On a Problem of Berlekamp

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At the 3rd Waterloo Conference on Combinatorics Berlekamp presented the following combinatorial problem, which originally arose arose in his studies on burst–error correcting convolution codes [1]. The problem will be illustrated with the following example also due to Berlekamp in the proceedings of that conference [6], pp. 341–342.

$$\begin{array}{rrrrrr} & & & & & 1 \\ & & & & 1 & 1 \\ & & & & 1 & 2 \\ & & & 1 & 1 & 3 \\ & & 1 & 1 & 2 & 7 \\ & & 1 & 2 & 5 & 19 \\ & 1 & 1 & 3 & 9 & 37 \\ 1 & 1 & 2 & 7 & 23 & 99 \\ 1 & 2 & 5 & 19 & 66 & 293 \end{array} \tag{1}$$

Berlekamp defines an array to be unitary if any square submatrix whose upper left corner falls on the boundary of the array has a determinant equal to 1. For instance, in the array above $\det \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 9 \end{pmatrix} = 1$. The problem then he states as follows: "... A periodic quasilinear boundary represents the best staircase approximation to a straight line of rational slope. ... Exact formulas are known for the values of the numbers in the unitary arrays generated by periodic quasilinear boundaries of slopes $1/n$ or n, but no such formulas are known (to me) for the values in the arrays with boundaries of slopes m/n where $1 < m < n$. The simplest such case is slope $2/3$" - this is shown above in (1).

We shall analyze an equivalent lattice path model and use a probablistic approach due to Gessel [3]. Especially, this allows to derive that the entries in the above array are determined by the 6 generating functions

$$g_1(x) = \sum_{n=0}^{\infty} \frac{1}{5n+1} \binom{5n+1}{2n} x^n - \frac{x[g_4(x)]^2}{2} = 1+2x+23x^2+..$$

$$g_2(x) = \sum_{n=0}^{\infty} \frac{1}{5n+1} \binom{5n+1}{2n} x^n + \frac{x[g_4(x)]^2}{2} = 1+3x+37x^2+..$$

$$g_3(x) = \sum_{n=0}^{\infty} \frac{1}{5n+2} \binom{5n+2}{2n+1} x^n = 1 + 5x + 66x^2 + ..$$

$$g_4(x) = \sum_{n=0}^{\infty} \frac{1}{5n+3} \binom{5n+3}{2n+1} x^n = 1 + 7x + 99x^2 + ..$$

$$g_5(x) = \sum_{n=0}^{\infty} \frac{1}{5n+4} \binom{5n+4}{2n+2} x^n - \frac{[g_3(x)]^2}{2} = 1+9x+136x^2+..$$

$$g_6(x) = \sum_{n=0}^{\infty} \frac{1}{5n+4} \binom{5n+4}{2n+2} x^n + \frac{[g_3(x)]^2}{2} = 2+19x+293x^2+..$$

In [5] also the entries of periodic quasilinear arrays of slope $2/5$ are determined. For further slopes, the analysis is more involved - methods for the analysis are provided in [4].

In order to compute the numbers in question, however, a low complexity algorithm was provided in [2]. We shall present a simplification of this algorithm, which allows the use of two-dimensional arrays similar to Pascal's triangle, where the entry $a(i,j)$ is obtained via the recursion $a(i,j) = a(i-1,j) + a(i,j-1)$. For slope $2/3$ three such arrays have to be considered, for instance in the array below the coefficients of $g_3(x)$ and $g_6(x)$ are the entries on the boundary.

$$\begin{array}{r|rrrrrr} 8 & & & & & & 293 \\ 7 & & & & & 66 & 293 \\ 6 & & & & & 66 & 227 \\ 5 & & & & 19 & 66 & 161 \\ 4 & & & 5 & 19 & 47 & 95 \\ 3 & & & 5 & 14 & 28 & 48 \\ 2 & & 2 & 5 & 9 & 14 & 20 \\ 1 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

References

[1] E. R. Berlekamp, "A class of convolutional codes", *Information and Control*, vol. 6, pp. 1–13, 1963.

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[4] U. Tamm, "Lattice paths not touching a given boundary", *J. Statist. Plann. Inference*, vol. 105, pp. 433–448, 2002.

[5] U. Tamm, "Size of downsets in the pushing order", *General Theory of Information Transfer and Combinatorics*, The Shannon Foundation, submitted.

[6] W. Tutte (ed.), *Recent Progress in Combinatorics*, Proceedings of the 3rd Waterloo Conference on Combinatorics, 1968.