A RAINBOW ERDŐS-ROTHSCHILD PROBLEM

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Abstract. We consider a multicolored version of a question posed by Erdős and Rothschild. For a fixed positive integer r and a fixed graph F, we look for n-vertex graphs that admit the maximum number of r-edge-colorings with the property that there is no copy of F for which all edges are assigned different colors. We show that, when F is a bipartite graph with at least three edges and $r \geq 3$, the number of r-edge colorings of an extremal configuration is close to the number of such edge-colorings of the complete graph K_n. On the other hand, for the rainbow pattern of $F = K_{k+1}$, the Turán graph $T_k(n)$ is the only extremal configuration for any $r \geq r_0(k)$ and large n.

1. Introduction

In this paper we consider a multicolored version of a problem introduced by Erdős and Rothschild [7], which in turn was motivated by the Turán problem [27]. To describe this classical problem, we say that a graph G is *F-free* if it does not contain some fixed graph F as a subgraph. The Turán problem for F asks for the maximum number $\text{ex}(n, F)$ of edges over all F-free n-vertex graphs and for the graphs that achieve this maximum, which are called *F-extremal*. This is a very popular problem in extremal graph theory and there is a vast literature related with it (more information may be found in Keevash [16], and the references therein).

The Turán problem was generlized to a multicolored setting by Keevash, Saks, Sudakov and Verstraëte [18]. They looked for n-vertex *multigraphs* with the largest number of edges which admit an r-edge-coloring such that all color classes induce simple graphs and there is no *rainbow copy* of the forbidden graph F, that is, no copy of F such that every edge is colored differently. Another multicolored extension of the Turán problem was introduced by Keevash, Mubayi, Sudakov and Verstraëte [17], who looked for graphs with the largest number of edges which admit a proper r-edge-coloring with no rainbow copy of F. More results in this direction have been obtained by Das, Lee and Sudakov [6]. We also refer to Bollobás, Keevash and Sudakov [5] for multicolored problems of a similar flavor in more general combinatorial structures.

In the remainder of this paper we focus on simple graphs. We consider a multicolored version of a problem introduced by Erdős and Rothschild [7] which has also been inspired by the Turán problem. Instead of looking for n-vertex graphs with the largest number of edges that satisfy some property, Erdős and Rothschild were interested in n-vertex graphs that admit the largest number of *r-edge-colorings* such that *every color class is F-free*. In particular, when F is the complete graph K_{k+1} on $k+1$ vertices, they conjectured that the number of F-free 2-colorings is maximized by the k-partite Turán graph $T_k(n)$ on n-vertices, that is, by the balanced, complete k-partite n-vertex graph. Note that F-extremal graphs are natural candidates, as any r-coloring is trivially F-free, which leads to $r^{\text{ex}(n,F)}$ such colorings.

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Yuster [28] provided an affirmative answer to the conjecture of Erdős-Rothschild for the triangle K_3 and any $n \geq 6$, while Alon, Balogh, Keevash and Sudakov [1] showed that, for $r \in \{2,3\}$ and $n \geq n_0$, where n_0 is a constant depending on r and k, the respective Turán graph is also optimal for the number of K_{k+1}-free r-colorings. However, they also proved that Turán graphs $T_k(n)$ are not optimal for any $r \geq 4$, but did not characterize the extremal graphs. For results about hypergraph versions of this problem, we refer to [20, 21, 22].

Balogh [3] considered another version of this problem, which involves edge-colorings of a graph avoiding a copy of F colored according to a *prescribed coloring*. He proved that, for $r = 2$ colors and any 2-coloring C of K_{k+1} that uses both colors, the graph $T_k(n)$ once again yields the largest number of 2-colorings avoiding C for $n \geq n_0$. However, he also remarked that, if we consider $r = 3$ and a rainbow-colored triangle, the complete graph on n vertices already admits $3 \cdot 2^{\binom{n}{2}} - 3$ colorings, by just choosing two of the three colors and coloring the edges of K_n arbitrarily with these two colors. This is more than $3^{n^2/4}$, which is an upper bound on the number of 3-colorings of the bipartite Turán graph.

Note that, if the number of colors satisfies $r \geq \binom{k+1}{2} + 1$ and C is an arbitrary r-edge coloring of K_{k+1}, the complete graph K_n admits the largest number of colorings avoiding C. Namely, the condition on r implies that some color does not appear in C, so that it can be used to extend any coloring of an n-vertex graph G which avoids C to a coloring of K_n avoiding C.

In the following, we modify Balogh's multicolored version of the Erdős-Rothschild problem by forbidding not only a prescribed coloring C, but all colorings that may be obtained from C by permuting the colors. More precisely, given a number r of colors and a graph F, an *r-pattern* P of F is a partition of its edge set into at most r classes, and an edge-coloring of a graph G is said to be *(F,P)-free* if G does not contain a copy of F in which the partition of the edge set induced by the coloring is isomorphic to P (for simplicity, this will be referred to as a copy of (F,P)). Observe that the result of Balogh [3] for $r = 2$ colors, which was mentioned above, implies that $T_k(n)$ yields the largest number of 2-colorings avoiding (K_{k+1}, P), as any coloring that avoids a pattern P must avoid any particular coloring that produces P.

Fix a positive integer r and a graph F, and let P be a pattern of F. Let $\mathcal{C}_{r,F,P}(G)$ be the set of all (F,P)-free r-colorings of a graph G. We write

$$c_{r,F,P}(n) = \max\{\, |\mathcal{C}_{r,F,P}(G)| \colon |V(G)| = n \,\},$$

and we say that an n-vertex graph G is *$\mathcal{C}_{r,F,P}$-extremal* if $|\mathcal{C}_{r,F,P}(G)| = c_{r,F,P}(n)$. A recent general result about the problem of computing $c_{r,F,P}(n)$ for $r \geq 3$ and the corresponding extremal graphs is that, if P is an arbitrary pattern of the complete graph, there is always a $\mathcal{C}_{r,F,P}$-extremal graph that is a complete multipartite graph. This was proved by Benevides, Sampaio and one of the current authors [4]. Pikhurko, Staden and Yilma [23] have independently obtained a similar result for a different extension of the original problem about monochromatic patterns, which leads to the same conclusion in the case of monochromatic patterns.

In this paper, our main objective is to study $\mathcal{C}_{r,F,P}$-extremal graphs where P is a rainbow pattern, that is, where every edge is assigned to a different class. (For simplicity, we shall write $F^R = (F,P)$ to refer to the rainbow pattern P of a graph F.) However, several of our results hold for more general patterns. For instance, we find a general approximate result for bipartite graphs, which implies that the complete graph K_n is not far, with respect to the number edge-colorings, from being $\mathcal{C}_{r,F,P}$-extremal when F is a bipartite graph and P contains at least three classes.

Theorem 1.1. *Let F be a bipartite graph, let P be a pattern of F with $t \geq 3$ nonempty classes, and fix a positive integer $r \geq t$. Given $\beta > 0$, there exists n_0 such that, for every $n \geq n_0$, we have $c_{r,F,P}(n) \leq (t-1)^{\binom{n}{2}+\beta n^2}$.*

Indeed, as we may trivially obtain $(t-1)^{\binom{n}{2}}$ distinct (F,P)-free colorings if we color the complete graph with a fixed set of $t-1$ colors, Theorem 1.1 implies that the complete graph is *almost* optimal for any pattern with at least three classes in a bipartite graph. In particular, this holds for rainbow patterns of bipartite graphs with at least three edges. We refer the reader to Section 5 for a discussion about possible extensions of Theorem 1.1.

On the other hand, the structure of $\mathcal{C}_{r,F,P}$-extremal graphs seems to be rather different for rainbow patterns when F is not bipartite. In this direction, we show that, in the case of rainbow patterns of complete graphs, the corresponding Turán graph is extremal if the number of colors is large.

Theorem 1.2. *Let r and $k \geq 2$ be positive integers such that $r \geq \binom{k+1}{2}^{8k+4}$. There is n_0 such that every graph of order $n > n_0$ has at most $r^{\mathrm{ex}(n,K_{k+1})}$ distinct K_{k+1}^R-free r-edge colorings. Moreover, the Turán graph $T_k(n)$ is the only graph on n vertices for which equality is achieved.*

Even though we believe that the conclusion of Theorem 1.2 should hold for smaller values of r, note that the Turán graph $T_k(n)$ cannot be extremal for arbitrary values of r, in particular, when the number of colors is $r < \binom{k+1}{2}$, the complete graph K_n yields more colorings than $T_k(n)$.

Beyond Theorem 1.2 itself, we believe that the main contribution of this paper is our modification of the general steps of the proof of Theorem 1.1 in Alon, Balogh, Keevash and Sudakov [1] (see also [3]) to non-monochromatic patterns of complete graphs. The main novelty of their method was applying the Regularity Lemma to obtain an exact result. This has been done in two steps: (i) Prove a stability result, establishing that any counterexample G to the desired result would be similar to $T_k(n)$, in the sense that its vertex set may be partitioned into k almost-balanced classes in such a way that there is only a small number of edges with both ends in the same class. (ii) Starting with a counterexample on n vertices, show that it is possible to find a counterexample on $n-1$ or $n-2$ vertices whose 'gap' to the desired optimal solution increases. A recursive application of this step leads to a counterexample whose number of colorings is too high to be feasible.

Here, we modify this strategy to derive results for more general patterns. For instance, we show that rainbow patterns are the only patterns for which the stability of part (i) holds for a large number $r \geq r_0$ of colors (see Lemma 4.6 and Remarks 4.2 and 4.8). To obtain smaller values for r_0, we apply a recent stability result due to Füredi [10]. Moreover, we generalize step (ii) by showing that this stability implies that the Turán graph $T_k(n)$ is actually $\mathcal{C}_{r,K_{k+1},P}$-extremal whenever the pattern P of K_{k+1} is *locally rainbow*, that is, whenever there is a vertex such that all edges incident with it lie in different classes (see Lemma 4.4 and Remark 4.5).

Our paper is structured as follows. In Section 2 we introduce the notation and the main preliminary results. This is used in Section 3 to prove Theorem 1.1 by applying the Regularity Lemma of Szemerédi [26]. The proof of Theorem 1.2 is the subject of Section 4, where we also treat the case of forbidden edge-color critical graphs. Advances in this and other related problems are discussed in Section 5.

2. Notation and Tools

In this section, we fix the notation and introduce basic concepts and results used to prove our main results. For simplicity, we shall assume that colors lie in sets $[r] := \{1, \ldots, r\}$.

2.1. **Regularity Lemma.** To prove our results we use an approach similar to the one from [1], which is based on the Szemerédi Regularity Lemma [26]. Let $G = (V, E)$ be a graph, and let A and B be two disjoint subsets of $V(G)$. If A and B are non-empty, define the density of edges between A and B by

$$d(A, B) = \frac{e(A, B)}{|A||B|},$$

where $e(A, B)$ is the number of edges with one vertex in A and the other in B. When $A = B$, we write $e(A, A) = e(A)$. For $\varepsilon > 0$ the pair (A, B) is called *ε-regular* if, for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

An *equitable partition* of a set V is a partition of V into pairwise disjoint classes $V_1, \ldots, V_m$ of almost equal size, i.e., $||V_i| - |V_j|| \leq 1$ for all i, j. An equitable partition of the set V of vertices of G into the classes $V_1, \ldots, V_m$ is called *ε-regular* if at most $\varepsilon\binom{m}{2}$ of the pairs (V_i, V_j) are not ε-regular.

We shall use the following standard property of regular pairs, whose simple proof has been added for completeness.

Lemma 2.1. *If (A, B) is an ε-regular pair with density $d > \varepsilon$ and $Y \subset B$ has size $|Y| \geq \varepsilon|B|$, then all but at most $\varepsilon|A|$ of the vertices of A have at least $(d - \varepsilon)|Y|$ neighbors in Y.*

Proof. If there is a subset $X \subseteq A$ of size $|X| \geq \varepsilon|A|$ vertices, where each vertex in X has less than $(d - \varepsilon)|Y|$ neighbors in Y, then we infer for the density of the pair (X, Y) that

$$d(X, Y) < \frac{|X| \cdot (d - \varepsilon)|Y|}{|X| \cdot |Y|} = d - \varepsilon,$$

which contradicts the ε-regularity of the pair (A, B). $\square$

We now state a version of the Regularity Lemma and of a colored version thereof [19] that will be particularly useful for our purposes.

Lemma 2.2. *For every $\varepsilon > 0$, there is an integer $M = M(\varepsilon) > 0$ such that for every graph G of order $n > M$ there is an ε-regular partition of the vertex set of G into m classes, for some $1/\varepsilon \leq m \leq M$.*

Lemma 2.3. *For every $\varepsilon > 0$ and every integer r, there exists an $M = M(\varepsilon, r)$ such that the following property holds. If the edges of a graph G of order $n > M$ are r-colored $E(G) = E_1 \cup \cdots \cup E_r$, then there is a partition of the vertex set $V(G) = V_1 \cup \cdots \cup V_m$, with $1/\varepsilon \leq m \leq M$, which is ε-regular simultaneously with respect to all graphs $G_i = (V, E_i)$ for $1 \leq i \leq r$.*

A partition as in Lemma 2.3 will be called a *multicolored ε-regular partition.* Given such a partition and given a color $c \in [r]$, we define the *cluster graph* associated with color c as follows. Given $\eta > 0$, the graph $H_c = H_c(\eta)$ has vertex set $[m]$ and $\{i, j\} \in E(H_c)$ if and only if (V_i, V_j) is an ε-regular pair with edge density at least η with respect to the subgraph of G induced by the edges of color c.

We may also define the *multicolored cluster graph H* associated with this partition: the vertex set is $[m]$ and $e = \{i, j\}$ is an edge of H if $e \in E(H_c)$ for some $c \in [r]$. Each edge e in H is assigned the list of colors $L_e = \{c \in [r] \mid e \in E(H_c)\}$. Given a colored graph F, we say that a multicolored cluster graph H contains F if H contains a copy of F for which the color of each edge of F is contained in the list of the corresponding edge in H. More generally, if F is a graph with color pattern P, we say that H contains (F, P) if it contains some colored copy of F with pattern P.

Given colored graphs F and H, a function $\psi\colon V(F) \to V(H)$ is called a *colored homomorphism* of F in H if, for every edge $e = \{i, j\} \in E(F)$, the pair $\{\psi(i), \psi(j)\}$ is an edge of H

with the color of e. If H is a multicolored cluster graph, it suffices that the color of e lies in the list associated with the edge $\{\psi(i), \psi(j)\}$.

In connection with these definitions, we may obtain the following embedding result.

Lemma 2.4. *For every $\eta > 0$ and all positive integers k and r, there exist $\varepsilon = \varepsilon(r, \eta, k) > 0$ and a positive integer $n_0(r, \eta, k)$ with the following property. Suppose that G is an r-colored graph on $n > n_0$ vertices with a multicolored ε-regular partition $V = V_1 \cup \cdots \cup V_m$ which defines the multicolored cluster graph $H = H(\eta)$. Let F be a k-vertex graph colored with $t \leq r$ colors. If there is a colored homomorphism ψ of F into H, then the graph G also contains F.*

Proof. The argument is quite standard and follows the proof of Theorem 2.1 in [19]. Let $u_1, \ldots, u_k$ be the vertices of F. The edges of F are denoted $e_{i,j} = \{u_i, u_j\}$ and their colors are denoted $c_{i,j}$. Let $N_c(v)$ denote the set of neighbors of v that are connected to v via an edge of color c.

We shall choose the vertices $v_1, \ldots, v_k$ that span the copy of F in G inductively. Based on the colored homomorphism ψ, we start with sets $Y_i^0 = V_{\psi(u_i)}$ for $i \in \{1, \ldots, k\}$. The idea is to choose $v_1 \in Y_1^0$ with the property that, for all i such that $e_{1,i} \in E(F)$, vertex v_1 has at least $(\eta - \varepsilon)|Y_i^0|$ neighbors in Y_i^0 that are joined by an edge in color $c_{1,i}$ to vertex v_1. By the definition of H and by Lemma 2.1, this can be done if $|Y_1^0| > (k-1)\varepsilon|Y_1^0| \geq d_F(u_1)\varepsilon|Y_1^0|$, where $d_F(u_1)$ is the degree of vertex u_1 in F. For $i \geq 2$ such that $e_{1,i} \in E(F)$, define $Y_i^1 = Y_i^0 \cap N_{c_{1,i}}(v_1)$, otherwise set $Y_i^1 = Y_i^0$. Note that $|Y_i^1| \geq (\eta - \varepsilon)|Y_i^0|$ for all $i \geq 2$.

Inductively, assume that $v_1, \ldots, v_{j-1}$ have been chosen and that we have defined sets $Y_i^{j-1} \subset V_{\psi(u_i)}$ for all $i \geq j$ such that $|Y_i^{j-1}| \geq (\eta - \varepsilon)^{j-1}|Y_i^0|$ and all vertices of Y_i^{j-1} are adjacent to v_ℓ with an edge of color $c_{\ell,i}$ provided that $\ell < j$ and $e_{i,\ell} \in E(F)$. As long as

$$|Y_j^{j-1}| - |\{i \colon i > j \text{ and } e_{i,j} \in E(F)\}| \cdot \varepsilon|Y_j^0| \geq j \tag{1}$$

and $|Y_i^{j-1}| > \varepsilon|Y_i^0|$ for all $i > j$, we may apply Lemma 2.1 to the ε-regular pair (Y_j^0, Y_i^0) and the subset $Y_i^{j-1} \subseteq Y_i^0$ to obtain $v_j \in Y_j^{j-1} \setminus \{v_1, \ldots, v_{j-1}\}$ with the property that

$$|Y_i^{j-1} \cap N_{c_{j,i}}(v_j)| \geq (\eta - \varepsilon)|Y_i^{j-1}| \geq (\eta - \varepsilon)^j |Y_i^0|$$

for all $i > j$ such that $e_{j,i} \in E(F)$. We then set $Y_i^j = Y_i^{j-1} \cap N_{c_{j,i}}(v_j)$ if $e_{j,i} \in E(F)$ and $Y_i^j = Y_i^{j-1}$ otherwise.

This procedure allows us to fully embed F into G if we fix $\varepsilon = \varepsilon(\eta, k) > 0$ small enough to satisfy

$$(\eta - \varepsilon)^{k-1} \geq k\varepsilon \tag{2}$$

and we let $n > n_0 \geq Mk/\varepsilon$, where $M = M(r, \varepsilon)$ is defined in Lemma 2.3. Indeed, condition (2) implies that, in the above process, $|Y_j^{j-1}| \geq k\varepsilon|Y_j^0|$ for all $j \leq k$. Moreover, the choice of n_0 guarantees that, for all j, we have $\varepsilon|Y_j^0| \geq \varepsilon n/M \geq k \geq j$. Then, we may use the condition $|Y_j^{j-1}| \geq k\varepsilon|Y_j^0|$, for all $j \leq k$, to ensure the validity of (1), as

$$\begin{aligned} |Y_j^{j-1}| &\geq k\varepsilon|Y_j^0| = \varepsilon|Y_j^0| + (k-1)\varepsilon|Y_j^0| \\ &\geq j + |\{i \colon i > j \text{ and } e_{i,j} \in E(F)\}| \cdot \varepsilon|Y_j^0|. \end{aligned}$$

□

2.2. **Stability.** Another concept that will be particularly useful in our paper are stability results for graphs. It will be convenient to use the following theorem by Füredi [10], which builds upon earlier work by Erdős and Simonovits [25].

Theorem 2.5. *Let $G = (V, E)$ be a K_{k+1}-free graph on m vertices. If $|E| = \mathrm{ex}(m, K_{k+1}) - t$ for some $t \geq 0$, then there exists a partition $V = V_1 \cup \cdots \cup V_k$ with $\sum_{i=1}^{k} e(V_i) \leq t$.*

We recall the following bounds on the number of edges in the Turán graph $T_k(n)$:

$$\frac{(k-1)n^2}{2k} - k < \mathrm{ex}(n, K_{k+1}) \leq \frac{(k-1)n^2}{2k}. \tag{3}$$

The *chromatic number* $\chi(G)$ of a graph G is the least positive integer t such that there is a coloring of the vertex set of G with t colors where the vertices of each edge have different colors. The following is a counterpart of Theorem 2.5 for arbitrary graphs, which is also due to Füredi [10].

Theorem 2.6. *Let F be a fixed graph with chromatic number $\chi(F) = k+1$. Let $\alpha > 0$ be fixed. Then, there exists m_1 such that for every F-free graph $G = (V, E)$ on $m \geq m_1$ vertices with $|E| \geq \mathrm{ex}(m, F) - \alpha m^2$, there exists a partition $V = V_1 \cup \cdots \cup V_k$ with $\sum_{i=1}^{k} e(V_i) \leq 4\alpha m^2$.*

For later use, we state the following fact about the size of the classes in a k-partite graph with a large number of edges.

Proposition 2.7. *Let $G = (V, E)$ be a k-partite graph on m vertices with k-partition $V = V_1 \cup \cdots \cup V_k$. If, for some $t \geq k^2$, the graph G contains at least $\mathrm{ex}(m, K_{k+1}) - t$ edges, then for $i \in \{1, \ldots, k\}$ we have*

$$\left| |V_i| - \frac{m}{k} \right| < \sqrt{2t}.$$

Proof. If $|V_k| = x$, then G contains at most

$$x(m-x) + \binom{k-1}{2} \cdot \left(\frac{m-x}{k-1}\right)^2$$

edges. For the second summand, we used that, when a sum $a_1 + \cdots + a_{k-1} = M$ is fixed, the value of $\sum_{1 \leq i < j \leq k-1} a_i a_j$ is maximum for $a_1 = \cdots = a_{k-1} = M/(k-1)$.

Since, using (3),

$$x(m-x) + \binom{k-1}{2} \cdot \left(\frac{m-x}{k-1}\right)^2 \geq \mathrm{ex}(m, K_{k+1}) - t \geq \frac{(k-1)m^2}{2k} - k - t,$$

we conclude that

$$x^2 - \frac{2}{k}mx + \frac{1}{k^2}m^2 - \frac{2(k-1)}{k} \cdot t - 2(k-1) \leq 0,$$

thus, $|x - m/k| \leq \sqrt{\frac{2(k-1)}{k}t + 2(k-1)} < \sqrt{2t}$ for $t \geq k^2$, as required. $\square$

2.3. **Entropy function.** Consider the entropy function $H\colon [0,1] \to [0,1]$ given by $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ with $H(0) = H(1) = 0$. Note that $\lim_{x \to 0^+} H(x) = 0$.

We will use the well-known inequality

$$\binom{n}{\alpha n} \leq 2^{H(\alpha)n} \tag{4}$$

for all $0 \leq \alpha \leq 1$.

We will also use the following upper bound on the entropy function for $x \leq 1/8$:

$$H(x) \quad \leq \quad -2x\log_2 x. \tag{5}$$

Namely, (5) is equivalent to $g(x) = x\ln x - (1-x)\ln(1-x) \leq 0$. Taking the derivative gives $g'(x) = \ln x + 2 + \ln(1-x) \leq 0$ for $x \leq 1/8$. With $g(1/8) < 0$ inequality (5) follows.

3. Colorings avoiding bipartite graphs with given patterns

In order to prove Theorem 1.1, we shall use the following embedding result, which is a simple consequence of Lemma 2.4. Recall that, given an edge e of a multicolored cluster graph H (defined by the cluster graphs $H_1, \dots, H_r$), L_e is the list of colors i such that $e \in E(H_i)$.

Lemma 3.1. *For every $\eta > 0$ and all positive integers k and r, there exist $\varepsilon = \varepsilon(r, \eta, k) > 0$ and a positive integer $n_0(r, \eta, k)$ with the following property. Suppose that G is an r-colored graph on $n > n_0$ vertices with a multicolored ε-regular partition $V = V_1 \cup \cdots \cup V_m$ which defines the multicolored cluster graph $H = H(\eta)$. Let F be a fixed k-vertex bipartite graph with a prescribed color pattern P on t classes, where $t \leq r$. If $|L_e| \geq t$ for some edge $e \in E(H)$, then G contains (F, P).*

Proof. To apply Lemma 2.4, fix an arbitrary coloring of the k-vertex bipartite graph F with pattern P such that the colors used lie in L_e. Then consider the colored homomorphism that maps all edges of F to e. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1: Fix a positive integer $r \geq 3$. Let F be a k-vertex bipartite graph and let P be a pattern of F with t classes, $3 \leq t \leq r$. We shall define our parameters implicitly in terms of a quantity $\eta > 0$. Let $\varepsilon = \varepsilon(r, \eta, k) > 0$ and $n_0(r, \eta, k)$ satisfy the assumption in Lemma 3.1, w.l.o.g. $\varepsilon < \eta/2$. Based on this $\varepsilon > 0$, define M as in Lemma 2.3. We will show that, for all $\beta > 0$, the number of (F, P)-free r-colorings of a graph G of order $n > \max\{n_0, M\}$ satisfies $|\mathcal{C}_{r,F,P}(G)| \leq (t-1)^{\binom{n}{2}+\beta n^2}$ if $\eta > 0$ is chosen appropriately.

Let G be such a graph and fix a (F, P)-free r-edge coloring of G. By Lemma 2.3 we obtain a partition $V = V_1 \cup \cdots \cup V_m$ of the vertex set V of G, for which each graph consisting of all edges in each of the r colors is ε-regular. For $1 \leq i \leq r$, consider the m-vertex cluster graph $H_i = H_i(\eta)$ and let H be the corresponding multicolored cluster graph.

There are at most $\varepsilon\binom{m}{2}$ irregular pairs with respect to the partition $V = V_1 \cup \cdots \cup V_m$ and with respect to some color, hence at most

$$r \cdot \varepsilon \cdot \binom{m}{2} \cdot \left(\frac{n}{m}\right)^2 \leq \frac{r\varepsilon}{2} n^2 \tag{6}$$

edges lie in these irregular pairs. By Lemma 2.2 and by the definition of an ε-regular partition, there are at most (using $m \geq 1/\varepsilon$)

$$m \cdot \left(\frac{n}{m}\right)^2 = \frac{n^2}{m} \leq \varepsilon n^2 \tag{7}$$

edges that are contained in some class V_i. Moreover, if we consider all the edges of the same color between classes V_i and V_j whose density is less than η, we obtain at most

$$r \cdot \eta \cdot \binom{m}{2} \cdot \left(\frac{n}{m}\right)^2 \leq \frac{r\eta}{2} \cdot n^2 \tag{8}$$

edges. Combining (6), (7) and (8) with $\varepsilon < \eta/2$ gives at most $r\eta n^2$ such edges, which we can choose in at most

$$\binom{\frac{n^2}{2}}{r\eta n^2}$$

ways. The number of colorings of this set of edges is at most $r^{r\eta n^2}$.

Clearly, the remaining edges, which lie in regular pairs and must be assigned colors that are dense with respect to this pair, may be colored in at most $\left(\prod_{e \in E(H)} |L_e|\right)^{(n/m)^2}$ ways.

With Lemma 3.1, we conclude that the number of r-edge colorings of G that give rise to the partition $V = V_1 \cup \cdots \cup V_m$ and the multicolored cluster graph H is bounded above by

$$\begin{aligned} \binom{\frac{n^2}{2}}{r\eta n^2} \cdot r^{r\eta n^2} \cdot \left(\prod_{e \in E(H)} |L_e| \right)^{\left(\frac{n}{m}\right)^2} &\leq \binom{\frac{n^2}{2}}{r\eta n^2} \cdot r^{r\eta n^2} \cdot \left((t-1)^{\binom{m}{2}} \right)^{\left(\frac{n}{m}\right)^2} \\ &\overset{(4)}{\leq} 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot (t-1)^{\binom{n}{2}}, \end{aligned}$$

where $H(x)$ denotes the entropy function. There are at most M^n partitions $V = V_1 \cup \cdots \cup V_m$, where $m \leq M$. For each partition there are at most $2^{rM^2/2}$ multicolored cluster graphs H. Thus, an upper bound on the number of (F, P)-free r-edge-colorings of G is

$$M^n \cdot 2^{rM^2/2} \cdot 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot (t-1)^{\binom{n}{2}}. \tag{9}$$

To conclude the proof, note that for $t \geq 3$ the product (9) is at most $(t-1)^{\binom{n}{2}(1+\beta)}$ for any fixed $\beta > 0$, as long as $\eta > 0$ is chosen sufficiently small. □

4. Colorings avoiding rainbow K_{k+1}

To prove that the Turán graph $T_k(n)$ is the unique graph maximizing the number of r-colorings that avoid a rainbow copy of K_{k+1}, whenever r is sufficiently large in terms of k and n is large, we need to proceed more carefully than in the previous section. Our strategy is to modify the general steps of the proof of Theorem 1.1 in Alon, Balogh, Keevash and Sudakov [1] (see also [3]) to our framework. The main novelty of their method was applying the Regularity Lemma to obtain an exact result. This involves proving a stability result showing that any extremal graph is not far from $T_k(n)$. The desired result is then obtained by contradiction: starting with a counterexample on n vertices, one shows that it is possible to find a counterexample on $n-1$ or $n-2$ vertices whose 'gap' to the desired optimal solution increases. A recursive application of this step would lead to an $\sqrt{n}$-vertex graph whose number of colorings is too high to be feasible.

Before stating the auxiliary result needed for our purposes, we give a preliminary definition, which distinguishes patterns that satisfy a stability result as above.

Definition 4.1. *Let F be a graph with chromatic number $\chi(F) = k+1 \geq 3$ and let P be a pattern of F. We say that the pair (F, P) satisfies the Color Stability Property for a positive integer r if, for every $\delta > 0$, there exists n_0 with the following property. If $n > n_0$ and G is an n-vertex graph such that $|\mathcal{C}_{r,F,P}(G)| \geq r^{\mathrm{ex}(n,F)}$, then there exists a partition $V(G) = V_1 \cup \cdots \cup V_k$ such that $\sum_{i=1}^{k} e(V_i) < \delta n^2$.*

Given a graph F and a pattern P of F, note that, if the pair (F, P) satisfies the Color Stability Property for a positive integer r, then for any $\delta > 0$ and n sufficiently large, it follows immediately that $|\mathcal{C}_{r,F,P}(G)| < r^{\mathrm{ex}(n,F)+\delta n^2}$.

In the remainder of this paper, we shall prove Theorem 1.2 in two steps. First we assume that, given $k \geq 2$, the pair (K_{k+1}, P) satisfies the Color Stability Property for a positive integer r and some particular pattern P of K_{k+1}. We show that $T_k(n)$ is the unique $\mathcal{C}_{r,K_{k+1},P}$-extremal graph on n vertices provided that n is sufficiently large. Next we show that K_{k+1}^R satisfies the Color Stability Property for $r \geq \binom{k+1}{2}^{8k+4}$.

Remark 4.2. We observe that the rainbow pattern is the only pattern of the complete graph K_{k+1}, where $k \geq 2$, for which the Color Stability Property holds for arbitrarily large r. Indeed, let P be a pattern of K_{k+1} that is *not* the rainbow pattern. For simplicity, assume that r is divisible by $\binom{k+1}{2}$ and split the set $[r]$ of colors into pairwise disjoint sets $C_1, \ldots, C_{\binom{k+1}{2}}$ of equal size. Associate each set C_j of colors with an edge of K_{k+1}. Consider the n-vertex graph H given by a blow-up of the vertex set of K_{k+1} into classes $V_1, \ldots, V_{k+1}$ of size $n/(k+1)$, where we assume that $k+1$ divides n to avoid technicalities. We look at colorings of H such that the edges between V_i and V_j are colored arbitrarily with colors from the set associated with the edge $\{i,j\}$ of K_{k+1}, so that all copies of K_{k+1} in H are rainbow-colored.

The number of such colorings of H is at least

$$\left(\frac{r}{\binom{k+1}{2}}\right)^{\frac{n^2}{(k+1)^2}\cdot\binom{k+1}{2}} = \left(\frac{r}{\binom{k+1}{2}}\right)^{\frac{k}{2(k+1)}n^2} > r^{\frac{k-1}{2k}\cdot n^2} \geq r^{\mathrm{ex}(n,K_{k+1})}$$

for $r > \binom{k+1}{2}^{k^2}$. Therefore, for any pattern of K_{k+1} that is not rainbow, the Color Stability Property does not hold provided that r is sufficiently large.

4.1. **Color Stability implies Extremality.** In this section, we prove that, under some condition on the sizes of r and n, if a rainbow pattern of the complete graph K_{k+1} satisfies the Color Stability Property for r colors, then the Turán graph $T_k(n)$ is the unique K_{k+1}^R-extremal graph. As a matter of fact, we prove this for a more general class of patterns: a pattern of a graph F is called *locally rainbow* if there is a vertex v such that all edges incident with v lie in different classes. We call such a vertex v a *locally rainbow vertex* of the pattern.

Before we state our main result of this section, we state a simple auxiliary lemma.

Lemma 4.3. *Let r and s be positive integers and let F be a colored graph with vertex set $\{v_1, \ldots, v_k\}$ such that each edge $\{v_i, v_j\} \in E(F)$ has color $c_{i,j} \in [r]$. Consider a graph G and mutually disjoint sets $W_1, \ldots, W_k \subseteq V(G)$ with the following property. For every $\{i,j\} \in E(F)$ and every pair of subsets $X_i \subseteq W_i$, $|X_i| \geq 10^{-sk}|W_i|$, and $X_j \subseteq W_j$, $|X_j| \geq 10^{-sk}|W_j|$, there are at least $10^{-s}|X_i||X_j|$ edges of color $c_{i,j}$ between X_i and X_j in G. Then G contains a copy of F with one vertex in each set W_i.*

Proof. We use induction on k. For $k=1$ and $k=2$ the statement is obviously true. Suppose that the result is true for $k-1$ and fix $W_1, \ldots, W_k \subseteq V(G)$ satisfying the conditions stated in the lemma.

For $1 \leq i \leq k-1$ such that $\{v_i, v_k\} \in E(F)\}$, let $W_k^i \subseteq W_k$ be the subset of all vertices in W_k that have less than $|W_i|/10^s$ neighbors in W_i via edges that are colored $c_{i,k}$. If $\{v_i, v_k\} \notin E(F)\}$, set $W_k^i = \emptyset$. Then we have $e(W_k^i, W_i) < |W_k^i||W_i|/10^s$, so that, by assumption, we have $|W_k^i| < 10^{-sk}|W_k|$. It follows that

$$\left|\bigcup_{i=1}^{k-1} W_k^i\right| < (k-1)10^{-sk}|W_k| < |W_k|.$$

and hence there exists a vertex $v \in W_k$ that is not contained in $\bigcup_{i=1}^{k-1} W_k^i$.

If $\{v_i, v_k\} \in E(F)\}$, let W_i' be the set of all neighbors of v in W_i that are adjacent to v by edges of color $c_{i,k}$, otherwise let $W_i' = W_i$. By the choice of v, the inequality $|W_i'| \geq |W_i|/10^s$ holds for all $1 \leq i \leq k-1$. Let $F' = F - v$. Note that, for all pairs (X_i, X_j), where $\{v_i, v_j\} \in E(F')$, with $X_i \subseteq W_i'$ and $X_j \subseteq W_j'$ and sizes $|X_i| \geq 10^{-s(k-1)}|W_i'| \geq 10^{-sk}|W_i|$ and $|X_j| \geq 10^{-s(k-1)}|W_j'| \geq 10^{-sk}|W_j|$, the graph G contains at least $10^{-s}|X_i||X_j|$ edges of color $c_{i,j}$ between X_i and X_j.

By induction, G contains a copy of F with vertex set $\{v_1, \ldots, v_{k-1}\}$ such that $v_i \in W_i'$, for $i \in \{1, \ldots, k-1\}$. Clearly, this may be extended to a copy of F containing v, where each edge $\{v, v_i\}$ has color $c_{i,k}$, as required. $\square$

The minimum degree $\delta_k(n)$ of the Turán graph $T_k(n)$ satisfies

$$\mathrm{ex}(n, K_{k+1}) = \mathrm{ex}(n-1, K_{k+1}) + \delta_k(n) \text{ and } \delta_k(n) = n - \lceil n/k \rceil.$$

We are now ready to prove the main result of this subsection.

Lemma 4.4. *Let $k \geq 2$ and let P be a locally rainbow pattern of K_{k+1} such that (K_{k+1}, P) satisfies the Color Stability Property of Definition 4.1 for a positive integer $r > e(k+1)$. Then there is n_0 such that every graph of order $n > n_0^2$ has at most $r^{\mathrm{ex}(n,K_{k+1})}$ distinct (K_{k+1}, P)-free r-edge colorings. Moreover, the only graph on n vertices for which the number of such colorings is $r^{\mathrm{ex}(n,K_{k+1})}$ is the Turán graph $T_k(n)$.*

Proof. The proof of this result uses the general strategy for proving Theorem 1.1 in [1]. Before we start, note that the number of colors r is at least as large as the number of classes of P. For a contradiction, we choose n_0 appropriately and we let $G \neq T_k(n)$ be a graph on $n > n_0^2$ vertices with at least $r^{\mathrm{ex}(n,K_{k+1})+m}$ distinct (K_{k+1}, P)-free r-edge colorings, for some $m \geq 0$. We will show that G contains a vertex x such that the graph $G - x$ obtained by deleting x has at least $r^{\mathrm{ex}(n-1,K_{k+1})+m+1}$ distinct (K_{k+1}, P)-free r-edge colorings, or it contains two vertices x and y such that $G - x - y$ has at least $r^{\mathrm{ex}(n-2,K_{k+1})+m+2}$ distinct (K_{k+1}, P)-free r-edge colorings. Repeating this argument iteratively, we obtain a graph on n_0 vertices whose number of (K_{k+1}, P)-free r-edge colorings is at least $r^{\mathrm{ex}(n_0,K_{k+1})+m+n-n_0} > r^{n_0^2}$. However, a graph on n_0 vertices has at most $n_0^2/2$ edges and hence the number of such colorings is at most $r^{n_0^2/2}$, which is the desired contradiction.

Since (K_{k+1}, P) satisfies the Color Stability Property for r colors, given $\delta = 10^{-8kr}$, there exists n_0 such that the assertion of Definition 4.1 holds for $n > n_0$. Let $G \neq T_k(n)$ be a graph on $n > n_0^2$ vertices with at least $r^{\mathrm{ex}(n,K_{k+1})+m}$ distinct (K_{k+1}, P)-free r-edge colorings, for some $m \geq 0$.

If G contains a vertex x of degree less than $\delta_k(n)$, then there are at most $r^{\delta_k(n)-1}$ ways to color the edges incident with x. This implies that $G - x$ must have at least $r^{\mathrm{ex}(n-1,K_{k+1})+m+1}$ distinct (K_{k+1}, P)-free r-edge colorings, which is precisely what we aimed to show. Henceforth we assume that the minimum degree of G is at least $\delta_k(n)$.

Consider a partition $V = V_1 \cup \cdots \cup V_k$ of the vertex set of G which minimizes $\sum_{i=1}^k e(V_i)$. Because (K_{k+1}, P) satisfies the Color Stability Property, and by our choice of n_0 and $\delta > 0$, we have $\sum_i e(V_i) < 10^{-8kr} n^2$. As G has at least $\mathrm{ex}(n, K_{k+1})$ edges, the k-partite subgraph induced by the partition $V = V_1 \cup \cdots \cup V_k$ contains at least $\mathrm{ex}(n, K_{k+1}) - 10^{-8kr} n^2$ edges. Thus, by Proposition 2.7 we know that $||V_i| - n/k| < \sqrt{2} \cdot 10^{-4kr} n$, for $i \in \{1, \ldots, k\}$.

Let $\mathcal{C}$ denote the set of all possible (K_{k+1}, P)-free r-edge colorings of G. First consider the case when there is some vertex with many neighbors in its own class of the partition, say $x \in V_1$ with $|N(x) \cap V_1| > n/(10^{3r}k)$. This also implies that $|N(x) \cap V_i| > n/(10^{3r}k)$ for $2 \leq i \leq k$, as otherwise we could reduce $\sum_i e(V_i)$ by moving x to another class, which contradicts our assumption that $\sum_{i=1}^k e(V_i)$ is minimized.

Let $\mathcal{C}_1$ be the subset of all the colorings for which there exists a choice of distinct colors $c_1, \ldots, c_k \in [r]$ such that there are subsets $W_i \subseteq V_i$ with $|W_i| \geq n/(10^{3r}kr)$, $i = 1, \ldots, k$, with the property that x is adjacent to each vertex in W_i via a c_i-colored edge. Let $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$.

Consider a coloring of G belonging to $\mathcal{C}_1$, and fix colors $c_1, \ldots, c_k$ and sets $W_1, \ldots, W_k$ as in the definition of $\mathcal{C}_1$. Fix a coloring $C(K_{k+1})$ of K_{k+1} with pattern P where the edges incident with some locally rainbow vertex v are colored $c_1, \ldots, c_k$. In particular, v could be mapped to x in an attempt to build a homomorphism of $C(K_{k+1})$ into G.

We say that a color c is *rare* for a pair (i, j) if there are subsets $X_i \subseteq W_i$ and $X_j \subseteq W_j$ with $|X_i| \geq 10^{-kr}|W_i| \geq n/(kr10^{(k+3)r})$ and $|X_j| \geq 10^{-kr}|W_j| \geq n/(kr10^{(k+3)r})$ such that there are fewer than $10^{-r}|X_i||X_j|$ edges of color c between them. Otherwise c is said to be *abundant* for the pair (i, j). Lemma 4.3 with $s = r$ ensures that there must be a color that is rare for some pair (i, j); indeed, if all colors in $[r]$ were abundant for all pairs (i, j), then all subsets $X_i \subseteq W_i$ and $X_j \subseteq W_j$ would satisfy the hypotheses of Lemma 4.3 for any assignment of colors, and hence it would be possible to extend the partial homomorphism of $C(K_{k+1})$ into G to a copy of K_{k+1} with pattern P as the number of colors is at least as large as the number of classes of P.

Once one rare color is fixed, the number of ways to color all edges between X_i and X_j is at most

$$\binom{|X_i||X_j|}{|X_i||X_j|/10^r}(r-1)^{|X_i||X_j|} < 2^{\frac{8r}{10^r}|X_i||X_j|}(r-1)^{|X_i||X_j|}.$$

Here we used (4) and (5) as follows:

$$\begin{array}{c} H(1/10^r) \leq \frac{2}{10^r}\log_2(10^r) < \frac{2}{10^r}\log_2(16^r) = \frac{8r}{10^r} \\ \binom{|X_i||X_j|}{|X_i||X_j|/10^r} \leq 2^{H(1/10^r)|X_i||X_j|} < 2^{\frac{8r}{10^r}|X_i||X_j|}. \end{array}$$

There are r choices for the rare color and at most 2^{2n} ways to choose the sets X_i and X_j. Since we know that there are at most $\mathrm{ex}(n, K_{k+1}) + 10^{-8kr}n^2 - |X_i||X_j|$ other edges in this graph, where $|X_i| \geq 10^{-kr}|W_i| \geq n/(kr10^{kr+3r})$, we infer that

$$\begin{aligned} |\mathcal{C}_1| &\leq r \cdot 2^{2n} \cdot r^{\mathrm{ex}(n,K_{k+1})+10^{-8kr}n^2-|X_i||X_j|} \cdot 2^{\frac{8r}{10^r}|X_i||X_j|} \cdot (r-1)^{|X_i||X_j|} \\ &\leq r^{\mathrm{ex}(n,K_{k+1})} \cdot r^{2\cdot 10^{-8kr}n^2} \cdot \left(2^{\frac{8r}{10^r}}\frac{(r-1)}{r}\right)^{|X_i||X_j|} \\ &\leq r^{\mathrm{ex}(n,K_{k+1})} \cdot r^{2\cdot 10^{-8kr}n^2} \cdot \left(2^{\frac{8r}{10^r}}\frac{(r-1)}{r}\right)^{\frac{n^2}{k^2r^2 10^{2kr+6r}}}. \end{aligned}$$

Note that

$$2^{8r}\left(\frac{r-1}{r}\right)^{10^r} \leq 2^{8r}e^{-\frac{10^r}{r}} \leq 2^{8r-\frac{10^r}{r}} \leq 2^{-2r} < \frac{1}{r} \text{ for } r \geq 2.$$

Because $k \geq 2$ and $k^2r^2 \leq 10^{kr}$, we have

$$|\mathcal{C}_1| \leq r^{\mathrm{ex}(n,K_{k+1})} \cdot r^{2\cdot 10^{-8kr}n^2} \cdot \left(\frac{1}{r^{\frac{1}{10^r}}}\right)^{10^{-6kr}n^2} \ll r^{\mathrm{ex}(n,K_{k+1})-1}.$$

By the above discussion, $\mathcal{C}_2$ contains $|\mathcal{C}| - |\mathcal{C}_1| \geq r^{\mathrm{ex}(n,K_{r+1})+m-1}$ colorings of G. Now we consider one of them. By the definition of $\mathcal{C}_2$ there is no k-tuple $(W_1, \ldots, W_k)$ as in the definition of $\mathcal{C}_1$. Let W_i^c be the set of all vertices of V_i that are adjacent to x through an edge of color c. Consider the bipartite graph with classes $[k]$ and $[r]$ where $\{i, c\}$ is an edge whenever $|W_i^c| \geq n/(10^{3r}rk)$. Since $|N(x) \cap V_i| > n/(10^{3r}k)$, it is impossible that, for some $i \in \{1, \ldots, k\}$, we have $|W_i^c| < n/(10^{3r}rk)$ for all $c \in [r]$ simultaneously. By Hall's Theorem, to avoid a locally rainbow distribution of colors as in the definition of $\mathcal{C}_1$, there exists h, $1 \leq h \leq k-1$, and pairwise distinct sets $V_{i_1}, \ldots, V_{i_{h+1}}$ such that for each $j = 1, \ldots, h+1$ we have $|W_{i_j}^c| \geq n/(10^{3r}rk)$ for at most h colors $c \in [r]$.

To construct such colorings, we fix h, and choose sets $V_{i_1}, \ldots, V_{i_{h+1}}$ in $\binom{k}{h+1}$ ways. Applying (4) and (5) again, there are at most

$$\left(2^{H(1/(10^{3r}kr))n}\right)^r \leq 2^{\frac{n}{10^{2r}}}$$

ways to select the at most $n/(10^{3r}kr)$ edges for each of the rare colors with respect to x and each V_{i_j}, $j = 1, \ldots, h+1$. The remaining edges are colored with the at most h abundant colors. We deduce that the number of ways to color edges between x and each V_{i_j} is bounded above by

$$2^{\frac{n}{10^{2r}}} \cdot h^{(\frac{1}{k}+\sqrt{2}\cdot 10^{-4kr})n}.$$

For all $\ell \in [k] \setminus \{i_1, \ldots, i_{h+1}\}$, the edges between x and V_ℓ can be colored in at most

$$r^{(\frac{1}{k}+\sqrt{2}\cdot 10^{-4kr})n}$$

ways. For large n, we conclude that the number N_x of colorings of edges incident with x satisfies

$$\begin{aligned} N_x &\leq \sum_{h=1}^{k-1} \binom{k}{h+1} \cdot \binom{r}{h}^{h+1} \cdot 2^{(h+1)/(10^{2r})n} \cdot h^{(h+1)(\frac{1}{k}+\sqrt{2}\cdot 10^{-4kr})n} \cdot r^{(k-h-1)(\frac{1}{k}+\sqrt{2}\cdot 10^{-4kr})n} \\ &\leq \sum_{h=1}^{k-1} 2^k \cdot 2^{kr} \cdot 2^{\frac{n}{10^r}} \cdot \left(h^{h+1} \cdot r^{k-h-1}\right)^{(\frac{1}{k}+\sqrt{2}\cdot 10^{-4kr})n}. \end{aligned} \tag{10}$$

Let $f(h) = h^{h+1} \cdot r^{k-h-1}$. Then, in case $k \geq 3$, for $h \leq k-2$ the inequality

$$\frac{f(h+1)}{f(h)} = \frac{\left(1+\frac{1}{h}\right)^h \cdot \frac{(h+1)^2}{h}}{r} \leq \frac{e(h+2+1/h)}{r} < 1$$

holds for $r > e(k+1)$. Thus, the summands are decreasing in h, hence we have

$$N_x \leq k \cdot 2^k \cdot 2^{kr} \cdot 2^{\frac{n}{10^r}} \cdot r^{(k-2)(\frac{1}{k}+\sqrt{2}\cdot 10^{-4kr})n} < r^{\delta_k(n) - \frac{n}{3k}}. \tag{11}$$

We already know that

$$|\mathcal{C}_2| \geq r^{\mathrm{ex}(n, K_{k+1})+m-1},$$

so that the number of (K_{k+1}, P)-free r-edge colorings of $G - x$ is at least

$$r^{\mathrm{ex}(n,K_{k+1})+m-1-\delta_k(n)+\frac{n}{3k}} \gg r^{\mathrm{ex}(n-1,K_{k+1})+m+1}.$$

This completes the induction step in the first case.

Now assume that every vertex has at most $n/(10^{3r}k)$ neighbors in its own class V_i, $i = 1, \ldots, k$. Recall that we may assume that G is not k-partite and that $|E(G)| > \mathrm{ex}(n, K_{k+1})$. Therefore, let $\{x, y\}$ be an edge with both ends in some class V_i, say, $x, y \in V_1$. In the following, let c be the color assigned to $\{x, y\}$.

Let $\mathcal{D}_1 \subseteq \mathcal{C}$ be the set of all (K_{k+1}, P)-free r-edge colorings of G for which there is a selection $c_{2,x}, \ldots, c_{k,x}, c_{2,y}, \ldots, c_{k,y} \in [r]$ of colors satisfying the following properties:

(i) The vertices x and y have at least $n/(10^{3r}kr)$ common neighbors in the class V_i that are adjacent to x via $c_{i,x}$-colored edges and adjacent to y via $c_{i,y}$-colored edges, for all $i \in \{2, \ldots, k\}$. Let $W_i \subseteq V_i$ be such a set of neighbors of x and y in V_i, respectively, that is, we have $|W_i| \geq n/(10^{3r}kr)$, $i = 2, \ldots, k$.

(ii) There exists a bijection between the set $\{x, y, W_2, \ldots, W_k\}$ and the vertex set of the graph K_{k+1}, where x is mapped to a locally rainbow vertex v of P and where, for the subgraph of G induced by $\{x, y\} \cup W_2 \cup \ldots \cup W_k$, the color pattern of all the edges incident to the vertices x and y coincides with the color pattern given by all edges incident with vertex v and one of its neighbors $w \neq v$ in (K_{k+1}, P).

Let $\mathcal{D}_2 = \mathcal{C} \setminus \mathcal{D}_1$.

Consider a coloring of G in $\mathcal{D}_1$ and fix sets $W_2, \ldots, W_k$ satisfying (ii). Fix a coloring of $K_{k-1} = K_{k+1} - v - w$ which, together with the coloring defined in (i), produces a pattern P. As in the first case, by Lemma 4.3 with $s = r$ there must be a pair (i, j) and subsets $X_i \subseteq W_i$, $X_j \subseteq W_j$ with $|X_i| \geq 10^{-(k-1)r}|W_i|$ and $|X_j| \geq 10^{-(k-1)r}|W_j|$ such that there is a rare color between X_i and X_j. With the arguments used before, we may prove that $|\mathcal{D}_1| < r^{\mathrm{ex}(n,K_{k+1})-1}$ and thus $|\mathcal{D}_2| > r^{\mathrm{ex}(n,K_{k+1})+m-1}$.

Next consider a coloring from $\mathcal{D}_2$, and let c be the color of the edge $\{x, y\}$. Since the coloring does not satisfy (i) or (ii), there must be an index $i \in \{2, \ldots, k\}$ and colors $c_x, c_y \in [r]$ such that the set $W_i = W_i^{c_x,c_y}$ of common neighbors of x and y such that all edges between x and W_i and between y and W_i have color c_x and c_y, respectively, satisfies $|W_i| \leq n/(10^{3r}kr)$.

To construct a coloring of this type, we may color edge $\{x,y\}$ in r ways, we may choose colors c_x and c_y in at most r^2 ways, we fix a set V_i in at most $k-1$ ways, and we choose a set W_i in at most $\binom{|V_i|}{n/(10^{3r}kr)}$ ways. As we are considering colorings in $\mathcal{D}_2$, every vertex has at most $n/(10^{3r}k)$ neighbors in its own class, where each class satisfies $|V_i - n/k| \leq \sqrt{2}\cdot 10^{-4kr}n \leq n/(10^{3r}k)$. Color the edges from x and y to their neighbors in their own class V_1 in at most $r^{2n/(10^{3r}k)}$ ways. Now consider every vertex z outside V_1, of which there are at most $n-|V_1| \leq \frac{k-1}{k}n+\sqrt{2}\cdot 10^{-4kr}n$. If z is not in V_i, the edges $\{x,z\}$ and $\{y,z\}$ may be colored in at most r^2 ways. If z is in $V_i \setminus W_i$, this number would be at most r^2-1. The condition on the degrees of x and y and the fact that they have fewer than $n/(10^{3r}k)$ neighbors within V_1 implies that each has at least $(k-1)n/k-n/(10^{3r}k)-\sum_{j\neq 1,i}|V_j| = |V_1|+|V_i|-n/k-n/(10^{3r}k)$ neighbors in V_i, thus their common neighborhood within class V_i has size at least

$$2|V_1|+2|V_i|-2n/k-2n/(10^{3r}k)-|V_i| \geq n/k-2n/(10^{3r}k)-3\sqrt{2}\cdot 10^{-4kr}n.$$

Under these conditions, the number of ways $N_{x,y}$ to color the edges of G incident with x and y is bounded above by

$$N_{x,y} \leq r^3(k-1)\cdot\binom{n}{\frac{n}{10^{3r}kr}}\cdot r^{\frac{2n}{10^{3r}k}}\cdot r^{\frac{2(k-1)n}{k}+2\sqrt{2}\cdot 10^{-4kr}n}\cdot\left(\frac{r^2-1}{r^2}\right)^{\frac{n}{k}-\frac{2n}{10^{3r}k}-3\sqrt{2}\cdot 10^{-4kr}n}.$$

The term $r^{2(k-1)n/k+2\sqrt{2}\cdot 10^{-4kr}n}$ comes from the fact that x and y have each at most $n-|V_1|$ neighbors outside V_1. The term $\left((r^2-1)/r^2\right)^{n/k-2n/(10^{3r}k)-3\sqrt{2}\cdot 10^{-4kr}n}$ is an adjustment for the fact that we are overcounting the number of ways to color the edges incident with vertices in V_i which are in the common neighborhood of x and y. We conclude that

$$\begin{aligned}
N_{x,y} &\overset{(4)}{\leq} 2^{H(1/(10^{3r}kr))n}\cdot(r^2-1)^{\frac{n}{k}-\frac{2n}{10^{3r}k}-3\sqrt{2}\cdot 10^{-4kr}n}\cdot r^{\frac{2(k-2)n}{k}+\frac{6n}{10^{3r}k}+8\sqrt{2}\cdot 10^{-4kr}n}\\
&\leq r^{\frac{n}{10^{2r}}}\cdot(r^2-1)^{\frac{n}{k}}\cdot r^{\frac{2(k-2)n}{k}+\frac{6n}{10^{3r}k}+8\sqrt{2}\cdot 10^{-4kr}n}\\
&\leq r^{\frac{2n}{k}}\cdot e^{-\frac{n}{kr^2}}\cdot r^{\frac{2(k-2)n}{k}+\frac{2n}{10^{2r}}}\\
&= r^{2\frac{k-1}{k}n}\cdot r^{-\frac{n}{kr^2\ln r}+\frac{2n}{10^{2r}}} \ll r^{2\frac{k-1}{k}n},
\end{aligned}$$

where we used that, for $r\geq 3$ and $r \geq k$,

$$\begin{aligned}
&\frac{6}{10^{3r}k}+\frac{8\sqrt{2}}{10^{4r}} < \frac{1}{10^{2r}}\\
&H(1/(10^{3r}k)) \overset{(5)}{\leq} (2/(10^{3r}k))\log_2(10^{3r}k) \leq (20rk)/(10^{3r}k) \leq 1/10^{2r}\\
&(r^2-1) \leq r^2e^{-1/r^2} = r^{2-1/(r^2\ln r)}\\
&kr^2\ln r < 10^{2r}/2.
\end{aligned}$$

We know that $|\mathcal{D}_2| \geq r^{\mathrm{ex}(n,K_{k+1})+m-1}$. Hence the number of (K_{k+1},P)-free r-edge colorings of $G-x-y$ is at least

$$\frac{r^{\mathrm{ex}(n,K_{k+1})+m-1}}{N_{x,y}} > r^{\mathrm{ex}(n-2,K_{k+1})+m+2}.$$

This completes two induction steps for the second case and proves the theorem. □

Remark 4.5. We remark that, in the case where the forbidden graph is a triangle, the lower bound $r > e(k+1)$ in the statement of Lemma 4.4 may be replaced by $r \geq 3$. Indeed, for $k=2$ the sum (10) contains a single term and the condition $r > e(k+1)$ is not needed. In the general case, the condition $r > e(k+1)$ may be replaced by other conditions, such as

$r > (k-1)^{k/(k-1)+\gamma}$, for some small $\gamma > 0$. Recall that the condition $r > e(k+1)$ has been used to bound (10); instead, we may define $f(h) = h^{h+1} \cdot r^{k-h-1}$, whose derivative satisfies $f'(h) = f(h) \cdot (\ln(eh) + 1/h - \ln r)$. Because $\ln(h) + 1/h$ is increasing in h, we derive that, for n large, the maximum summand in (10) occurs either for $h = 1$ or $h = k-1$. The case $h = 1$ may be treated easily, while for $h = k-1$ we obtain

$$N_x \quad \leq \quad k \cdot 2^k \cdot 2^{kr} \cdot 2^{\frac{n}{10^r}} \cdot (k-1)^{k(\frac{1}{k}+\sqrt{2}\cdot 10^{-4kr})n} \ll r^{\delta_k(n)}$$

for $r > (k-1)^{\frac{k}{k-1}+\gamma}$. This bound is better than $r > e(k+1)$ for some values of k.

Another observation is that Theorem 1.2 in [4] implies that, for $r \geq 3$ and $k \geq 2$ and any locally rainbow pattern P of K_{k+1}, *all* $\mathcal{C}_{r,K_{k+1},P}$ extremal graphs are complete multipartite. This may be used to simplify the proof of Lemma 4.4 in a way that avoids the case where every vertex has a small number of neighbours in its own class of the partition (see [4] for more details). We opted to use the current argument because it can easily be extended to locally rainbow patterns of other graphs F in Lemma 4.11.

We also remark that our proof of Lemma 4.4 fails for all patterns that are not locally rainbow if r is large (in comparison to k). The problem occurs when we consider colorings in $\mathcal{C}_2$, as we cannot bound the use of abundant colors in terms of k, as we did for locally rainbow colorings using Hall's Theorem. Indeed, we may split the set of r colors into k disjoint sets of size r/k (ignoring divisibility issues) and associate each such set with one of the classes $V_1, \ldots, V_k$. We then use the colors in each set to color the edges joining x with its neighbors in the corresponding class. In all colorings of this type, the vertex x would be locally rainbow, and thus could not be a part of the forbidden pattern, while the bound $(r/k)^{|N(x)|}$ prevents us from deriving a bound such as (11) if the number $|N(x)|$ of neighbors of x is not far from $n-1$ and r is large.

4.2. **Color Stability of K^R_{k+1}.** To conclude our proof of Theorem 1.2, we prove the following stability result for graphs with a large number of colorings avoiding K^R_{k+1}.

Lemma 4.6. *Let $k \geq 2$ and $r \geq \binom{k+1}{2}^{8k+4}$ be positive integers. Then for each $\delta > 0$ there is a positive integer n_0 such that the following statement is true. If G is a graph of order $n > n_0$ which has at least $r^{\text{ex}(n,K_{k+1})}$ distinct K^R_{k+1}-free r-edge colorings, then there is a partition $V(G) = W_1 \cup \cdots \cup W_k$ such that $\sum_{i=1}^k e(W_i) < \delta n^2$.*

Proof. Fix positive integers $k \geq 2$ and $r \geq \binom{k+1}{2}^{8k+4}$ and fix $\delta > 0$. Fix $\eta > 0$ satisfying $\eta < \delta/(2r)$ and $(4k+3) \cdot (4H(2r\eta) + 4r\eta) < \min\{\delta/2, (4k+3)/(8k^2)\}$, where $H(x)$ is the entropy function. This is possible as $H(x) \to 0$ with $x \to 0$. Let $n_0 = n_0(r, \eta, k)$ and $\varepsilon = \varepsilon(r, \eta, k) > 0$ satisfy the assumption in Lemma 2.4 and $\varepsilon < \eta/2$. Consider $M = M(\varepsilon, r)$ given by Lemma 2.3.

Let G be an n-vertex graph with $n > \max\{n_0, M\}$ for which the number of K^R_{k+1}-free r-edge colorings is at least $r^{\text{ex}(n,K_{k+1})}$. We will show that the vertex set of G may be partitioned as in the statement of the lemma. Fix a K^R_{k+1}-free r-edge coloring of G. By Lemma 2.3 we obtain a partition $V = V_1 \cup \cdots \cup V_m$ of the vertex set V of G for which each graph consisting of all edges in each of the r colors is ε-regular. For $i \in \{1, \ldots, r\}$, let $H_i = H_i(\eta)$ be the m-vertex cluster graph for color i and let H be the corresponding multicolored cluster graph.

Just as in the proof of Theorem 1.1, we may argue that there are at most $r\varepsilon n^2/2$ edges in irregular pairs, at most εn^2 edges with both endpoints in the same class and at most $r\eta n^2/2$ edges with endpoints in distinct classes whose color has density less than η between these two classes. Hence, there are at most $r\eta n^2$ such edges, which we can choose in at most $\binom{n^2/2}{r\eta n^2}$ ways. The number of colorings of this set of edges is at most $r^{r\eta n^2}$.

Let $E_j(H) = \{e \in E(H) : |L_e| = j\}$ be the sets of all edges with color lists of size j in the multicolored cluster graph H, and let $e_j = |E_j(H)|$ be their corresponding cardinalities, $j \in \{1, \dots, r\}$. The number of r-edge colorings of G that give rise to the partition $V = V_1 \cup \cdots \cup V_m$ and the multicolored cluster graph H is bounded above by

$$\binom{\frac{n^2}{2}}{r\eta n^2} \cdot r^{r\eta n^2} \cdot \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2} \overset{(4)}{\leq} 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2}. \quad (12)$$

There are at most M^n partitions $V = V_1 \cup \cdots \cup V_m$, where $m \leq M$. Summing (12) over all possible partitions and all possible multicolored cluster graphs, the number of K_{k+1}^R-free r-edge-colorings of G is bounded above by

$$\begin{aligned} & M^n \cdot 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot \sum_H \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2} \\ \leq \; & r^{H(2r\eta)n^2 + r\eta n^2} \cdot \sum_H \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2}. \end{aligned} \quad (13)$$

By Lemma 2.4, a multicolored cluster graph obtained from a K_{k+1}^R-free r-edge-coloring cannot contain a copy of K_{k+1} for which all edges have lists of size at least $\binom{k+1}{2}$, as this would lead to a copy of K_{k+1}^R in G, hence by Turán's Theorem we have

$$\sum_{i=\binom{k+1}{2}}^{r} e_i(H) \leq \mathrm{ex}(m, K_{k+1}) \leq \frac{k-1}{2k} \cdot m^2. \quad (14)$$

In the following our aim is to show that the sum $\sum_{i=\binom{k+1}{2}}^{r} e_i(H)$ is very close to $\mathrm{ex}(m, K_{k+1})$, as otherwise there will be too few colorings of G.

First assume that, for some $\beta \geq 1/(8k^2)$, $e_{\binom{k+1}{2}}(H) + \cdots + e_r(H) \leq \mathrm{ex}(m, K_{k+1}) - \beta m^2$ for *all* multicolored cluster graphs H arising from K_{k+1}^R-free r-edge colorings of G. Because $e_1(H) + \cdots + e_r(H) \leq \binom{m}{2}$, we deduce that (13) is at most

$$\begin{aligned} & r^{H(2r\eta)n^2 + r\eta n^2} \cdot \sum_H \left(\prod_{j=1}^{\binom{k+1}{2}-1} j^{e_j(H)} \cdot \prod_{j=\binom{k+1}{2}}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2} \\ \leq \; & r^{H(2r\eta)n^2 + r\eta n^2} \cdot \sum_H \left(\left(\binom{k+1}{2} - 1\right)^{\binom{m}{2} - (e_{\binom{k+1}{2}}(H) + \cdots + e_r(H))} \cdot r^{e_{\binom{k+1}{2}}(H) + \cdots + e_r(H)}\right)^{\left(\frac{n}{m}\right)^2} \\ \leq \; & 2^{rM^2/2} \cdot r^{H(2r\eta)n^2 + r\eta n^2} \cdot \left(\left[\binom{k+1}{2} - 1\right]^{\binom{m}{2} - (\mathrm{ex}(m,K_{k+1}) - \beta m^2)} \cdot r^{\mathrm{ex}(m,K_{k+1}) - \beta m^2}\right)^{\left(\frac{n}{m}\right)^2} \\ \leq \; & 2^{rM^2/2} \cdot r^{H(2r\eta)n^2 + r\eta n^2} \cdot \left[\binom{k+1}{2} - 1\right]^{\frac{n^2}{2k} + \beta n^2} \cdot r^{\mathrm{ex}(n,K_{k+1}) - \beta n^2} \\ \leq \; & 2^{rM^2/2} \cdot r^{H(2r\eta)n^2 + r\eta n^2} \cdot r^{\mathrm{ex}(n,K_{k+1}) - \frac{\beta}{2}n^2} \ll r^{\mathrm{ex}(n,K_{k+1})} \end{aligned}$$

for $r \geq \binom{k+1}{2}^{8k+4}$ with $0 < 4H(2r\eta) + 4r\eta \leq 1/(8k^2) \leq \beta$. In the above equation, we ignored rounding effects of order $r^{O(n)}$ related with the values of $\mathrm{ex}(n, K_{k+1})$ and $\mathrm{ex}(m, K_{k+1})$, but they are negligible owing to the upper bounds with exponentials of power $\Omega(n^2)$ used in other terms of the equation. The second to last inequality follows from

$$\frac{\left[\binom{k+1}{2}-1\right]^{\frac{1}{2k}+\beta}}{r^{\frac{\beta}{2}}} \leq \frac{\left[\binom{k+1}{2}-1\right]^{\frac{1}{2k}+\beta}}{\left[\binom{k+1}{2}^{8k+4}\right]^{\frac{\beta}{2}}} = \frac{\left[\binom{k+1}{2}-1\right]^{\frac{1}{2k}}}{\binom{k+1}{2}^{4k\beta}} \cdot \frac{\left[\binom{k+1}{2}-1\right]^{\beta}}{\binom{k+1}{2}^{2\beta}} < 1.$$

Since this number of colorings is small, there *exists* a multicolored cluster graph H with $e_{\binom{k+1}{2}}(H)+\cdots+e_r(H) \geq \mathrm{ex}(m,K_{k+1})-\beta m^2$ for some $\beta \leq 1/(8k^2)$. Let us first assume that there are only cluster graphs H such that $\beta \geq 4H(2r\eta)+4r\eta$. For such a multicolored cluster graph H let H' be the graph with vertex set $[m]$ obtained from H by deleting all edges in $E_1(H)\cup\cdots\cup E_{\binom{k+1}{2}-1}(H)$. As H' is K_{k+1}-free and

$$e(H') = \sum_{i=\binom{k+1}{2}}^{r} e_i(H) = \mathrm{ex}(m,K_{k+1}) - \left(\mathrm{ex}(m,K_{k+1}) - \sum_{i=\binom{k+1}{2}}^{r} e_i(H)\right),$$

by Theorem 2.5 there is a partition $U_1\cup\cdots\cup U_k=[m]$ with

$$\sum_{i=1}^{k} e_{H'}(U_i) \leq \mathrm{ex}(m,K_{k+1}) - \sum_{i=\binom{k+1}{2}}^{r} e_i(H). \tag{15}$$

Let H'' be the k-partite subgraph of H' with partition $U_1\cup\cdots\cup U_k$, whose number of edges is at least

$$\mathrm{ex}(m,K_{k+1}) - 2\cdot\left(\mathrm{ex}(m,K_{k+1}) - \sum_{i=\binom{k+1}{2}}^{r} e_i(H)\right) = 2\cdot\sum_{i=\binom{k+1}{2}}^{r} e_i(H) - \mathrm{ex}(m,K_{k+1}).$$

We use the following lemma from [2].

Lemma 4.7. *Let $0<\gamma\leq 1/(4k^2)$ be fixed and let H'' be a k-partite graph on m vertices with partition $V(H'')=U_1\cup\cdots\cup U_k$ with at least $\mathrm{ex}(m,K_{k+1})-\gamma m^2$ edges. If we add at least $(2k+1)\gamma m^2$ new edges to H'', then in the resulting graph there is a copy of K_{k+1} with exactly one new edge connecting two vertices of K_{k+1} in the same vertex class U_i of H''.*

We include its simple proof for completeness.

Proof. Let $V=U_1\cup\cdots\cup U_k$ be a k-partition for H''. If we add a set N with at least $(2k+1)\gamma m^2$ new edges to H'', one class, say U_1, contains at least $2\gamma m^2$ of the new edges. By MAXCUT we can obtain a bipartite subgraph H''_N induced by the new edges within U_1 with more than γm^2 edges. The sum of the number of new edges with the number of edges in H'' is larger than $\mathrm{ex}(m,K_{k+1})$, hence there exists a K_{k+1} containing exactly one new edge, as H''_N is bipartite. □

As $\beta\leq 1/(8k^2)$, when we set $\gamma=2\beta$ in Lemma 4.7, all assumptions of Lemma 4.7 are satisfied, and with it and Lemma 2.4, we immediately infer that

$$\sum_{i=1}^{\binom{k+1}{2}-1} e_i(H) \leq e(H)-e(H'') \leq (4k+2)\cdot\left(\mathrm{ex}(m,K_{k+1}) - \sum_{i=\binom{k+1}{2}}^{r} e_i(H)\right). \tag{16}$$

Assume first that we have

$$\sum_{i=\binom{k+1}{2}}^{r} e_i(H) := \mathrm{ex}(m, K_{k+1}) - \beta m^2 \leq \mathrm{ex}(m, K_{k+1}) - 4H(2r\eta)m^2 - 4r\eta m^2$$

for *all* multicolored cluster graphs H. Then, for n sufficiently large, (13) is at most

$$\begin{aligned}
& r^{H(2r\eta)n^2+r\eta n^2} \cdot \sum_H \left(\prod_{j=1}^{r} j^{e_j(H)} \right)^{\left(\frac{n}{m}\right)^2} \\
\leq \quad & r^{H(2r\eta)n^2+r\eta n^2} \cdot \sum_H \left(\left[\binom{k+1}{2} - 1 \right]^{\sum_{i=1}^{\binom{k+1}{2}-1} e_i(H)} \cdot r^{\sum_{i=\binom{k+1}{2}}^{r} e_i(H)} \right)^{\left(\frac{n}{m}\right)^2} \\
\overset{(16)}{\leq} \quad & r^{H(2r\eta)n^2+r\eta n^2} \times \\
& \times \sum_H \left(\left[\binom{k+1}{2} - 1 \right]^{(4k+2)\cdot \mathrm{ex}(m,K_{k+1})} \cdot \left(\frac{r}{\left[\binom{k+1}{2} - 1\right]^{4k+2}} \right)^{\sum_{i=\binom{k+1}{2}}^{r} e_i(H)} \right)^{\left(\frac{n}{m}\right)^2} \\
\leq \quad & r^{H(2r\eta)n^2+r\eta n^2} \times \\
& \times \sum_H \left(\left[\binom{k+1}{2} - 1 \right]^{(4k+2)\cdot \mathrm{ex}(m,K_{k+1})} \cdot \left(\frac{r}{\left[\binom{k+1}{2} - 1\right]^{4k+2}} \right)^{\mathrm{ex}(m,K_{k+1})-(4H(2r\eta)+4r\eta)m^2} \right)^{\left(\frac{n}{m}\right)^2} \\
\leq \quad & 2^{\frac{rM^2}{2}} \cdot r^{H(2r\eta)n^2+r\eta n^2} \cdot r^{\mathrm{ex}(n,K_{k+1})-(2H(2r\eta)+2r\eta)n^2} \qquad \text{for } r \geq \binom{k+1}{2}^{8k+4} \\
\leq \quad & 2^{\frac{rM^2}{2}} \cdot r^{-H(2r\eta)n^2-r\eta n^2} \cdot r^{\mathrm{ex}(n,K_{k+1})} \\
\ll \quad & r^{\mathrm{ex}(n,K_{k+1})}.
\end{aligned}$$

This contradicts our choice of G in that there are too few colorings. Therefore, there must be a multicolored cluster graph H for which $\beta \leq 4H(2r\eta) + 4r\eta$. As above, by Theorem 2.5, we obtain a partition $V(H) = U_1 \cup \cdots \cup U_k$ of the vertex set of the cluster graph H with $\sum_{i=1}^{k} e_{H'}(U_i) \leq \beta m^2$. For $W_i = \bigcup_{j \in U_i} V_j$, where $i \in \{1, \ldots, k\}$, we obtain

$$\begin{aligned}
\sum_{i=1}^{k} e_G(W_i) \quad & \leq \quad r\eta n^2 + \left(\frac{n}{m}\right)^2 \cdot \left(\sum_{i=1}^{k} e_{H'}(U_i) + \sum_{i=1}^{\binom{k+1}{2}-1} e_i(H) \right) \\
& \overset{(15)}{\leq} \quad r\eta n^2 + \beta n^2 + \left(\frac{n}{m}\right)^2 \cdot \sum_{i=1}^{\binom{k+1}{2}-1} e_i(H). \\
& \overset{(16)}{\leq} \quad r\eta n^2 + (4k+3)\beta n^2 \\
& < \quad \frac{\delta}{2} n^2 + (4k+3)(4H(2r\eta) + 4r\eta)n^2 \\
& < \quad \frac{\delta}{2} n^2 + \frac{\delta}{2} n^2 = \delta n^2,
\end{aligned}$$

by our choice of $\eta > 0$. □

Remark 4.8. In the proof of Lemma 4.6, it is possible to reduce slightly the lower bound $r \geq \binom{k+1}{2}^{8k+4}$ using more careful arguments. We used the current bound for clarity of presentation, as we believe that the conclusion of Lemma 4.6 would still hold for much smaller values of r. One way of obtaining a better lower bound is using MAX-s-CUT-type arguments. To clarify this suggestion, consider first the case of a rainbow K_4. Let $E_5'(H) \subseteq E_5(H)$ be a maximum subset of edges that is 3-partite. Then, it must be the case that $|E_5'(H) \cup E_6(H) \cup \ldots \cup E_r(H)| \leq \mathrm{ex}(m, K_4)$, as otherwise we have a rainbow K_4 in G. By MAX-3-CUT we know that $|E_5'(H)| \geq \frac{2}{3}|E_5(H)|$, from which we derive the inequality

$$\frac{2}{3}e_5(H) + \sum_{i=6}^{r} e_i(H) \leq \mathrm{ex}(m, K_4).$$

Moreover, consider a maximum subset $E'(H) \subseteq E_4(H) \cup E_5(H)$ of edges that is bipartite. Then, it must be the case that $|E'(H) \cup E_6(H) \cup \ldots \cup E_r(H)| \leq \mathrm{ex}(m, K_4)$, as otherwise we have a rainbow K_4 in G. By MAX-2-CUT we know that $|E'(H)| \geq \frac{1}{2}(|E_4(H)| + |E_5(H)|)$, from which we derive the inequality

$$\frac{e_4(H) + e_5(H)}{2} + \sum_{i=6}^{r} e_i(H) \leq \mathrm{ex}(m, K_4),$$

as otherwise we have a rainbow K_4. If we add on both inequalities to (14) for $k = 3$, we are able to replace the above lower bound $r \geq 6^{28}$ by $r \geq 3^{33}$.

In general, for $s = 2, \ldots, k$ take a maximum subset $E'(H) \subseteq E_{\mathrm{ex}(k+1,K_{s+1})}(H) \cup \cdots \cup E_{\binom{k+1}{2}-1}(H)$ of edges of the multicolored cluster graph H that is s-partite, so that

$$|E'(H)| \geq \frac{s-1}{s} \cdot |E_{\mathrm{ex}(k+1,K_{s+1})}(H) \cup \cdots \cup E_{\binom{k+1}{2}-1}(H)|.$$

The set $E'(H) \cup (E_{\binom{k+1}{2}}(H) \cup \cdots \cup E_r(H))$ does not contain a copy of K_{k+1}, as otherwise we have a rainbow K_{k+1} in G, hence

$$\frac{s-1}{s} \cdot \left(e_{\mathrm{ex}(k+1,K_{s+1})}(H) + \cdots + e_{\binom{k+1}{2}-1}(H)\right) + e_{\binom{k+1}{2}}(H) + \cdots + e_r(H) \leq \mathrm{ex}(m, K_{k+1}).$$

If we add on these inequalities to (14), we are able to replace the base $\binom{k+1}{2}$ in the above lower bound by a slightly better value. However, it seems hard to get an explicit expression for all values of k, and we therefore refrain from further calculations in this paper.

On the other hand, Remark 4.2 implies that the stability of Lemma 4.6 cannot hold for patterns of K_{k+1} that are not rainbow if $r > \binom{k+1}{2}^{k^2}$. As it turns out, this lower bound can be modified to $r > k^{2k^2/(k+1)}$ if P is not locally rainbow. Indeed, to prove this with the strategy in Remark 4.2, it suffices to partition the r colors into k sets of size r/k and associate each such set with some edge incident with a vertex of K_{k+1}, which we denote v_1 (we ignore divisibility constraints). The other edges of K_{k+1} are assigned all of the r colors. As before, we consider a blow-up H of the vertex set of K_{k+1} with classes $V_1, \ldots, V_{k+1}$ and we create edge-colorings of H in such a way that the edges between V_i and V_j use colors assigned to the edge $\{v_i, v_j\}$ of K_{k+1}. All copies of K_{k+1} in such colorings are locally rainbow (because of their vertex in V_1), so that they cannot produce P. For $r > k^{2k^2/(k+1)}$, the conclusion follows from

$$\left(\frac{r}{k}\right)^{\frac{k \cdot n^2}{(k+1)^2}} r^{\frac{n^2}{(k+1)^2}\binom{k}{2}} = \left(\frac{r^{\frac{k}{2(k+1)}}}{k^{\frac{k}{(k+1)^2}}}\right)^{n^2} > r^{\frac{k-1}{2k} \cdot n^2} \geq r^{\mathrm{ex}(n, K_{k+1})}.$$

4.3. **Color Stability of Edge-Color Critical Graphs.** A graph F is called *edge-color critical* if there is an edge e in F such that the chromatic number of $F-e$ satisfies $\chi(F-e) < \chi(F)$. Graphs with this property are sometimes called *weakly edge-color-critical.*

The authors of [1] showed that the strategy to prove that the Turán graph $T_k(n)$ is $\mathcal{C}_{r,K_{k+1},P}$-extremal for $r \in \{2,3\}$ and $k \geq 2$, where P is the monochromatic pattern, can be adapted to colorings that forbid a monochromatic copy of an edge-color critical graph F with chromatic number $\chi(F) = k+1$. A similar result may be obtained in the rainbow setting.

Theorem 4.9. *Let $k \geq 2$ be an integer and let F be an edge-color critical graph with $v(F)$ vertices, $e(F)$ edges and chromatic number $\chi(F) = k+1$. Fix a positive integer $r \geq r_0 = e(F)^{6k2^{v(F)}e(F)}$. There is n_0 such that every graph of order $n > n_0$ has at most $r^{\mathrm{ex}(n,K_{k+1})}$ distinct F^R-free r-edge colorings. Moreover, the Turán graph $T_k(n)$ is the only graph on n vertices for which equality is achieved.*

We prove the following stability result for graphs with a large number of colorings avoiding a rainbow F for a fixed edge-color critical graph F.

Lemma 4.10. *Let $k \geq 2$ be an integer, let F be an edge-color critical graph with $v(F)$ vertices, $e(F)$ edges and chromatic number $\chi(F) = k+1$ and let $r \geq r_0 = e(F)^{6k2^{v(F)}e(F)}$ be a positive integer. Then, for each $\delta > 0$ there is a positive integer n_0 such that the following statement is true. If G is a graph of order $n > n_0$ which has at least $r^{\mathrm{ex}(n,F)}$ distinct F^R-free r-edge colorings, then there is a partition $V(G) = W_1 \cup \cdots \cup W_k$ such that $\sum_{i=1}^{k} e(W_i) < \delta n^2$.*

Proof. The beginning of the proof is almost identical to the proof of Lemma 4.6; however, we cannot apply Lemma 4.7, and therefore we argue differently in the final part of the proof. Fix positive integers $k \geq 2$ and $r \geq r_0$ and let $\delta > 0$. We let $\eta > 0$ be small enough to ensure that $\max\{12k2^{v(F)}e(F)(H(2r\eta) + r\eta), r\eta + 16H(2r\eta) + 16r\eta\} < \delta/2$ and $4H(2r\eta) + 4r\eta \ll 1/(90k^2)$. Fix m_1 according to Theorem 2.6 for $\alpha = 4H(2r\eta) + 4r\eta$ with the additional property that $m_1 \geq \max\{k/\sqrt{5\alpha}, 2kv(F)\}$. Let $n_0 = n_0(r,\eta,k)$ and $\varepsilon = \varepsilon(r,\eta,k) > 0$ satisfy the assumptions of Lemma 2.4, where we assume that $\varepsilon < \min\{\eta/2, 1/m_1\}$. Define $M = M(\varepsilon, r)$ as in Lemma 2.3.

Let $G = (V,E)$ be an n-vertex graph with $n > \max\{n_0, M\}$ for which the number of F^R-free r-edge colorings is at least $r^{\mathrm{ex}(n,K_{k+1})}$. We will show that G may be partitioned as in the statement of the lemma. Fix an F^R-free r-edge coloring of G. By Lemma 2.3 we obtain a partition $V = V_1 \cup \cdots \cup V_m$ that is ε-regular with respect to every subgraph of G induced by the edges in each of the r colors. For $i \in \{1,\ldots,r\}$, let $H_i = H_i(\eta)$ be the m-vertex cluster graph for color i and let H be the corresponding multicolored cluster graph. Note that $m \geq 1/\varepsilon > m_1$.

Just as in the proof of Lemma 4.6, summing over all possible partitions and all possible multicolored cluster graphs H, the number of F^R-free r-edge-colorings of G is at most

$$M^n \cdot 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot \sum_H \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2}. \tag{17}$$

By Lemma 2.4, a multicolored cluster graph obtained from an F^R-free r-edge-coloring cannot contain a copy of F for which all edges have lists of size at least $e(F) = |E(F)|$, as this would lead to a copy of F^R in G. By Simonovits [25, Theorem 1], we have

$$e_{|E(F)|}(H) + \cdots + e_r(H) \leq \mathrm{ex}(m,F) = \mathrm{ex}(m,K_{k+1}) \leq \frac{k-1}{2k} \cdot m^2. \tag{18}$$

First assume that, for some $\beta \geq 1/(90k^2)$, the inequality $e_{|E(F)|}(H) + \cdots + e_r(H) \leq \mathrm{ex}(m,F) - \beta m^2$ holds for *all* multicolored cluster graphs H arising from F^R-free r-edge colorings of G. As $e_1(H)+\cdots+e_r(H)\le\binom{m}{2}$, we deduce that (17) is at most

$$\begin{aligned}
& r^{H(2r\eta)n^2+r\eta n^2}\cdot\sum_H\left(\prod_{j=1}^{e(F)-1} j^{e_j(H)}\cdot\prod_{j=e(F)}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2}\\
\le\ & r^{H(2r\eta)n^2+r\eta n^2}\cdot\sum_H\left((e(F)-1)^{\binom{m}{2}-(e_{|E(F)|}(H)+\cdots+e_r(H))}\cdot r^{e_{|E(F)|}(H)+\cdots+e_r(H)}\right)^{\left(\frac{n}{m}\right)^2}\\
\le\ & 2^{rM^2/2}\cdot r^{H(2r\eta)n^2+r\eta n^2}\cdot\left((e(F)-1)^{\frac{m^2}{2}-\left(\mathrm{ex}(m,F)-\beta m^2\right)}\cdot r^{\mathrm{ex}(m,F)-\beta m^2}\right)^{\left(\frac{n}{m}\right)^2}\\
\le\ & 2^{rM^2/2}\cdot r^{H(2r\eta)n^2+r\eta n^2}\cdot(e(F)-1)^{\frac{n^2}{2k}+\beta n^2}\cdot r^{\mathrm{ex}(n,F)-\beta n^2}\\
\le\ & 2^{rM^2/2}\cdot r^{H(2r\eta)n^2+r\eta n^2}\cdot r^{\mathrm{ex}(n,F)-\frac{\beta}{2}n^2} \qquad (19)\\
\ll\ & r^{\mathrm{ex}(n,F)}
\end{aligned}$$

for $r\ge r_0\ge e(F)^{90k+2}$ and small enough $0<4H(2r\eta)+4r\eta\ll 1/(90k^2)\le\beta$. Inequality (19) follows from

$$\frac{(e(F)-1)^{\frac{1}{2k}+\beta}}{r^{\frac{\beta}{2}}}\le\frac{(e(F)-1)^{\frac{1}{2k}+\beta}}{\left(e(F)^{45k+1}\right)^{\beta}}\le\frac{(e(F)-1)^{\frac{1}{2k}}}{e(F)^{45k\beta}}\cdot\frac{(e(F)-1)^{\beta}}{e(F)^{\beta}}<1.$$

Since this number of colorings is small, there *exists* a multicolored cluster graph H with $e_{|E(F)|}(H)+\cdots+e_r(H)\ge \mathrm{ex}(m,F)-\beta m^2$ for some $\beta<1/(90k^2)$. Next assume that $e_{|E(F)|}(H)+\cdots+e_r(H)\le\mathrm{ex}(m,F)-(4H(2r\eta)+4r\eta)m^2$ for *all* multicolored cluster graphs H. Given such H, let $V=V_1\cup\cdots\cup V_m$ and $H_1,\ldots,H_r$ be the partition and the cluster graphs that define H.

Let H' be the graph with vertex set $[m]$ obtained from the multicolored cluster graph H by deleting all edges in $E_1(H)\cup\cdots\cup E_{e(F)-1}(H)$. As H' is F-free and $m>m_1$, by Theorem 2.6 there is a partition $U_1\cup\cdots\cup U_k=[m]$ such that

$$\sum_{i=1}^{k} e_{H'}(U_i)\le 4\beta m^2.$$

Let H'' be the subgraph of H' containing all edges from H' with endvertices in different sets U_i, and write $u_i=|U_i|$, where $i\in\{1,\ldots,k\}$. Since the number of edges in H'' is at least $\mathrm{ex}(m,F)-5\beta m^2$, by Proposition 2.7 we know that, since $5\beta m^2\ge 5\beta m_1^2\ge k^2$ by our choice of m_1,

$$\left|u_i-\frac{m}{k}\right| \ < \ \sqrt{10\beta}\cdot m,\ i=1,\ldots,k. \qquad (20)$$

Next we provide an upper bound on $\sum_{i=1}^{|E(F)|-1} e_i(H)$. For $i\in\{1,\ldots,k\}$, let A_i be a set of edges $e\in E(H)$ with both endpoints in U_i whose color lists L_e have size at most $|E(F)|-1$. For $A=A_1\cup\cdots\cup A_k$, let H_A be the graph obtained by adding A to H''. Since F is edge-color critical, there is a partition of the vertex set of F into classes $X_1\cup\cdots\cup X_k$ such that there is a single edge $e=\{a,b\}$ with both endpoints in the same class, which we assume to be X_k. Assume that X_i contains $f_i\ge 1$ vertices of F. We say that an assignment ϕ of vertices of F to vertices of $V(H)$ is a *potential embedding* of F with respect to the partition $U_1\cup\cdots\cup U_k$ if ϕ maps vertices in X_i into U_i for $i\in\{1,\ldots,k\}$. Note that no such potential embedding generates an actual embedding of F into G, as this would lead to a copy of F^R in G.

Given $\{a,b\}\in A$, we let $\Phi_{a,b}$ be the family of potential embeddings such that the single edge of F with both ends in the same class is mapped onto $\{a,b\}$. To prove that the number of edges in A is small, we shall use double counting: we consider the sum

$$\sum_{\{a,b\}\in A_k}\sum_{\phi\in\Phi_{a,b}} |E(F)| - |E(H_A[a,b,\phi(X_1),\ldots,\phi(X_{k-1}),\phi(X_k)\setminus\{a,b\}])|\,, \tag{21}$$

which, for each edge $\{a,b\}$ in A_k, counts the number of edges missing from any potential embedding with respect to the partition $U_1\cup\cdots\cup U_k$ such that the edge with both ends in the same class is mapped to $\{a,b\}$.

On the one hand, because there must be an edge missing from H_A for any potential embedding $\phi\in\Phi_{a,b}$ to avoid a copy of F^R (and this edge is not $\{a,b\}$, as we are summing over edges in A_k, hence it must be an edge of H''), the sum in (21) is bounded below by

$$\begin{aligned} 2|A_k|\cdot(u_k-f_k+1)^{f_k-2}\cdot\prod_{i=1}^{k-1}(u_i-f_i+1)^{f_i} &\geq \frac{2|A_k|}{u_k^2\cdot 2^{v(F)-2}}\cdot\prod_{i=1}^{k}u_i^{f_i}\\ &\geq \frac{|A_k|}{\max\{u_i\}^2\cdot 2^{v(F)-3}}\cdot\prod_{i=1}^{k}u_i^{f_i}. \end{aligned}$$

This is because there are two ways to fix the endpoints of $\{a,b\}$ in ϕ, and $(f_k-2)!\cdot\binom{u_k-2}{f_k-2}$ ways to map the remaining vertices of X_k into U_k and $f_i!\cdot\binom{u_i}{f_i}$ ways to map vertices of X_i into U_i. Moreover, $\binom{u}{f}\geq(u-f+1)^f/f!$ and our choice of m implies that $u_i-f_i+1\geq u_i/2$ for every i.

On the other hand, all missing edges counted above have endpoints in different sets U_i. One such missing edge e, say, between U_i and U_j, $1\leq i<j\leq k$, is counted at most

$$(e(F)-1)\cdot u_k^{f_k}\cdot u_i^{f_i-1}\cdot u_j^{f_j-1}\cdot\prod_{\ell=1;\ell\neq i,j}^{k-1}u_\ell^{f_\ell}\leq\frac{(e(F)-1)}{\min\{u_i\}^2}\cdot\prod_{i=1}^{k}u_i^{f_i}$$

times in (21). To reach this number, note that e could play the role of any edge between X_i and X_j in F, and hence it may be chosen in at most $(e(F)-1)$ different ways (it cannot be the edge $\{a,b\}$), while the remaining terms in the product account for the number of ways of choosing vertices other than the endpoints of e in the embedding.

Because $e(H'')\geq \mathrm{ex}(m,K_{k+1})-5\beta m^2$, we infer with (20) that

$$\begin{aligned} |A_k| &\leq 5\beta m^2\cdot e(F)\cdot 2^{v(F)-3}\cdot\frac{\max_i\{u_i\}^2}{\min_i\{u_i\}^2}\\ &\leq \left(\frac{1+k\sqrt{10\beta}}{1-k\sqrt{10\beta}}\right)^2\cdot 5\beta m^2\cdot e(F)\cdot 2^{v(F)-3}. \end{aligned}$$

Since $\beta<1/(90k^2)$, we have

$$\left(\frac{1+k\sqrt{10\beta}}{1-k\sqrt{10\beta}}\right)^2\leq 4,$$

and hence

$$|A_k|\leq\frac{5}{2}\cdot e(F)\cdot 2^{v(F)}\cdot\beta m^2.$$

The same argument applies to edges contained in other sets A_i, leading to

$$\sum_{i=1}^{k}|A_i|\leq\frac{5}{2}\cdot k\cdot e(F)\cdot 2^{v(F)}\cdot\beta m^2.$$

Moreover, we know that the number of missing edges between distinct sets U_i is at most $5\beta m^2 \leq k \cdot e(F) \cdot 2^{|V(F)|-1} \cdot \beta m^2$, and hence

$$\sum_{i=1}^{e(F)-1} e_i(H) \quad \leq \quad 3k \cdot e(F) \cdot 2^{v(F)} \cdot \beta m^2. \tag{22}$$

Recall our assumption that $\sum_{i=|E(F)|}^{r} e_i(H) := \text{ex}(m, F) - \beta m^2 \leq \text{ex}(m, F) - 4H(2r\eta)m^2 - 4r\eta m^2$ for *all* multicolored cluster graphs H. Then (17) is at most

$$\begin{aligned}
& r^{H(2r\eta)n^2+r\eta n^2} \cdot \sum_H \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2} \\
\leq\ & r^{H(2r\eta)n^2+r\eta n^2} \cdot \sum_H \left((e(F)-1)^{\sum_{i=1}^{e(F)-1} e_i(H)} \cdot r^{\sum_{i=e(F)}^{r} e_i(H)}\right)^{\left(\frac{n}{m}\right)^2} \\
\overset{(22)}{\leq}\ & 2^{\frac{rM^2}{2}} \cdot r^{H(2r\eta)n^2+r\eta n^2} \cdot \left((e(F)-1)^{3k2^{v(F)}e(F)(4H(2r\eta)+4r\eta)m^2} \cdot r^{\text{ex}(m,F)-(4H(2r\eta)+4r\eta)m^2}\right)^{\left(\frac{n}{m}\right)^2} \\
\leq\ & 2^{\frac{rM^2}{2}} \cdot r^{H(2r\eta)n^2+r\eta n^2} \cdot \left(r^{\text{ex}(m,F)-(2H(2r\eta)+2r\eta)m^2}\right)^{\left(\frac{n}{m}\right)^2} \qquad (\text{as } r \geq r_0 = e(F)^{6k2^{v(F)}e(F)}) \\
\leq\ & 2^{\frac{rM^2}{2}} \cdot r^{-H(2r\eta)n^2-r\eta n^2} \cdot \left(r^{\text{ex}(m,F)}\right)^{\left(\frac{n}{m}\right)^2} \\
\ll\ & r^{\text{ex}(n,F)}.
\end{aligned}$$

This contradicts our choice of G, as there are too few F^R-free colorings of G.

Therefore, there must be a multicolored cluster graph H for which $\beta \leq 4H(2r\eta) + 4r\eta$. As above, by Theorem 2.6, we obtain a partition $U_1 \cup \cdots \cup U_k$ of the vertex set of the cluster graph H' with $\sum_{i=1}^{k} e_{H'}(U_i) \leq 16H(2r\eta)m^2 + 16r\eta m^2$ and we let $W_i = \bigcup_{j\in U_i} V_j$, $i = 1, \ldots, k$. Then,

$$\begin{aligned}
\sum_{i=1}^{k} e_G(W_i) \quad &\leq \quad r\eta n^2 + \left(\frac{n}{m}\right)^2 \left(\sum_{i=1}^{k} e_{H'}(U_i) + \sum_{i=1}^{e(F)-1} e_i(H)\right) \\
&\leq \quad r\eta n^2 + 16H(2r\eta)n^2 + 16r\eta n^2 + \left(\frac{n}{m}\right)^2 \sum_{i=1}^{e(F)-1} e_i(H). \\
&\overset{(22)}{<} \quad \frac{\delta}{2} \cdot n^2 + 3k \cdot 2^{v(F)} \cdot e(F) \cdot (4H(2r\eta) + 4r\eta) \cdot n^2 \\
&< \quad \frac{\delta}{2} \cdot n^2 + \frac{\delta}{2} \cdot n^2 = \delta n^2,
\end{aligned}$$

by our choice of $\eta > 0$. □

To obtain our result for edge-color critical graphs, we combine the previous lemma with the following, which may be proved with arguments as in Lemma 4.4. Even though we just need it for rainbow patterns, our proof holds for any locally rainbow pattern with the additional property that an edge e for which $\chi(G-e) < \chi(G)$ is incident with a vertex which is locally rainbow.

Lemma 4.11. *Let $k \geq 2$, let F be an edge-color critical graph with $\chi(F) = k+1$, and let P be a locally rainbow pattern of F, where the maximum degree of a vertex in F whose incident edges are rainbow-colored is d. Let $k \geq 2$, let F be an edge-color critical graph with $\chi(F) = k+1$, and let P be a pattern of F such that an edge e for which $\chi(F-e) < \chi(F)$*

is incident with a locally rainbow vertex of degree d. Assume that (F, P) satisfies the Color Stability Property of Definition 4.1 for a positive integer $r > e(d+1)$. Then there is n_0 such that every graph of order $n > n_0$ has at most $r^{\mathrm{ex}(n,F)}$ distinct (F, P)-free r-edge colorings. Moreover, the only n-vertex graph for which the number of such colorings is $r^{\mathrm{ex}(n,F)}$ is the Turán graph $T_k(n)$.

The proof of Lemma 4.11 is an easy adaptation of the proof of Lemma 4.4. Indeed, the case in which there is a vertex x with degree less than $\delta_k(n)$ is identical. The case where there is a vertex x with large degree in its own class (in the partition given by the Color Stability Property) may be treated as follows: recall that in the proof of Lemma 4.4, we try to find a partial embedding of F where x is one of the vertices. If we succeed, we argue that there must be large sets X_i and X_j in two different classes such that some color does not appear often on edges between them. If we fail, we argue that $G - x$ must have a large number of colorings. The same may be done here, but we need to consider a good embedding of F such that x is locally rainbow and an edge e with the property that $\chi(F - e) = k$ is incident with x. The latter ensures that the sets X_i and X_j lie in different classes in the partition. Also note that the condition $r \geq e(d+1)$ naturally replaces the condition $r \geq e(k+1)$ of Lemma 4.4. The case in which there is no vertex with large degree in its own class may also be treated as in Lemma 4.4, but we must look for an embedding of F such that $\{x, y\}$ plays the role of an edge e as above. Details are left to the reader.

Note that Theorem 4.9 follows from Lemma 4.11 and Lemma 4.10, where we use that $e(F)^{6k2^{v(F)}e(F)} \geq e(d+1)$ holds for any edge color critical graph F with $\chi(F) \geq 3$. Further note that Lemma 4.11 might be of interest only for values of d that are not too large compared to k, see Remarks 4.2 and 4.8.

5. Final remarks and open problems

In this paper, we studied n-vertex graphs with the maximum number of r-edge-colorings avoiding the occurrence of a subgraph F colored according to a pattern P. We showed that, whenever F is a bipartite graph and P has at least three classes, the complete graph K_n is *almost* extremal, in the sense that no other n-vertex graph may beat the number of (F, P)-free r-edge colorings of K_n by more than a multiplicative factor of $C^{o(n^2)}$, where C is a constant. We also proved that this behavior is not shared with arbitrary patterns of non-bipartite graphs. In fact, if the number r of colors is large with respect to $k \geq 2$ and P is the rainbow pattern of the complete graph K_{k+1}, we showed that the Turán graph $T_k(n)$ is the unique graph maximizing the number of rainbow K_{k+1}-free r-edge colorings.

Our results raise several natural questions. For instance, if F is bipartite and P is a pattern of F with at least three classes, we may ask if, for large n, the complete graph K_n is indeed extremal for (F, P)-free r-edge colorings. Two of the authors [12] gave a positive answer to this question for patterns of matchings with at least three classes, while Sanches and the current authors [15] proved that the same holds for patterns of stars $S_\ell = K_{1,\ell}$ with at least three classes. However, this has not been proved for other classes of bipartite graphs, and the arguments used to prove equality in [12] and [15] have a heavy dependency on the structure of the forbidden bipartite graph.

Another natural improvement on our result about bipartite graphs would be to drop the condition on the number of classes in the forbidden pattern. However, this cannot be done in general: it is shown in [12] that there is an infinite family of matching patterns with two classes for which K_n is not extremal (and is not even almost extremal in the sense of Theorem 1.1). On the other hand, the arguments in the current work imply that, for any fixed $\beta > 0$, $r \geq 2$ and any pattern P with $t = 2$ classes in a bipartite graph F, we have $c_{r,F,P}(n) \leq 2^{\beta n^2}$ if n is sufficiently large. In the case of monochromatic colorings, Ramsey's Theorem implies that, for n large, the complete graph K_n does not admit any (F, P)-free r-edge coloring when P has a single color class (and hence when we forbid monochromatic copies of F). More information about results in the monochromatic case may be found in [11].

For rainbow patterns in complete graphs, the most natural question would be to weaken considerably the condition $r \geq \binom{k+1}{2}^{8k+4}$ in Theorem 1.2. As we remarked in and at the end of its proof, it is possible to reduce the basis of the exponent as well as the exponent in this lower bound using more careful arguments, but we believe that the same conclusion would still hold for much smaller values of r. For triangles [14], the lower bound in Theorem 1.2 has been improved to $r \geq 5$ (provided that n is sufficiently large) and to $r \geq 10$ (also for small values of n) using more careful arguments. On the other hand, we know that Theorem 1.2 cannot hold for arbitrary r as the complete graph would be trivially extremal if we had $r < \binom{k+1}{2}$. In fact, if we use only $\binom{k+1}{2} - 1$ colors, the complete graph K_n may be colored in $(\binom{k+1}{2} - 1)^{\binom{n}{2}}$ ways without creating a rainbow K_{k+1}, so that it has more colorings than the Turán graph $T_k(n)$ whenever $(\binom{k+1}{2} - 1)^{\binom{n}{2}} > r^{\mathrm{ex}(n,K_{k+1})}$, which happens if $r < (\binom{k+1}{2} - 1)^{k/(k-1)}$. It would be interesting to determine whether the complete graph is extremal in this case. We should mention that there is some recent work on problems of this type applying modern techniques in Extremal Combinatorics, such as graph limits and the container method [9].

Furthermore, Remarks 4.2 and 4.8 imply that the stability of Lemma 4.6 cannot hold for patterns P of K_{k+1} that are not rainbow if $r > \binom{k+1}{2}^{k^2}$, and even for smaller values of r if P is not locally rainbow. For monochromatic patterns, the results in [1] imply that the Color Stability Property does not hold for all $r \geq 4$, but holds for $r \in \{2, 3\}$. The results in [3, 4] also imply that a large class of patterns of complete graphs with $t = 2$ classes satisfy the stability property for $r \in \{2, 3\}$. We wonder whether the stability of Lemma 4.6 holds for other locally rainbow patterns P of complete graphs K_{k+1} for some values of r satisfying $t \leq r \leq \binom{k+1}{2}^{k^2}$, where t is the number of classes of P. By Lemma 4.4, this would automatically imply that $T_k(n)$ is (K_{k+1}, P)-extremal for n sufficiently large.

One may also ask whether, for some values of r, Lemma 4.4 holds for locally rainbow patterns of graphs F with $\chi(F) \geq 3$ that are not complete. Advances in this direction would be naturally intertwined with the investigation of patterns of such general graphs F that satisfy the stability of Lemma 4.6 for certain values of r.

In general, the problem of determining $c_{r,F,P}(n)$ and the n-vertex graphs that achieve this extremal value has only been solved in very special cases. Even for the original problem, in which $F = K_{k+1}$ and P is monochromatic, it is known [1, 28] that $T_k(n)$ is the only extremal graph for all values of $k \geq 2$ if $r \in \{2, 3\}$, but that it is never extremal for $r \geq 4$. The only other extremal configurations that are known are for the cases $r = 4$ and $F \in \{K_3, K_4\}$, which were obtained by Pikhurko and Yilma [24], but good approximations to $c_{r,K_{k+1},P}(n)$ for all values of r and k have been provided in [23] in terms of solutions to optimization problems. One interesting feature of the monochromatic case is the emergence of extremal graphs that are neither complete nor isomorphic to the Turán graph. For more general patterns, other extremal structures appear for patterns with two classes in matchings [12], but there are no known examples of alternative extremal configurations for rainbow patterns.

Another natural pattern of complete graphs comes from the *lexical* colorings of the classical Canonical Ramsey Theorem of Erdős and Rado [8]: to construct such an edge-coloring of K_{k+1}, assume that the vertex set is given by $[k+1]$ and assign distinct colors $c_1, \ldots, c_k$ to the first k vertices, respectively. Any edge $\{i, j\}$ of K_{k+1}, where $i < j$, is assigned color c_i. Note that this pattern is locally rainbow, and hence Lemma 4.4 applies.

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