot\left(\nabla_{e_i}e_j\cdot\nabla_{e_j}\varphi + e_j\cdot\nabla_{e_i}\nabla_{e_j}\varphi\right) = \sum_{i,j=1} e_i\cdot e_j\cdot\nabla_{e_i}\nabla_{e_j}\varphi \\ &= -\sum_{i=1}^n\nabla_{e_i}\nabla_{e_i}\varphi + \sum_{i<j} e_i\cdot e_j\cdot\left(\nabla_{e_i}\nabla_{e_j}\varphi - \nabla_{e_j}\nabla_{e_i}\varphi\right)\,. \end{aligned}$$

Since $(\nabla e_i)_p = 0$ and $[e_i,e_j]_p = (\nabla_{e_i}e_j - \nabla_{e_j}e_i)_p = 0$ (the Levi-Civita connection is, by definition, torsionfree), this is equal to

$$\begin{aligned} &-\sum_{i=1}^n\left(\nabla_{e_i}\nabla_{e_i}\varphi - \nabla_{\nabla_{e_i}e_i}\varphi\right) + \sum_{i<j} e_i\cdot e_j\cdot\left(\nabla_{e_i}\nabla_{e_j}\varphi - \nabla_{e_j}\nabla_{e_i}\varphi - \nabla_{[e_i,e_j]}\varphi\right) \\ =&\Delta^\Sigma\varphi + \sum_{i<j} e_i\cdot e_j\cdot R^{\Sigma M}(e_i,e_j)\varphi = \Delta^\Sigma\varphi + \frac{1}{2}\sum_{i,j=1}^n e_i\cdot e_j\cdot R^{\Sigma M}(e_i,e_j)\varphi\,. \end{aligned}$$

It remains to show that the second term on the right hand side is equal to $1/4\operatorname{scal}\varphi$. By Proposition 3.33 this term is

$$\begin{aligned} &\frac{1}{8}\sum_{i,j,k=1}^n e_i\cdot e_j\cdot e_k\cdot R(e_i,e_j)e_k\cdot\varphi = \frac{1}{8}\sum_{i,j,k,l=1} g(R(e_i,e_j)e_k,e_l)e_i\cdot e_j\cdot e_k\cdot e_l\cdot\varphi \\ =&\frac{1}{8}\sum_{l=1}^n\Bigg(\frac{1}{3}\sum_{\substack{i,j,k\\ \text{p.w. dist.}}} g(R(e_i,e_j)e_k + R(e_j,e_k)e_i + R(e_k,e_i)e_j,e_l)e_i\cdot e_j\cdot e_k\cdot \\ &+\sum_{i,j=1}^n g(R(e_i,e_j)e_i,e_l)e_i\cdot e_j\cdot e_i\cdot + \sum_{i,j=1}^n g(R(e_i,e_j)e_j,e_l)e_i\cdot e_j\cdot e_j\cdot\Bigg)e_l\cdot\varphi\,. \end{aligned}$$

By the first Bianchi-identity for the Riemannian curvature tensor, the first sum vanishes and we are left with

$$\begin{aligned} &\frac{1}{8}\sum_{l=1}^n\left(\sum_{i,j=1}^n g(R(e_i,e_j)e_l,e_i)e_j\cdot e_i\cdot e_i\cdot + \sum_{i,j=1}^n g(R(e_j,e_i)e_l,e_j)e_i\cdot e_j\cdot e_j\cdot\right)e_l\cdot\varphi \\ =&-\frac{1}{4}\sum_{i,l=1}^n\operatorname{Ric}(e_i,e_l)e_i\cdot e_l\cdot\varphi = -\frac{1}{4}\sum_{i=1}^n\operatorname{Ric}(e_i,e_i)e_i\cdot e_i\cdot\varphi = \frac{1}{4}\operatorname{scal}\varphi\,, \end{aligned}$$

where we have used the symmetry properties of the curvature tensor, the Ricci curvature and the Clifford relations. □

Remark 3.38. *If we are considering the twisted spinor bundle $\Sigma M\otimes E$ over the spin manifold M, then the first part of the proof of Theorem 3.37 shows that*

$$D_E^2 = \Delta^{\Sigma M\otimes E} + \mathfrak{R}\,, \tag{3.7}$$

where $\mathfrak{R}\in\Gamma_{C^\infty}(M,\mathsf{End}(\Sigma M\otimes E))$ is given by

$$\mathfrak{R}(\sigma) = \frac{1}{2}\sum_{i,j=1}^n e_i\cdot e_j\cdot R^{\Sigma M\otimes E}(e_i,e_j)(\sigma)\,,$$

with $(e_1,\dots,e_n)$ a local ONB. Formula (3.7) *is an example of a* Weitzenböck formula, *also called a* Bochner identity.

Corollary 3.39. *Let (M,g) be a connected, compact Riemannian spin manifold with fixed spin-structure. Assume that* $\operatorname{scal}\geqslant 0$ *and that there exists a point $p\in M$ such that* $\operatorname{scal}(p)>0$. *Then there do not exist any nontrivial harmonic spinors, i.e., the equation*

$$D\varphi = 0\,,\qquad \varphi\in\Gamma(M,\Sigma M)$$

has only the trivial solution.

Proof. Let $\varphi \in \Gamma(M, \Sigma M)$ be a harmonic spinor. Then $D^2\varphi = 0$ and so

$$0 = (D^2\varphi, \varphi) = (\Delta^\Sigma \varphi, \varphi) + \tfrac{1}{4}(\operatorname{scal}\varphi, \varphi)\,,$$

that is,

$$-|\nabla\varphi|^2 = -(\nabla\varphi, \nabla\varphi) = -(\Delta^\Sigma\varphi, \varphi) = \tfrac{1}{4}(\operatorname{scal}\varphi, \varphi) = \tfrac{1}{4}\int_M \operatorname{scal}\|\varphi\|^2 \mathrm{d}\mu_g\,.$$

The right-hand side is nonnegative, so we must have $\nabla\varphi = 0$. Since the spinor connection is metric, this implies that $\|\varphi\|^2$ is constant,

$$X\|\varphi\|^2 = X\langle\varphi, \varphi\rangle = \langle\nabla_X\varphi, \varphi\rangle + \langle\varphi, \nabla_X\varphi\rangle = 0 + 0 \quad \text{for all} \quad X \in T_xM, x \in M\,.$$

By assumption $\operatorname{scal}(p) > 0$ which means we must have $\operatorname{scal} > 0$ on an open neighborhood of p. This implies $\|\varphi\|^2 = 0$ for otherweise the integral on the right hand-side was positive. □

3.2. **Special Spinors and Geometry.** We constructed the spinor bundle and its covariant derivative using the metric and the Levi-Civita connection. This means that the geometry of the spinor bundle is closely related to the geometry of the underlying manifold, a fact which can be seen in the formula for the curvature tensor of ΣM or in the Lichnerowicz-formula. It comes as no surprise that the existence of spinors satisfying certain field equations has strong geometric implications.

Definition 3.40. *Let (M, g) be a Riemannian spin manifold with a fixed spin-structure. Then a spinor $\varphi \in \Gamma(M, \Sigma M)$ is called* parallel *if*

$$\nabla\varphi = 0\,,$$

that is, if $\nabla_X\varphi = 0$ for all $X \in \mathcal{V}(M)$.

Lemma 3.41. *If M is connected and $\varphi \in \Gamma(M, \Sigma M)$ parallel, then the function $\|\varphi\|$ is constant.*

Proof. We have for every $X \in \mathcal{V}(M)$,

$$X\|\varphi\|^2 = X\langle\varphi, \varphi\rangle = \langle\nabla_X\varphi, \varphi\rangle + \langle\varphi, \nabla_X\varphi\rangle = 0 + 0\,.$$

Hence, $\|\varphi\|^2$ is constant and then so is $\|\varphi\|$. □

Theorem 3.42. *Let (M, g) be a connected Riemannian spin manifold with a fixed spin-structure. If there exists a nontrivial parallel spinor $\varphi \in \Gamma(M, \Sigma M)$, then (M, g) is Ricci-flat, i.e.,* $\operatorname{Ric} = 0$.

Proof. Let $\varphi \in \Gamma(M, \Sigma M)$ be nontrivial and parallel. By definition of the curvature tensor $R^{\Sigma M}$, we have

$$R^{\Sigma M}(X, Y)\varphi = 0 \quad \text{for all} \quad X, Y \in \mathcal{V}(M)\,.$$

Fix a point $x \in M$, let $(e_1, \dots, e_n)$ be an ONB of T_xM and $X \in T_xM$. A calculation similar to the one in the proof of the Lichnerowicz formula yields

$$0 = \sum_{i=1}^n e_i \cdot R_x^{\Sigma M}(e_i, X)\varphi(x) = \frac{1}{2}\operatorname{ric}_x(X) \cdot \varphi(x)\,.$$

The previous lemma assures $\varphi(x) \neq 0$. Hence, $\operatorname{ric}_x(X) = 0$ for all $X \in T_xM$, i.e., $\operatorname{ric}_x = 0$. □

A more general notion than that of a parallel spinor is given in the following definition.

Definition 3.43. *Let (M, g) be a Riemannian spin manifold with a fixed spin-structure. A spinor $\varphi \in \Gamma(M, \Sigma M)$ for which there exists a number $\zeta \in \mathbb{C}$ such that*

$$\nabla_X\varphi = \zeta X \cdot \varphi \quad \textit{for all} \quad X \in \mathcal{V}(M)$$

is called a Killing spinor with Killing number ζ.

Remark 3.44. *The defining equation for a Killing spinor is in general well overdetermined. Indeed, if M has dimension n the spinor bundle has rank $2^{\lfloor n/2 \rfloor}$. Hence, locally, $\nabla_X\varphi = \zeta X \cdot \varphi$ is a system of $2^{\lfloor n/2 \rfloor}$ equations in n variables. As we will see in the following propositions, neccessary conditions for Killing spinors to exist are quite restrictive.*

Proposition 3.45. *Let (M, g) be a connected Riemannian spin manifold with a fixed spin-structure and $\varphi \in \Gamma(M, \Sigma M)$ a Killing spinor with Killing number $\zeta \in \mathbb{C}$. Then*

(i) if φ is nontrivial, then $\varphi(x) \neq 0$ for all $x \in M$,
(ii) $D(\varphi) = -n\zeta\varphi$, i.e., φ is an eigenspinor for the Dirac operator with eigenvalue $-n\zeta$.

Proof. (i): Since we already handled the case of parallel spinors, we can assume $\zeta \neq 0$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be any smooth curve and let $\psi : (-\varepsilon, \varepsilon) \ni t \mapsto \varphi(\gamma(t)) \in \Sigma M$. Since φ is a Killing spinor we then have

$$\frac{\nabla}{\mathrm{d}t}\psi(t) = (\nabla_{\gamma'(t)}\varphi)_{\gamma(t)} = \zeta\gamma'(t) \cdot \varphi(\gamma(t)) = \zeta\gamma'(t) \cdot \psi(t)\,,$$

i.e., ψ satisfies a first order ordinary linear differential equation. By uniqueness of solutions of ODEs, $\psi(0) = \varphi(\gamma(0)) = 0$ would imply $\psi \equiv 0$. Since γ was arbitrary, this in turn implies $\varphi \equiv 0$.
(ii): Locally, we have

$$D\varphi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\varphi = \sum_{i=1}^{n} e_i \cdot \zeta e_i \cdot \varphi = -n\zeta\varphi\,.$$

□

Definition 3.46. *Let (M, g) be a Riemannian manifold. A vector field $X \in \mathcal{V}(M)$ is a* Killing (vector) field *if*

$$\mathcal{L}_X g = 0\,,$$

where the Lie-derivative on 2-tensors is given by

$$(\mathcal{L}_X h)(Y, Z) := Xh(Y, Z) - h(\mathcal{L}_X Y, Z) - h(Y, \mathcal{L}_X Z)$$

for all $X, Y, Z \in \mathcal{V}(M)$.

Remark 3.47. *The vector field $X \in \mathcal{V}(M)$ is Killing if and only if*

$$\begin{aligned} 0 &= Xg(Y, Z) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g([X, Y], Z) - g(Y, [X, Z]) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X)\,, \end{aligned}$$

i.e., if and only if $Y \mapsto \nabla_Y X$ is a skew-symmetric endomorphism of the tangent bundle.

Remark 3.48. *Let (M, g) be a Riemannian manifold and assume for simplicity that M is compact. The diffeomorphism group* $\mathrm{Diff}(M)$ *of M is an infinite-dimensional (Fréchet-) Lie group and $\mathcal{V}(M)$ together with the Lie-bracket $[\cdot, \cdot]$ on vector fields is its Lie algebra. This can be seen as follows. Suppose we are given a one-parameter group $t \mapsto \Phi^t$ of diffeomorphisms Φ^t of M with $\Phi^0 = \mathrm{id}_M$. Then $p \mapsto X_p := {}^{\mathrm{d}}\!/_{\mathrm{d}t}{}_{|t=0}\Phi^t(p)$ clearly is a vector field of M. On the other hand, given any $X \in \mathcal{V}(M)$, then, by compactness, X is complete, i.e., for any starting point $p \in M$ the flow $\Phi^t_X(p)$ exists for all time $t \in \mathbb{R}$. In particular, $t \mapsto \Phi^t_X$ is a one-parameter group of diffeomorphisms with $\Phi^0 = \mathrm{id}_M$.*

Inside $\mathrm{Diff}(M)$ *we have the isometry group*

$$\mathrm{Isom}(M, g) := \{\Phi \in \mathrm{Diff}(M) \mid \mathrm{d}\Phi_x : (T_x M, g_x) \to (T_{\Phi(x)}M, g_{\Phi(x)}) \text{ is an isometry for all } x \in M\}\,.$$

This is a (finite-dimensional) Lie group as in Section 1.1. While for a generic *Riemannian metric g on M the isometry group* $\mathrm{Isom}(M, g)$ *will be trivial, there are Riemannian manifolds whose isometry group has dimension $\geqslant 1$. The most prominent example is of course (S^n, g_{round}) with isometry group* $\mathrm{Isom}(S^n, g_{round}) = \mathrm{O}(n+1)$*. A noncompact example is the hyperbolic plane $(\mathbb{H}, g_{hyp})$, where $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ and $g_{hyp} = {}^1\!/_{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$, with isometry group* $\mathrm{Isom}(\mathbb{H}, g_{hyp}) = \mathrm{PSL}(2; \mathbb{R})$ *acting by Möbius transformations.*

A Killing field X is a vector field for which the associated flow Φ^t_X is a one-parameter group of isometries *of (M, g), i.e., for each $t \in \mathbb{R}$ the map $M \ni p \mapsto \Phi^t_X(p) \in M$ is an isometry. Thus, the existence of a Killing field $X \in \mathcal{V}(M)$ on a Riemannian manifold (M, g) is equivalent to the assertion that the isometry group* $\mathrm{Isom}(M, g)$ *has positive dimension. Killing fields are sometimes called* infinitesimal isometries*.*

A typical Killing field on the round sphere can be obtained by differentiating the one-parameter group of rotations around a fixed axis. An example of a Killing field on the hyperbolic plane is $\frac{\partial}{\partial x}$ which corresponds to the one-parameter group of translations along lines parallel to the x-axis.

Proposition 3.49. *Let (M, g) be a connected Riemannian spin manifold with a fixed spin-structure and $\varphi \in \Gamma(M, \Sigma M)$ a Killing spinor with Killing number $\zeta \in \mathbb{R}$. Then the vector field*

$$X := \sum_{j=1}^{n} \mathrm{i}\langle\varphi, e_j \cdot \varphi\rangle e_j \in \mathcal{V}(M)\,,$$

where $(e_1, \ldots, e_n)$ is a local ONB, is a (possibly vanishing) Killing field of (M, g).

Proof. Let $p \in M$ and $(e_1, \dots, e_n)$ a local ONB in a neighborhood of p with $(\nabla e_j)_p = 0$ for all $j = 1, \dots, n$. Let $Y \in T_pM$. Then, at p, we have

$$\begin{aligned}
\nabla_Y X &= \mathrm{i} \sum_{j=1}^n \big(Y(\langle \varphi, e_j \cdot \varphi \rangle) e_j + \langle \varphi, e_j \cdot \varphi \rangle \nabla_Y e_j \big) \\
&= \mathrm{i} \sum_{j=1}^n \big(\langle \nabla_Y \varphi, e_j \cdot \varphi \rangle + \langle \varphi, \nabla_Y (e_j \cdot \varphi) \rangle \big) \, e_j \\
&= \mathrm{i} \sum_{j=1}^n \big(\langle \nabla_Y \varphi, e_j \cdot \varphi \rangle + \langle \varphi, \nabla_Y e_j \cdot \varphi \rangle + \langle \varphi, e_j \cdot \nabla_Y \varphi \rangle \big) \, e_j \\
&= \mathrm{i}\zeta \sum_{j=1}^n \big(\langle Y \cdot \varphi, e_j \varphi \rangle + \langle \varphi, e_j \cdot Y \cdot \varphi \rangle \big) \, e_j \\
&= \mathrm{i}\zeta \sum_{j=1}^n \langle \varphi, e_j \cdot Y \cdot \varphi - Y \cdot e_j \cdot \varphi \rangle e_j \, ,
\end{aligned}$$

so that

$$\begin{aligned}
g(\nabla_Y X, Z) &= \mathrm{i}\zeta \sum_{j=1}^n \langle \varphi, e_j \cdot Y \cdot \varphi - Y \cdot e_j \cdot \varphi \rangle g(e_j, Z) = \mathrm{i}\zeta \sum_{j=1}^n \langle \varphi, g(e_j, Z)(e_j \cdot Y \cdot \varphi - Y \cdot e_j \cdot \varphi) \rangle \\
&= \mathrm{i}\zeta \langle \varphi, Z \cdot Y \cdot \varphi - Y \cdot Z \cdot \varphi \rangle \, ,
\end{aligned}$$

which is skew-symmetric in (Y, Z), i.e., $Y \mapsto \nabla_Y X$ is a skew-symmetric endomorphism of the tangent bundle TM. By the last remark, X is a Killing field. ☐

Proposition 3.50. *Let (M, g) be a connected Riemannian spin manifold with a fixed spin-structure. Assume there exists a Killing spinor $\varphi \in \Gamma(M, \Sigma M)$ with Killing number $\zeta \in \mathbb{C}$. Then we have:*

(i) $\mathrm{ric}(X) = 4(n-1)\zeta^2 X$. *In particular, (M, g) is an Einstein manifold with $\zeta^2 = \frac{1}{4} \frac{\mathrm{scal}}{n(n-1)}$ and $\zeta \in \mathbb{R}$ or $\zeta \in \mathrm{i}\mathbb{R}$.*

(ii) If $\zeta \neq 0$ then (M, g) is locally irreducible, i.e., no point admits a neighborhood U such that $(U, g_{|U})$ is isometric to a Riemmanian product $(V, g_V) \times (W, g_W)$.

Proof. By definition of the curvature tensor we have

$$\begin{aligned}
R^{\Sigma M}(X, Y)\varphi &= \nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{[X,Y]} \varphi = \nabla_X(\zeta Y \cdot \varphi) - \nabla_Y(\zeta X \cdot \varphi) - \zeta [X, Y] \varphi \\
&= \zeta \left(\nabla_X Y \cdot \varphi + Y \cdot \nabla_X \varphi - \nabla_Y X \cdot \varphi - X \cdot \nabla_Y \varphi - [X, Y] \cdot \varphi \right) \\
&= \zeta \left(\nabla_X Y - \nabla_Y X - [X, Y] \right) \varphi + \zeta \left(Y \cdot \zeta X \cdot \varphi - X \cdot \zeta Y \cdot \varphi \right) \\
&= \zeta^2 (Y \cdot X - X \cdot Y) \varphi \, .
\end{aligned}$$

As before we also have

$$\begin{aligned}
\mathrm{ric}(X) \cdot \varphi &= -2 \sum_{i=1}^n e_i \cdot R^{\Sigma M}(X, e_i)\varphi = -2\zeta^2 \sum_{i=1}^n e_i \cdot (e_i \cdot X - X \cdot e_i)\varphi = -2\zeta^2 \sum_{i=1}^n (e_i^2 \cdot X - e_i \cdot X \cdot e_i)\varphi \\
&= -2\zeta^2 \sum_{i=1}^n (e_i^2 \cdot X + e_i^2 \cdot X + 2g(X, e_i)e_i)\varphi = 4(n-1)\zeta^2 X \cdot \varphi \, .
\end{aligned}$$

By Proposition 3.45(i), φ is nowhere zero, which implies $\mathrm{ric}(X) = 4(n-1)\zeta^2 X$, or, equivalently, $\mathrm{Ric}(X, Y) = 4(n-1)\zeta^2 g(X, Y)$. A straightforward calculation yields

$$\mathrm{scal} = \sum_{i=1}^n \mathrm{Ric}(e_i, e_i) = \sum_{i=1}^n 4(n-1)\zeta^2 g(e_i, e_i) = 4n(n-1)\zeta^2 \, .$$

To see (ii) assume $U \subseteq M$ is open and that $(U, g_{|U})$ is isometric to the Riemannian product $(V, g_V) \times (W, g_W)$ by an orientation preserving isometry f. We give $(V \times W, g_{V \times W} = g_V \oplus g_W)$ the spin-structure induced by f so that the spinor bundles over U and $V \times W$ are isomorphic by a vector bundle isomorphism which preseres bundle metrics and covariant derivatives. We now view φ as a spinor on $V \times W$. Let $(x, y) \in V \times W$, $X \in$

$T_xV\backslash\{0\}$, $Y \in T_yW\backslash\{0\}$, so that $X+Y \in T_xV \oplus T_yW \cong T_{(x,y)}V \times W$. Then $R^{V\times W}(X,Y)Z = 0$ for all $Z \in T_xV \oplus T_yW$.

From the above we have on the one hand

$$R^{\Sigma(V\times W)}(X,Y)\varphi(x,y) = \frac{1}{4}\sum_{i=1}^{n} e_i \cdot R^{V\times W}(X,Y)e_i \cdot \varphi(x,y) = 0$$

and on the other hand

$$R^{\Sigma(V\times W)}(X,Y)\varphi(x,y) = \zeta^2(Y\cdot X - X\cdot Y)\varphi(x,y)\,.$$

Since $\zeta \neq 0$ and $g_{V\times W}(X,Y) = 0$ this implies

$$X\cdot Y\cdot \varphi(x,y) = 0\,.$$

But Clifford multiplication by a nonzero vector is an isomorphism ($X\cdot X\cdot\varphi(x,y) = -\|X\|^2\varphi(x,y)$), hence $\varphi(x,y) = 0$, which contradicts Proposition 3.45(i). □

Corollary 3.51. *Let (M,g) be a connected Riemannian spin manifold with a fixed spin-structure. Assume there exists a Killing spinor $\varphi \in \Gamma(M,\Sigma M)$ with Killing number $\zeta \neq 0$.*

(i) If ζ is real and (M,g) complete, then M is compact.
(ii) If ζ is imaginary, M is noncompact.

Proof. By the last proposition we have $\mathrm{Ric} = 4(n-1)\zeta^2 g$. If ζ is real, $4(n-1)\zeta^2 > 0$, and Myers' theorem asserts that M is compact.

If ζ is imaginary, we have $\zeta^2 < 0$ and by Proposition 3.45(ii), φ is an eigenspinor of D^2 with eigenvalue $n^2\zeta^2 < 0$. Assuming M is compact implies

$$0 \leqslant (D\varphi, D\varphi) = (D^2\varphi,\varphi) = n^2\zeta^2(\varphi,\varphi) < 0\,,$$

a contradiction. Hence, M must be noncompact. □

Remark 3.52. *In dimensions three every Einstein manifold has constant sectional curvature. In dimension four, one can show that a Riemannian spin manifold possessing a nontrivial Killing spinor with nonzero Killing number has vanishing Weyl tensor. Since such a manifold is an Einstein space, it follows that its sectional curvature is constant. Thus, Killing spinors become interesting only in dimension $\geqslant 5$.*

Remark 3.53. *Our next goal is an eigenvalue estimate for the Dirac operator. Since the spinor Laplacian is a nonnegative operator, the Lichnerowicz formula tells us that any eigenvalue λ of the Dirac operator on a closed Riemannian manifold (M,g) satisfies $\lambda^2 \geqslant \frac{\mathrm{scal}_0}{4}$, where $\mathrm{scal}_0 := \inf_{x\in M}\mathrm{scal}(x)$. Indeed, let λ be an eigenvalue of D with a corresponding L^2-normalized smooth eigenspinor $\varphi \in \Gamma(M,\Sigma M)$. On the one hand, we have*

$$(D^2\varphi,\varphi) = \lambda^2(\varphi,\varphi) = \lambda^2\,,$$

and on the other hand

$$(D^2\varphi,\varphi) = (\Delta\varphi,\varphi) + \left(\tfrac{1}{4}\,\mathrm{scal}\,\varphi,\varphi\right) \geqslant \tfrac{1}{4}\,\mathrm{scal}_0(\varphi,\varphi) = \tfrac{1}{4}\,\mathrm{scal}_0\,.$$

As the next theorem shows, this inequality is not sharp and we can do better.

Theorem 3.54 (Friedich's inequality)**.** *Let (M^n,g) be closed Riemannian spin manifold with fixed spin-structure. Then every eigenvalue λ of the Dirac operator D satisfies*

$$\lambda^2 \geqslant \frac{n}{n-1}\frac{\mathrm{scal}_0}{4}\,.$$

Moreover, if $\lambda = \pm\frac{1}{2}\sqrt{\frac{n}{n-1}\,\mathrm{scal}_0}$ is an eigenvalue of the Dirac operator with corresponding eigenspinor φ, then φ is a Killing spinor with Killing number $\mp\frac{1}{2}\sqrt{\frac{1}{n(n-1)}\,\mathrm{scal}_0}$. In particular, the scalar curvature is constant.

Remark 3.55. *Friedrich's inequality is sharp. Indeed, equality is attained on, e.g., the sphere where we have $\mathrm{scal}_0 = \mathrm{scal} = n(n-1)$.*

Proof of Theorem 3.54. Let $\zeta \in \mathbb{C}$ and consider the twisted connection

$$\nabla^{\zeta}_X \varphi := \nabla_X \varphi - \zeta X \cdot \varphi\,, \qquad \varphi \in \Gamma(M, \Sigma M), X \in \mathcal{V}(M)\,.$$

$$\begin{aligned}
\langle \nabla^{-\zeta}\varphi, \nabla^{-\zeta}\varphi\rangle &= \sum_{j=1}^{n} \langle \nabla^{-\zeta}_{e_j}\varphi, \nabla^{-\zeta}_{e_j}\varphi\rangle = \sum_{j=1}^{n}\langle \nabla_{e_j}\varphi + \zeta e_j\cdot\varphi, \nabla_{e_j}\varphi + \zeta e_j \cdot \varphi\rangle \\
&= \sum_{j=1}^{n}\Big(\langle \nabla_{e_j}\varphi, \nabla_{e_j}\varphi\rangle + \zeta\langle e_j\cdot\varphi, \nabla_{e_j}\varphi\rangle + \zeta\langle \nabla_{e_j}\varphi, e_j\cdot\varphi\rangle + \zeta^2\langle e_j\cdot\varphi, e_j\cdot\varphi\rangle\Big) \\
&= \sum_{j=1}^{n}\Big(\langle \nabla_{e_j}\varphi, \nabla_{e_j}\varphi\rangle - \zeta\langle \varphi, e_j\cdot\nabla_{e_j}\varphi\rangle - \zeta\langle e_j\cdot\nabla_{e_j}\varphi, \varphi\rangle + \zeta^2\langle \varphi, \varphi\rangle\Big) \\
&= \langle \nabla\varphi, \nabla\varphi\rangle - \zeta\langle \varphi, D\varphi\rangle - \zeta\langle D\varphi, \varphi\rangle + n\zeta^2\langle\varphi,\varphi\rangle\,.
\end{aligned}$$

Integrating this yields

$$(\nabla^{-\zeta}\varphi, \nabla^{-\zeta}\varphi) = (\nabla\varphi, \nabla\varphi) - 2\zeta(D\varphi, \varphi) + n\zeta^2(\varphi,\varphi)\,. \tag{3.8}$$

We also have

$$(D-\zeta)^2\varphi = (D-\zeta)(D\varphi - \zeta\varphi) = D^2\varphi - 2\zeta D\varphi + \zeta^2\varphi\,.$$

Integrating and using the Lichnerowicz formula and Proposition 3.35 we obtain

$$\begin{aligned}
((D-\zeta)^2\varphi,\varphi) = (D^2\varphi - 2\zeta D\varphi + \zeta^2\varphi, \varphi) &= (\Delta\varphi,\varphi) + ((1/4\,\mathrm{scal} + \zeta^2)\varphi,\varphi) - 2\zeta(D\varphi,\varphi) \\
&= (\nabla\varphi, \nabla\varphi) + ((1/4\,\mathrm{scal} + \zeta^2)\varphi,\varphi) - 2\zeta(D\varphi,\varphi)\,.
\end{aligned} \tag{3.9}$$

Let λ be an eigenvalue of D with corresponding eigenspinor $\varphi \in \Gamma(M, \Sigma M)$. Set $\zeta := \lambda/n$. From (3.8) we obtain

$$(\nabla^{-\lambda/n}\varphi, \nabla^{-\lambda/n}\varphi) = (\nabla\varphi,\nabla\varphi) - 2\frac{\lambda^2}{n}(\varphi,\varphi) + n\frac{\lambda^2}{n^2}(\varphi,\varphi) = (\nabla\varphi,\nabla\varphi) - \frac{\lambda^2}{n}(\varphi,\varphi)\,.$$

Combining this with (3.9) yields

$$\begin{aligned}
\left(\lambda - \frac{\lambda}{n}\right)^2(\varphi,\varphi) = ((D-\lambda/n)^2\varphi,\varphi) &= (\nabla\varphi,\nabla\varphi) + \left(\left(\frac{1}{4}\,\mathrm{scal} + \frac{\lambda^2}{n^2}\right)\varphi,\varphi\right) - 2\frac{\lambda^2}{n}(\varphi,\varphi) \\
&= (\nabla^{-\lambda/n}\varphi, \nabla^{-\lambda/n}\varphi) + \left(\frac{\lambda^2}{n^2} - \frac{\lambda^2}{n}\right)(\varphi,\varphi) + \frac{1}{4}(\mathrm{scal}\,\varphi,\varphi)\,.
\end{aligned}$$

Substracting $\lambda^2(1-n)/n^2(\varphi,\varphi)$ from both sides we obtain

$$\lambda^2\frac{n-1}{n}(\varphi,\varphi) = (\nabla^{-\lambda/n}\varphi, \nabla^{-\lambda/n}\varphi) + \frac{1}{4}(\mathrm{scal}\,\varphi,\varphi) \geqslant \frac{\mathrm{scal}_0}{4}(\varphi,\varphi)\,, \tag{3.10}$$

which is the desired inequality.

Now assume that $\lambda = \pm\frac{1}{2}\sqrt{\frac{n}{n-1}\,\mathrm{scal}_0}$. Then we have equality in (3.10), which implies $\nabla^{\lambda/n}\varphi = 0$, i.e., φ is a Killing spinor with Killing number $\lambda/n = \mp\frac{1}{2}\sqrt{\frac{1}{n(n-1)}\,\mathrm{scal}_0}$ and the scalar curvature is automatically constant. □

Appendix A: topological spin structures

Let $\mathrm{GL}_+(n;\mathbb{R})$ be the group of invertible $n \times n$-matrices with real entries and positive determinant. The group $\mathrm{SO}(n) \subseteq \mathrm{GL}_+(n;\mathbb{R})$ is a deformation retract. This can be seen by noting that the Gram-Schmidt-algorithm $\mathrm{GS} : \mathrm{GL}_+(n;\mathbb{R}) \to \mathrm{SO}(n)$ is a continuous map and that $\iota \circ \mathrm{GS} : \mathrm{GL}_+(n;\mathbb{R}) \to \mathrm{GL}_+(n;\mathbb{R})$, where $\iota : \mathrm{SO}(n) \to \mathrm{GL}_+(n;\mathbb{R})$ is the inclusion, is homotopic to the identity.

It follows that $\pi_1(\mathrm{GL}_+(2;\mathbb{R})) = \mathbb{Z}$ and $\pi_1(\mathrm{GL}_+(n \geqslant 3;\mathbb{R})) = \mathbb{Z}_2$. We denote with $\widetilde{\mathrm{GL}_+(n;\mathbb{R})}$ the double cover group (which is the universal cover for $n \geqslant 3$) and with $\Lambda : \widetilde{\mathrm{GL}_+(n;\mathbb{R})} \to \mathrm{GL}_+(n;\mathbb{R})$ the corresponding covering map, which is automatically a Lie group homomorphism.

Definition 3.56. *Let M be a smooth oriented manifold and denote with $(\mathrm{GL}_+(M), \pi_{\mathrm{GL}^+(M)}; \mathrm{GL}_+(n;\mathbb{R}))$ the $\mathrm{GL}_+(n;\mathbb{R})$-principle fibre bundle of oriented frames of M.*

(i) A topological spin-structure *is a pair (P,π) consisting of a $\widetilde{\mathrm{GL}_+(n;\mathbb{R})}$-principle fibre bundle $(P, \pi_P; \widetilde{\mathrm{GL}_+(n;\mathbb{R})})$ over M and a two-sheeted covering $\pi : P \to \mathrm{GL}_+(M)$ such that*

(a) $\pi_{\mathrm{GL}^+(M)} \circ \pi = \pi_P$,

(b) $\pi(p \cdot g) = \pi(p) \cdot \Lambda(g)$ *for all* $p \in P, g \in \widetilde{\mathrm{GL}_+(n;\mathbb{R})}$.

$$\begin{array}{ccccc}
P \times \widetilde{\mathrm{GL}_+(n;\mathbb{R})} & \xrightarrow{\quad\cdot\quad} & P & & \\
\big\downarrow{\scriptstyle \pi\times\Lambda} & & \big\downarrow{\scriptstyle \pi} & \searrow^{\pi_P} & \\
\mathrm{GL}_+(M) \times \mathrm{GL}_+(n;\mathbb{R}) & \xrightarrow{\ \cdot\ } & \mathrm{GL}_+(n;\mathbb{R}) & \xrightarrow{\pi_{\mathrm{GL}_+(M)}} & M
\end{array}$$

(ii) Two topological spin-structures (P_1, π_1) *and* (P_2, π_2) *on* M *are* equivalent *if there exists a* $\widetilde{\mathrm{GL}_+(n;\mathbb{R})}$*-principal fibre bundle isomorphism* $\Phi : P_1 \to P_2$ *such that* $\pi_{P_2} \circ \Phi = \pi_{P_1}$.

Now let M be an oriented manifold with a topological spin-structure (P, π). We choose a Riemannian metric g on M. The Riemannian metric induces a reduction of the bundle of oriented frames to the group $\mathrm{SO}(n)$,

$$\begin{array}{ccc}
P & & \\
\big\downarrow{\scriptstyle \pi} & & \\
\mathrm{GL}_+(M) & \xleftarrow{\ \supseteq\ } & \mathrm{SO}(M,g) \\
\big\downarrow{\scriptstyle \pi_{\mathrm{GL}^+(M)}} & \swarrow_{\pi_{\mathrm{SO}(M,g)}} & \\
M & &
\end{array}$$

Defining $Q := Q^g := \pi^{-1}(\mathrm{SO}(M,g))$, we obtain a 2-sheeted covering,

$$\begin{array}{ccc}
P & \xleftarrow{\ \supseteq\ } & Q^g \\
\big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi_{|Q}} \\
\mathrm{GL}_+(M) & \xleftarrow{\ \supseteq\ } & \mathrm{SO}(M,g) \\
\big\downarrow{\scriptstyle \pi_{\mathrm{GL}^+(M)}} & \swarrow_{\pi_{\mathrm{SO}(M,g)}} & \\
M & &
\end{array}$$

which is then automaically a spin-structure as in Definition 3.1, cmp. Proposition 3.6. From the above it is evident that $\mathrm{SO}(M,g)$ is a deformation retract of $\mathrm{GL}_+(M)$ and one can show that Q^g is a deformation retract of P. Setting the question of the existence of P aside, the choice of a spin-structure Q^g thus uniquely determines the topological spin-structure P and thereby a unique spin-structure Q^h for every other Riemannian metric h.

The above raises the question why we do not define spin-structures as topological spin-structures. Unfortunately, the group $\widetilde{\mathrm{GL}_+(n;\mathbb{R})}$ does not posses any finite-dimensional representations that are not lifts from $\mathrm{GL}_+(n;\mathbb{R})$, [LM89, Ch. II, Lemma 5.23]. It is only after we reduce to the compact group $\mathrm{SO}(n;\mathbb{R})$ and cosidering its compact double cover $\mathrm{Spin}(n)$ that we obtain the fundamental spin-representation.

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