ON CONFIGURATIONS OF GENERALIZED ERDŐS-ROTHSCHILD PROBLEMS

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Abstract. In this paper a generalization of a problem of Erdős and Rothschild is considerd. Given an integer $r \geq 1$ and a graph F, an *r-pattern* P of F is a partition of its edge set into at most r nonempty classes. Let $\mathcal{P}$ be a *pattern family*, which is an arbitrary non-empty family whose elements are of the form (F, P), where F is a graph and P is an r-pattern of F. An r-coloring of a graph G is *$\mathcal{P}$-free* if G does not contain any copy of F for which the r-pattern P' induced by the coloring is isomorphic to some r-pattern P where $(F, P) \in \mathcal{P}$. Let $\mathcal{C}_{r,\mathcal{P}}(G)$ be the set of all $\mathcal{P}$-free r-colorings of a graph G. Let $c_{r,\mathcal{P}}(n) = \max \{ |\mathcal{C}_{r,\mathcal{P}}(G)| \colon |V(G)| = n \}$. An n-vertex graph G is *$(r, \mathcal{P})$-extremal* if $|\mathcal{C}_{r,\mathcal{P}}(G)| = c_{r,\mathcal{P}}(n)$. We wish to characterize the n-vertex graphs that admit the largest number of $\mathcal{P}$-free r-colorings.

It is shown that, for some choices of r and $\mathcal{P}$, and for every positive integer n, there exists an $(r, \mathcal{P})$-extremal n-vertex graph that is a complete multipartite graph. Moreover, it is shown, that in some cases, all $(r, \mathcal{P})$-extremal n-vertex graphs must be complete multipartite. In the case of pattern families with a single element, this extends recent results of Benevides, Hoppen and Sampaio [4] who proved that there is an $(r, \mathcal{P})$-extremal n-vertex complete multipartite graph for any $\mathcal{P} = \{(K_k, P)\}$ where $k \geq 3$ and P is a pattern of the complete graph K_k.

1. Introduction

In the last decade there has been growing interest in an extremal problem about graphs (or other combinatorial structures) that admit a large number of edge-colorings* that satisfy some restrictions. This became known as the Erdős-Rothschild problem. In this paper, we consider a generalized version of this extremal problem and discuss properties of the configurations that achieve extremality.

For us, an r-(edge)-coloring of a graph G is just a function $f \colon E(G) \longrightarrow [r]$ that associates a color in $[r] = \{1, \dots, r\}$ with each edge of G. Given an integer $r \geq 2$ and a fixed graph F, we say that an r-coloring $E = E_1 \cup \cdots \cup E_r$ of the edge-set of a host graph $G = (V, E)$ is F-free if the graphs $G_i = (V, E_i)$ do not contain F as a subgraph, for all $i \in [r]$. The problem originally addressed by Erdős and Rothschild [6] was to characterize the n-vertex graphs that admit the largest number of F-free r-colorings, where n is an integer. In other words, they considered edge-colorings that avoid *monochromatic copies* of a fixed graph F.

Erdős and Rothschild conjectured that, for all $n \geq n_0(k)$, the number of K_k-free 2-colorings is maximized by the Turán graph $T_{k-1}(n)$, namely the balanced, complete, $(k-1)$-partite graph on n vertices, and that this maximum is unique up to isomorphism.

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*The edge-colorings in this paper are not necessarily proper.

As usual, a graph $G = (V, E)$ is *complete s-partite* if there is a partition $V = V_1 \cup \cdots \cup V_s$ of its vertex set such that $\{v, w\} \in E$ if and only if $v \in V_i$, $w \in V_j$ and $i \neq j$. This partition is *balanced* if $||V_i| - |V_j|| \leq 1$ for all $i, j \in [s]$.

According to the Erdős-Rothschild Conjecture, finding an n-vertex graph with the largest number of K_k-free 2-colorings turns out to be the same as finding an n-vertex graph with the largest number of edges and no copy of K_k as a subgraph, the well-known Turán problem [16]. In general, for any fixed F, we write $\mathrm{ex}(n, F)$ to denote the maximum number of edges over all n-vertex graphs that do not contain F as a subgraph, and we say that an n-vertex graph G is *F-extremal* if it has $\mathrm{ex}(n, F)$ edges and does not contain F as a subgraph. Observe that F-extremal graphs on n vertices have the largest number of edges among all graphs that may be colored arbitrarily without producing a monochromatic copy of F, which leads to $r^{\mathrm{ex}(n,F)}$ colorings. The number of colorings might increase if we have more than $\mathrm{ex}(n, F)$ edges to color, but extra edges lead to copies of F, creating restrictions on how to color them.

Yuster [17] proved the Erdős-Rothschild Conjecture for $k = 3$ and any $n \geq 6$. Later, Alon, Balogh, Keevash and Sudakov [1] proved that, for $r \in \{2, 3\}$ and $n \geq n_0$, where n_0 is a constant depending on r and k, the Turán graph $T_{k-1}(n)$ is the unique optimal n-vertex graph for the number of K_k-free r-colorings. An interesting feature of their proof was to apply the Szemerédi Regularity Lemma [15] to obtain an exact result, which, on the other hand, required n_0 to be very large. The value of n_0 has been recently improved by Hàn and Jiménez [7] using the Container Method. When the number of colors satisfies $r \geq 4$, the problem has shown to be much harder and it is known that $T_{k-1}(n)$ cannot maximize the number of K_k-free r-colorings. Pikhurko and Yilma [13] determined, for large n, the graphs that admit the largest number of such colorings for $r = 4$ and $k \in \{3, 4\}$. For $k = 3$, Botler et al. [5] characterized these graphs for $r = 6$, and have an approximate result for $r = 5$. As it turns out, the extremal configurations obtained so far are always some balanced complete multipartite graph $T_\ell(n)$, but $\ell \geq k$ for $r \geq 4$. Interestingly, even in this small sample of results, the value of ℓ is not a monotone non-decreasing function of r.

This problem has been generalized in a few different ways, by considering colorings where the size of the forbidden clique may vary according to the color class [12], or colorings where the forbidden graph is not colored according to some given coloring [2] or according to some given coloring pattern [4, 9]. Here, we consider an extension of this last version.

Given an integer $r \geq 1$ and a graph F, an *r-pattern* P of F is a partition of its edge set into at most r nonempty classes. Let $\mathcal{P}$ be a *pattern family*, namely an arbitrary non-empty family whose elements are of the form (F, P), where F is a graph and P is an r-pattern of F. We say that an r-coloring of a graph G is *$\mathcal{P}$-free* if G does not contain any copy of F for which the r-pattern P' induced by the coloring is isomorphic to some r-pattern P where $(F, P) \in \mathcal{P}$. Let $\mathcal{C}_{r,\mathcal{P}}(G)$ be the set of all $\mathcal{P}$-free r-colorings of a graph G. We write $c_{r,\mathcal{P}}(n) = \max\{\, |\mathcal{C}_{r,\mathcal{P}}(G)| \colon |V(G)| = n \,\}$, and we say that an n-vertex graph G is *$(r, \mathcal{P})$-extremal* if $|\mathcal{C}_{r,\mathcal{P}}(G)| = c_{r,\mathcal{P}}(n)$. In other words, we wish to characterize the n-vertex graphs that admit the largest number of $\mathcal{P}$-free r-colorings.

In the version of [4, 9], the family $\mathcal{P}$ contains a single pair (F, P), while in the original Erdős-Rothschild problem this single pattern P is monochromatic, that is, contains a single class. Moreover, if the family $\mathcal{P}$ contains all possible r-patterns of a fixed graph F with at least one edge, where $r \geq 2$, then $\mathcal{C}_{r,\mathcal{P}}(G) \neq \emptyset$ if and only if G is F-free, and hence G is $(r, \mathcal{P})$-extremal if and only if it is F-extremal. This means that the problem considered here generalizes the Turán problem, further illustrating the connection between the Turán and the Erdős-Rothschild problems.

The following result collects other immediate consequences of the definition. To state it, we need to introduce some additional notation. The number of classes in a pattern P is denoted by $\gamma(P)$ and, for a pattern family $\mathcal{P}$, we have $\gamma_{min}(\mathcal{P}) = \min\{\gamma(P) \colon (F, P) \in \mathcal{P}\}$. Given a pattern family $\mathcal{P}$, let $\chi_{min}(\mathcal{P}) = \min\{\chi(F) \colon (F, P) \in \mathcal{P}\}$, where $\chi(F)$ denotes the (vertex) chromatic number of F.

Proposition 1.1. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be pattern families, and let $r \geq 2$ be an integer.*

(a) *If $r < \gamma_{min}(\mathcal{P}_1)$, then $c_{r,\mathcal{P}_1}(n) = r^{\binom{n}{2}}$ and K_n is the unique $(r, \mathcal{P}_1)$-extremal graph.*
(b) *If $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $c_{r,\mathcal{P}_1}(n) \geq c_{r,\mathcal{P}_2}(n)$. Moreover, if $s = \chi_{min}(\mathcal{P}_2) - 1$ and the s-partite Turán graph $T_s(n)$ is $(r, \mathcal{P}_1)$-extremal, then $T_s(n)$ is also $(r, \mathcal{P}_2)$-extremal.*

Proof. Part (a) is trivial, as no r-coloring can ever produce a forbidden pattern.

For part (b), fix $r \geq 2$ and consider pattern families $\mathcal{P}_1 \subseteq \mathcal{P}_2$. For any graph G, every $(r, \mathcal{P}_2)$-free coloring is also $(r, \mathcal{P}_1)$-free, so that $|\mathcal{C}_{r,\mathcal{P}_1}(G)| \geq |\mathcal{C}_{r,\mathcal{P}_2}(G)|$. This immediately implies $c_{r,\mathcal{P}_1}(n) \geq c_{r,\mathcal{P}_2}(n)$. Assume that the Turán graph $T_s(n)$ is $(r, \mathcal{P}_1)$-extremal, where $s = \chi_{min}(\mathcal{P}_2) - 1$. This choice of s implies that no forbidden pattern can be produced by coloring the edges of $T_s(n)$. As a consequence, for any n-vertex graph G, we have

$$|\mathcal{C}_{r,\mathcal{P}_2}(T_s(n))| = r^{\mathrm{ex}(n,K_{s+1})} = |\mathcal{C}_{r,\mathcal{P}_1}(T_s(n))| \geq |\mathcal{C}_{r,\mathcal{P}_1}(G)| \geq |\mathcal{C}_{r,\mathcal{P}_2}(G)|,$$

and therefore $T_s(n)$ is $(r, \mathcal{P}_2)$-extremal. □

The aim of this paper is to show that, for some choices of r and $\mathcal{P}$, and for every positive integer n, there exists an $(r, \mathcal{P})$-extremal n-vertex graph that is a complete multipartite graph. Moreover, we show that, in some cases, all $(r, \mathcal{P})$-extremal n-vertex graphs must be complete multipartite. In the case of pattern families with a single element, Benevides, Hoppen and Sampaio [4] proved that there is an $(r, \mathcal{P})$-extremal n-vertex complete multipartite graph for any $\mathcal{P} = \{(K_k, P)\}$ where $k \geq 3$ and P is a pattern of K_k. If P is monochromatic, this is also implied by Pikhurko, Staden and Yilma [12] (whose result also holds for a different generalization of the original Erdős Rothschild problem).

Theorem 1.2. *Fix integers $r \geq 2$ and $n > k \geq 3$, and let P be an r-pattern of the complete graph K_k. Then there exists an n-vertex (r, P)-extremal graph that is a complete multipartite graph.*

Regarding instances where *all* $(r, \mathcal{P})$-extremal n-vertex graphs are necessarily complete multipartite, the following is known for singletons $\mathcal{P} = \{(K_k, P)\}$.

Theorem 1.3. [4, Theorem 1.2] *Let $r \geq 2$ and $k \geq 3$ be integers and let P be an r-pattern of the complete graph K_k which is not monochromatic and is different from the pattern T_0. Also assume that if $r = 2$ then P is different from the pattern P_2 (see Figure 1). If $\mathcal{P} = \{(K_k, P)\}$, then* every *$(r, \mathcal{P})$-extremal graph is a complete multipartite graph.*

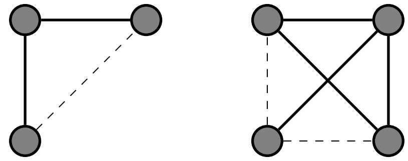

FIGURE 1. Some special 2-patterns: T_0 a triangle K_3 colored by exactly two colors and P_2 a complete K_4 colored by exactly two colors as shown
.

One of our main results is that Theorem 1.2 may be generalized to any pattern family of complete graphs.

Theorem 1.4. *Let $\mathcal{P}$ be a pattern family of complete graphs and let $r \geq 2$ be an integer. For any positive integer n, there exists an n-vertex complete multipartite graph G^* that is $(r, \mathcal{P})$-extremal. Moreover, for any $(r, \mathcal{P})$-extremal n-vertex graph G, there is one such n-vertex complete multipartite graph G^* such that $|E(G^*)| \geq |E(G)|$.*

Our proof of Theorem 1.4 has the following useful consequence.

Theorem 1.5. *Let $\mathcal{P}$ be a pattern family of complete graphs and $n, r \geq 2$ be integers. If there exists an $(r, \mathcal{P})$-extremal graph that is not complete multipartite, then there exist at least two non-isomorphic $(r, \mathcal{P})$-extremal complete multipartite graphs on n vertices.*

Regarding extensions of Theorem 1.3, we show that it holds for a class of pattern families. Given integers $k \geq 3$ and $1 \leq s \leq \binom{k}{2}$, let $\mathcal{P}_{k,s}$ be the pattern family containing all patterns P of K_k such that $\gamma(P) \geq s$.

Theorem 1.6. *Let $n, r \geq 2$ and $k \geq 3$ be integers, and fix $k \leq s \leq \binom{k}{2}$. If G is an n-vertex $(r, \mathcal{P}_{k,s})$-extremal graph, then G is a complete multipartite graph.*

2. Extremal configurations for pattern families of complete graphs

The results in this paper may be derived with the approach in [4], which in turn was influenced by the Zykov Symmetrization proof of Turán's Theorem.

Let $\mathcal{I}$ be a non-empty, and possibly infinite, set of indices, and let

$$\mathcal{P} = \{(K_{k_i}, P_i) \colon i \in \mathcal{I}\}$$

be a pattern family, where $k_i \geq 3$ and P_i is a pattern of K_{k_i}, for each $i \in \mathcal{I}$. We shall prove that, given positive integers n and $r \geq 2$, there is an $(r, \mathcal{P})$-extremal n-vertex graph that is a complete multipartite graph.

For a vector $\vec{x}$ with coordinates indexed by a set T, we will denote by $x(t)$ the value of x at coordinate t, where $t \in T$. We will use $\|\vec{x}\|_p$ to denote the ℓ_p-norm of $\vec{x}$, so for $p \in (0, \infty)$ we have

$$\|x\|_p = \left(\sum_{t \in T} |x(t)|^p \right)^{1/p}.$$

Moreover, for a sequence of vectors $\vec{x_1}, \ldots, \vec{x_s}$, each indexed by T, we will denote their pointwise product by $\prod_{k=1}^{s} \vec{x}_k$, that is, the vector $\vec{y}$ such that for each $t \in T$ we have $y(t) = \prod_{k=1}^{n} x_k(t)$.

Definition 2.1. *Let H be a graph and let $\mathcal{P}$ be a family of r-patterns. If H is a subgraph of a graph G and $\widehat{H}$ is a $\mathcal{P}$-free r-coloring of H, we denote by $c_{r,\mathcal{P}}(G \mid \widehat{H})$ the number of ways to r-color the edges in $E(G) - E(H)$ in such a way that the resulting coloring is still $\mathcal{P}$-free. For a single vertex $v \in V(G) - V(H)$, we use the notation $c_{r,\mathcal{P}}(v, \widehat{H})$ for the number of ways to r-color the edges from v to $V(H)$ (again avoiding $\mathcal{P}$). We also define $\vec{v}_{H,r,\mathcal{P}}$ as the vector indexed by all $\mathcal{P}$-free r-colorings of H, whose coordinate corresponding to a coloring $\widehat{H}$ is given by $\vec{v}_{H,r,\mathcal{P}}(\widehat{H}) = c_{r,\mathcal{P}}(v, \widehat{H})$.*

The following proposition is a simple consequence of the fact that all graphs in the pattern family are complete.

Proposition 2.2. *If H is an induced subgraph of G such that $S = V(G) - V(H)$ is an independent set in G, and $\widehat{H}$ is a $\mathcal{P}$-free r-coloring of H, then*

$$c_{r,\mathcal{P}}(G \mid \widehat{H}) = \prod_{v \in S} c_{r,\mathcal{P}}(v, \widehat{H}).$$

We shall also use the following consequence of Hölder's inequality.

Lemma 2.3. *Let $\vec{x}_1, \ldots, \vec{x}_s$ be complex-valued vectors indexed by the same set T. We have*

$$\left\| \prod_{k=1}^{s} \vec{x}_k \right\|_1 \leq \prod_{k=1}^{s} \|\vec{x}_k\|_s .$$

Equality happens if and only if, for every $i, j \in [s]$, there exists $\alpha_{i,j}$ with the property that $x_i(t)^s = \alpha_{i,j} x_j(t)^s$ for all $t \in T$.

Definition 2.4. *Two vertices of a graph are said to be* twins *if they are non-adjacent and have the same neighborhood.* Cloning *a vertex v of a graph G means to create a new graph $\widetilde{G}$ whose vertex set is $V(G) \cup \{\widetilde{v}\}$ where $\widetilde{v}$ is a new vertex which is a twin of v.*

For the next lemma we consider the following operation: take an independent set S of a graph G, select a particular vertex $v \in S$, delete all vertices in $S - v$ and add $s - 1$ new twins of v. For some vertex $v \in S$, we produce a new graph which has at least as many good colorings as G.

Lemma 2.5. *Let $\mathcal{P}$ be a family of r-patterns. Let G be a graph on n vertices, $\emptyset \neq S \subset V(G)$ be an independent set with $s = |S|$, $H = G - S$, and $A = V(G) - S$. There exists a vertex $v \in S$ with the following property: if we construct the graph $\widetilde{G}$ with $V(\widetilde{G}) = V(H) \cup \widetilde{S}$, where $\widetilde{S}$ is an independent set on s vertices, each of which is a twin of v, and $\widetilde{G}[A] = G[A]$, then:*

1. *$c_{r,\mathcal{P}}(\widetilde{G}) \geq c_{r,\mathcal{P}}(G)$;*
2. *if G is $(r, \mathcal{P})$-extremal, then for each vertices $u, w \in S$ we must have $\vec{u}_{H,r,\mathcal{P}} = \vec{w}_{H,r,\mathcal{P}}$.*

Proof. Let S be any independent set in G, and let $H = G - S$. For each vertex $u \in S$, consider the vector $\vec{u}_{H,r,\mathcal{P}}$ as in Definition 2.1. By Proposition 2.2, the total number of $\mathcal{P}$-free r-colorings of G is

$$c_{r,\mathcal{P}}(G) = \sum_{\widehat{H}} c_{r,\mathcal{P}}(G \mid \widehat{H}) = \sum_{\widehat{H}} \prod_{u \in S} c_{r,\mathcal{P}}(u, \widehat{H}) = \left\| \prod_{u \in S} \vec{u}_{H,r,\mathcal{P}} \right\|_1 ,$$

where the sums are taken over all possible $\mathcal{P}$-free r-colorings $\widehat{H}$ of H. (For the last equality we also used that every coordinate of $\vec{u}_{H,r,\mathcal{P}}$ is non-negative).

Let v be a vertex in S for which $\|\vec{v}_{H,r,\mathcal{P}}\|_s$ is maximum. This fact, together with Hölder's Inequality (Lemma 2.3), gives us:

$$\left\|\prod_{u\in S}\vec{u}_{H,r,\mathcal{P}}\right\|_1 \leq \prod_{u\in S}\|\vec{u}_{H,r,\mathcal{P}}\|_s \leq \|\vec{v}_{H,r,\mathcal{P}}\|_s^s. \tag{1}$$

On the other hand, for the graph $\widetilde{G}$ defined in the statement of this lemma, we have:

$$c_{r,\mathcal{P}}(\widetilde{G}) = \sum_{\widehat{H}} c_{r,\mathcal{P}}(v,\widehat{H})^s = \|\vec{v}_{H,r,\mathcal{P}}\|_s^s.$$

Therefore, $c_{r,\mathcal{P}}(\widetilde{G}) \geq c_{r,\mathcal{P}}(G)$, proving part (1).

To prove part (2), assume G is extremal and $\widetilde{G}$ is as above. Since $c_{r,\mathcal{P}}(\widetilde{G}) \geq c_{r,\mathcal{P}}(G)$, we must have $c_{r,\mathcal{P}}(\widetilde{G}) = c_{r,\mathcal{P}}(G)$. Therefore, we must also have equality in both inequalities in (1). From the second one, it follows that for every vertex $u \in S$, we must have $\|\vec{u}_{H,r,\mathcal{P}}\|_s = \|\vec{v}_{H,r,\mathcal{P}}\|_s$. From the first one, where we may use the equality conditions of Lemma 2.3, the fact that $\|\vec{u}_{H,r,\mathcal{P}}\|_s = \|\vec{v}_{H,r,\mathcal{P}}\|_s$, together with the fact that all our vectors have only non-negative entries, implies that $\vec{u}_{H,r,\mathcal{P}} = \vec{v}_{H,r,\mathcal{P}}$. □

Corollary 2.6. *If G is an $(r,\mathcal{P})$-extremal graph, and $u, v \in V(G)$ are any non-adjacent vertices, then deleting v and cloning u produces a graph that is also extremal.*

Proof. Since G is extremal, by Lemma 2.5 part (2) with $S = \{u, v\}$ and $G_{uv} = G - \{u, v\}$, we must have $\vec{u}_{G_{uv},r,\mathcal{P}} = \vec{v}_{G_{uv},r,\mathcal{P}}$, therefore replacing v by a twin of u (or u by a twin of v) does not change the number of $(r,\mathcal{P})$-free colorings of the graph. □

By repeatedly applying Corollary 2.6 above, we shall show that Theorem 1.4 holds, namely that *there exists* a complete multipartite graph on n vertices that is $(r,\mathcal{P})$-extremal.

Proof of Theorem 1.4. Let G be any $(r,\mathcal{P})$-extremal graph on n vertices. We will build a sequence $G_0, G_1, \ldots, G_t$ of extremal graphs, each on n vertices, where $G_0 = G$, and $G_t = G^*$ is the desired complete multipartite graph. We do it in such a way that, for $i \geq 1$, we have $V(G_i) = S_1 \cup \cdots \cup S_i \cup R_i$, where for every $j \in \{1, \ldots, i\}$, the set S_j is an independent set and every vertex in S_j is adjacent to every vertex outside S_j (including those in R_i), but we have no control of the edges inside R_i. It will also hold that $R_t \subset R_{t-1} \subset \cdots \subset R_1 \subset V(G)$, and R_t will be independent.

To simplify the notation, we also define $R_0 = V(G_0) = V(G)$. Assume that we have constructed G_i, for some $i \geq 0$. If R_i is an independent set, we have found a complete multipartite graph which is extremal, so we can set $t = i$ and stop. Otherwise, let v_i be any vertex of R_i that has a neighbor in R_i. Note that, by the definition of G_i, all non-neighbors of v_i belong to R_i. Let $\overline{d}_i$ be the number of non-neighbors of v_i. We can obtain G_{i+1} by applying Corollary 2.6 successively $\overline{d}_i$ times, deleting each non-neighbor of v_i and adding twins of v_i (one by one). Let S_{i+1} be the set formed by v_i and its new twins and let R_{i+1} to be the set of neighbors of v_i in R_i. Observe that R_{i+1} is strictly smaller than R_i since it does not contain v_i. It is also important to notice that, when we use Corollary 2.6 in each step we apply it to the entire graph G_i and not only to $G_i[R_i]$.

To construct a graph G^* with the property that $|E(G^*)| \geq |E(G)|$, it suffices to choose always a vertex $v_i \in R_i$ with maximum degree. □

Proposition 2.7. *Let G be a graph with at least three vertices, which is not complete multipartite. Then, there exist three vertices vertices u, v, w such that $uv, uw \notin E(G)$ and $vw \in E(G)$.*

Proof. Let $V(G) = V_1 \cup \cdots \cup V_t$ be a partition of the vertex set of G, where each class is an independent set and t is minimum among all such partitions. As long as there exists a vertex $v \in V_j$ which has no neighbors in class V_i, $i < j$, we move v to V_i. Then, every vertex $v \in V_j$ has at least one neighbor in every class V_i for all $i < j$. As G is not complete multipartite, there exist vertices $u \in V_g$ and $v \in V_h$ for some $g \neq h$, which do not form an edge. As v has at least one neighbor $w \in V_g - u$ it is $uw \notin E(G)$, as V_i is an independent set. □

The following result, in combination with Proposition 2.7, implies that any $(r, \mathcal{P})$-extremal graph may also be turned into an extremal multipartite graph by removing edges.

Lemma 2.8 (Edge deletion lemma)**.** *Let $\mathcal{P}$ be a pattern family of complete graphs and let $r \geq 2$ be an integer. Let G be an $(r, \mathcal{P})$-extremal graph. For any vertices u, v, w such that $uv, uw \notin E(G)$ and $vw \in E(G)$, if we delete the edge vw, then the resulting graph is still extremal.*

Proof. Let G be a graph as in the statement. Fix vertices u, v, w such that $uv, uw \notin E(G)$ and $vw \in E(G)$.

Let $H = G - \{u, v, w\}$, and $H^x = G[V(H) \cup x]$ for $x \in \{u, v, w\}$. Let G' be the graph obtained from G by deleting the edge vw (but not the vertices u or v), and let G^* be the graph obtained from H^u by adding another two clones of u, say u_1 and u_2. By Corollary 2.6, the graph G^* is also extremal, as we may first apply the replacement operation to the pair u, v (deleting v and adding u_1) and apply it again to the pair u, w. Therefore, $c_{r,\mathcal{P}}(G) = c_{r,\mathcal{P}}(G^*)$.

Applying Proposition 2.2 to G^* with $S = \{u, u_1, u_2\}$, we have

$$c_{r,\mathcal{P}}(G^*) = \sum_{\widehat{H}} c_{r,\mathcal{P}}(G^* \mid \widehat{H}) = \sum_{\widehat{H}} c_{r,\mathcal{P}}(u, \widehat{H})^3 = \|\vec{u}_{H,r,\mathcal{P}}\|_3^3\,,$$

where the sum is taken over all $\mathcal{P}$-free r-colorings of H.

Observe that, with an analogous computation, if we start from H and add three clones of w instead of u, the resulting graph has $\|\vec{w}_{H,r,\mathcal{P}}\|_3^3$ good colorings. However, Proposition 2.2 cannot be applied, and we do not know if this graph is extremal, so we have only

$$\|\vec{w}_{H,r,\mathcal{P}}\|_3^3 \leq \|\vec{u}_{H,r,\mathcal{P}}\|_3^3\,. \tag{2}$$

On the other hand, since there are no edges from u to $\{v, w\}$, we can compute $c_{r,\mathcal{P}}(G)$ as follows:

$$\begin{aligned} c_{r,\mathcal{P}}(G) &= \sum_{\widehat{H}} \left(c_{r,\mathcal{P}}(u, \widehat{H}) \cdot c_{r,\mathcal{P}}(G - u \mid \widehat{H}) \right) \\ &= \sum_{\widehat{H}} \left(c_{r,\mathcal{P}}(u, \widehat{H}) \cdot \left(\sum_{\widehat{H^w} \mid \widehat{H}} c_{r,\mathcal{P}}(v, \widehat{H^w}) \right) \right). \end{aligned} \tag{3}$$

Here, the inner sum is taken over the good colorings of H^w that extend a given good coloring of H, that is, over the colorings of the edges from w to H, for which the resulting coloring is good. By Lemma 2.5 (2), since G is extremal and $uv \notin E(G)$, we have $\vec{v}_{H^w,r,\mathcal{P}} = \vec{u}_{H^w,r,\mathcal{P}}$, that is $c_{r,\mathcal{P}}(v, \widehat{H^w}) = c_{r,\mathcal{P}}(u, \widehat{H^w})$ for every $\widehat{H^w}$. Finally, note that $c_{r,\mathcal{P}}(u, \widehat{H^w})$ does not depend on the colors of the edges from w to H, so $c_{r,\mathcal{P}}(u, \widehat{H^w}) = c_{r,\mathcal{P}}(u, \widehat{H})$. Therefore,

$$\begin{aligned} c_{r,\mathcal{P}}(G) &= \sum_{\widehat{H}} \left(c_{r,\mathcal{P}}(u, \widehat{H}) \left(\sum_{\widehat{H^w} \mid \widehat{H}} c_{r,\mathcal{P}}(u, \widehat{H}) \right) \right) \\ &= \sum_{\widehat{H}} \left(c_{r,\mathcal{P}}(u, \widehat{H}) c_{r,\mathcal{P}}(u, \widehat{H}) \sum_{\widehat{H^w} \mid \widehat{H}} 1 \right) \\ &= \sum_{\widehat{H}} c_{r,\mathcal{P}}(u, \widehat{H})^2 c_{r,\mathcal{P}}(w, \widehat{H}) \\ &\leq \|\vec{u}_{H,r,\mathcal{P}}\|_3 \, \|\vec{u}_{H,r,\mathcal{P}}\|_3 \, \|\vec{w}_{H,r,\mathcal{P}}\|_3 & (4) \\ &\leq \|\vec{u}_{H,r,\mathcal{P}}\|_3^3 . & (5) \end{aligned}$$

Notice that to get (4) we used Hölder's Inequality (Lemma 2.3), and (5) follows from (2). Finally, since $c_{r,\mathcal{P}}(G) = \|\vec{u}_{H,r,\mathcal{P}}\|_3^3$, we must have equality in both (4) and (5), which in turn leads to $\|\vec{u}_{H,r,\mathcal{P}}\|_3 = \|\vec{w}_{H,r,\mathcal{P}}\|_3$. The equality condition in Lemma 2.3 implies that $\vec{u}_{H,r,\mathcal{P}} = \vec{w}_{H,r,\mathcal{P}}$. Analogously, $\vec{u}_{H,r,\mathcal{P}} = \vec{v}_{H,r,\mathcal{P}}$. It follows that

$$c_{r,\mathcal{P}}(G^*) = \sum_{\widehat{H}} c_{r,\mathcal{P}}(u, \widehat{H}) c_{r,\mathcal{P}}(v, \widehat{H}) c_{r,\mathcal{P}}(w, \widehat{H}) = c_{r,\mathcal{P}}(G').$$

□

Combining Theorem 1.4 and Lemma 2.8, we derive Theorem 1.5.

Proof of Theorem 1.5. Assume that G is an $(r, \mathcal{P})$-extremal graph that is not complete multipartite. By Theorem 1.4, we may produce an $(r, \mathcal{P})$-extremal complete multipartite graph G_1^* such that $|E(G_1^*)| \geq |E(G)|$. By applying Proposition 2.7 and Lemma 2.8, starting with G, we may produce an $(r, \mathcal{P})$-extremal complete multipartite graph G_2^* such that $|E(G_2^*)| < |E(G)|$. □

3. Extremal configurations for the pattern family $\mathcal{P}_{k,s}$

Recall that $\mathcal{P}_{k,s}$ is the pattern family containing all patterns P of K_k such that $\gamma(P) \geq s$. In this section, we will prove Theorem 1.6, which states that *every* n-vertex $(r, \mathcal{P}_{k,s})$-extremal graph is complete multipartite.

Proof of Theorem 1.6. Let $n, r \geq 2$ and $k \geq 3$ be integers, and fix an integer s such that $k \leq s \leq \binom{k}{2}$. Let G be an n-vertex $(r, \mathcal{P}_{k,s})$-extremal graph and assume for a contradiction that G is not a complete multipartite graph. By Proposition 2.7, there exist vertices u, v, w such that $uv, uw \notin E(G)$ and $vw \in E(G)$.

Let $H = G - \{u, v, w\}$, and $H^x = G[V(H) \cup x]$ for $x \in \{u, v, w\}$. At the end of the proof of Lemma 2.8, we concluded that $\vec{u}_{H,r,\mathcal{P}_{k,s}} = \vec{w}_{H,r,\mathcal{P}_{k,s}} = \vec{v}_{H,r,\mathcal{P}_{k,s}}$, so for every coloring $\widehat{H}$ of H we have $c_{r,\mathcal{P}_{k,s}}(u, \widehat{H}) = c_{r,\mathcal{P}_{k,s}}(w, \widehat{H}) = c_{r,\mathcal{P}_{k,s}}(v, \widehat{H})$. We also observed that, for every extension of $\widehat{H}$ to a coloring $\widehat{H^w}$, we must have $c_{r,\mathcal{P}_{k,s}}(u, \widehat{H^w}) = c_{r,\mathcal{P}_{k,s}}(u, \widehat{H})$. Finally, since u and v are not adjacent, by Lemma 2.5, we have $\vec{u}_{H^w,r,\mathcal{P}} = \vec{v}_{H^w,r,\mathcal{P}}$, that is, $c_{r,\mathcal{P}_{k,s}}(u, \widehat{H^w}) = c_{r,\mathcal{P}_{k,s}}(v, \widehat{H^w})$ for every good coloring $\widehat{H^w}$ of H^w. It follows that, for every $\mathcal{P}_{k,s}$-free extension $\widehat{H^w}$ of $\widehat{H}$, we must have

$$c_{r,\mathcal{P}_{k,s}}(v, \widehat{H^w}) = c_{r,\mathcal{P}_{k,s}}(v, \widehat{H}). \tag{6}$$

We will get a contradiction from this fact (which implies that such G cannot exist). We only need to find an r-coloring of $\widehat{H}$ and an extension of it to H^w, which is $\mathcal{P}_{k,s}$-free and such that equation (6) does not hold. Since $s \geq k \geq 3$, the monochromatic coloring $\widehat{H}$ of H such that all edges are blue is a good coloring, which can be extended to a fully blue coloring $\widehat{H^w}$ of H^w. Let $\mathcal{H}(v)$ and $\mathcal{H}^w(v)$ be the classes of all good colorings that extend $\widehat{H}$ to H^v and $\widehat{H^w}$ to $G-u$, respectively. We show that there is an injective mapping $\phi\colon \mathcal{H}(v) \to \mathcal{H}^w(v)$ that is not surjective.

Given a coloring $\widehat{H^v} \in \mathcal{H}(v)$, let $\phi(\widehat{H^v})$ be the coloring of $G - u$ that extends $\widehat{H^w}$ by assigning to any edge e between v and $V(H)$ the color of e in $\widehat{H^v}$ and by assigning the color blue to $\{v, w\}$. The function ϕ is clearly injective. We claim that $\phi(\mathcal{H}(v)) \subseteq \mathcal{H}^w(v)$. To see why this is true, suppose that $\phi(\mathcal{H}(v))$ contains a copy $\widehat{K_k}$ of K_k whose pattern lies in $\mathcal{P}_{k,s}$. Clearly, this copy involves v and w, otherwise it would also occur in $\widehat{H^v}$, a contradiction. Moreover, the only edges in this copy that are not necessarily blue are edges $\{v, x\}$, where $x \in V(H)$. The number of such edges is $k-2$, so that at most $k-1$ colors appear in $\widehat{K_k}$, contradicting the hypothesis that $s \geq k$ and establishing our claim.

On the other hand, the coloring of $G-u$ where $\{v, w\}$ is red and all other edges are blue is clearly in $\mathcal{H}^w(v)$, as $s \geq 3$. However, it does not lie in $\phi(\mathcal{H}(v))$, as ϕ always assigns color blue to $\{v, w\}$. So ϕ is not surjective and we have reached the desired contradiction. □

Another class of families is the following: given a pattern P of K_k, we say that a vertex v in K_k is *bicolored* if all edges incident with v lie in at most two classes of P. Let $\mathcal{P}_2$ be the pattern family containing all patterns of the form (K_k, P) such that $k = 3$ and P is the rainbow pattern, or such that $k \geq 4$ and P contains at least two vertices that are not bicolored.

Theorem 3.1. *Let $n, r \geq 2$ and let $\mathcal{P} \subseteq \mathcal{P}_2$. If G is an n-vertex $(r, \mathcal{P})$-extremal graph, then G is complete multipartite.*

Proof. The proof follows the ideas above and uses the same blue colorings $\widehat{H}$ and $\widehat{H^w}$, with the corresponding classes $\mathcal{H}(v)$ and $\mathcal{H}^w(v)$. When defining $\phi(\widehat{H^v})$, we proceed as before. We argue that this cannot create a forbidden pattern. If it did, the forbidden pattern (K_k, P) would involve v, w and $k-2$ vertices $v_1, \ldots, v_{k-2}$ of $V(H)$. If $k = 3$, this pattern cannot be (K_3, P_R), where P_R is the rainbow pattern, as both $\{v, w\}$ and $\{v_1, w\}$ are blue. If $k \geq 4$, all vertices in the copy are bicolored, with the possible exception of v, contradicting the fact that P must contain at least two vertices that are not bicolored. On the other hand, the coloring of $G - u$ where $\{v, w\}$ is red and all other edges are blue is clearly in $\mathcal{H}^w(v)$, so that the function is not surjective. $\square$

Remark 3.2. As in this last example, in the statement of Theorem 1.6 we could have defined the family $\mathcal{P}^*$ of all patterns of the form (K_k, P), where $k \geq 3$ and $\gamma(P) \geq k$. The proof would hold for any $\mathcal{P} \subseteq \mathcal{P}^{*\dagger}$.

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†Actually, for any $\mathcal{P} \subseteq \mathcal{P}^* \cup \mathcal{P}_2$.

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