A Deterministic Polynomial Time Algorithm for Heilbronn's Problem in Dimension Three*

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Abstract

Heilbronn conjectured that among arbitrary n points in the 2-dimensional unit square $[0,1]^2$, there must be three points which form a triangle of area at most $O(1/n^2)$. This conjecture was disproved by a nonconstructive argument of Komlós, Pintz and Szemerédi [14] who showed that for every n there is a configuration of n points in the unit square $[0,1]^2$ where all triangles have area at least $\Omega(\log n/n^2)$. Here we will consider a 3-dimensional analogue of this problem and we will give a deterministic polynomial time algorithm which finds n points in the unit cube $[0,1]^3$ such that the volume of every tetrahedron among these n points is at least $\Omega(\ln n/n^3)$.

1 Introduction

An old conjecture of Heilbronn states that for every distribution of n points in the 2-dimensional unit square $[0,1]^2$ (or unit disc) there exist three distinct points which form a triangle of area at most c/n^2 for some fixed constant $c > 0$. Erdös observed that this conjecture, if true, would be best possible, as the points $(i \bmod n, i^2 \bmod n)_{i=0,\ldots,n-1}$ on the moment-curve in the $n \times n$ grid would show after rescaling, cf. [2]. However, Komlós, Pintz and Szemerédi [14] disproved Heilbronn's conjecture by proving that for every n there exists a configuration of n points in the unit square $[0,1]^2$ with every three points forming a triangle of area at least $c' \cdot \log n/n^2$, where $c' > 0$ is constant. Using techniques from derandomization, this existence argument was made constructive in [5], where a polynomial time algorithm was given, which finds n points in $[0,1]^2$ achieving the lower bound $\Omega(\log n/n^2)$ on the minimum triangle area. Upper bounds on Heilbronn's triangle problem were given by Roth [17], [18], [19], [20], [21] and Schmidt [23] in a series of papers, cf. Rothschild and Straus [22] for related results, and the currently best upper bound is due to Komlós, Pintz and Szemerédi [13] and is of the order $n^{-8/7+\varepsilon}$ for every fixed $\varepsilon > 0$. On the other hand, using arguments from Kolmogorov complexity, recently Jiang, Li and Vitány [11] proved that if n points are dropped uniformly at random and independently of each other in the unit square $[0,1]^2$, then the expected value of the smallest area of a triangle among these n points is $\Theta(1/n^3)$.
Also recently, Barequet [3] considered a k-dimensional version of Heilbronn's problem. For a subset $S = \{p_0, \ldots, p_k\} \subset \mathbb{R}^k$ of $(k+1)$ points, the set $S^* = \{p_0 + \sum_{i=1}^{k} \lambda_i \cdot (p_i - p_0) \mid \sum_{i=1}^{k} \lambda_i \leq 1; \lambda_1, \ldots, \lambda_k \in [0,1]\}$ is called a *simplex*. If $k = 3$, then S^* is called a *tetrahedron*. The *volume* of the simplex $S^* \subset \mathbb{R}^k$ is defined by $\mathrm{vol}(S^*) := 1/k \cdot h \cdot \mathrm{vol}(S')$,

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where h is the distance of p_k to the affine space generated by $p_0, \ldots, p_{k-1}$ and S' is in this space the simplex generated by $p_0, \ldots, p_{k-1}$.
For given dimension $k \geq 3$ Barequet showed, that for every n there exist n points in the k-dimensional unit cube $[0,1]^k$ such that the minimum volume of every simplex spanned by any $(k+1)$ of these n points is at least $\Omega(1/n^k)$. Barequet gave three different approaches for proving his lower bound. The first one, for dimension $k = 3$, uses a Greedy-type argument (cf. also [23] for the case $k = 2$) and he obtained a configuration of n points in the 3-dimensional unit cube $[0,1]^3$ such that the minimum volume of every tetrahedron is at least $\Omega(1/n^4)$. The second approach yields a better lower bound and was worked out for every fixed dimension $k \geq 3$ and uses a random argument: $2n$ points are dropped uniformly at random and independently of each other in the k-dimensional unit cube $[0,1]^k$. The expected number of $(k+1)$-point simplices with volume at most $B := c_k/n^k$ is at most n for some constant $c_k > 0$. Deleting one point from every such small simplex with volume at most B yields the existence of n points in $[0,1]^k$ with every $(k+1)$-point simplex having minimum volume at least $\Omega(1/n^k)$. The third approach however is similar to Erdös' construction, namely taking the points $P_l = (l^j \bmod n/n)_{j=1,\ldots,k}$ for $l = 0, 1, \ldots, n-1$ on the moment-curve. The volume of any $(k+1)$-point simplex is given by a Vandermonde determinant rescaled by the factor $\Theta(1/n^k)$ and this gives a minimum value for the volume of any $(k+1)$ points of these n points on the moment-curve of at least $\Omega(1/n^k)$.
The corresponding problem for dimension $k = 1$ is trivial as there are always n points in the unit interval $[0,1]$ with minimum distance between two distinct points at least $\Omega(1/n)$ and this bound cannot be improved.
In [15] Barequet's lower bound was improved by a factor $\Theta(\ln n)$ for dimensions $k \geq 3$, using a probabilistic existence argument based on a variant of Theorem 1.2. For the proof the continuous structure of the unit cube $[0,1]^k$ was crucial.

Theorem 1.1. [15] *For every fixed integer $k \geq 2$ and for every n there exists a configuration of n points in the k-dimensional unit cube $[0,1]^k$ such that the volume of any simplex spanned by any $(k+1)$ points is at least $\Omega(\ln n/n^k)$.*

Here we will give for dimension $k = 3$ a deterministic polynomial time algorithm for the result in Theorem 1.1:

Theorem 1.2. *For every positive integer n one can find deterministically in polynomial time a configuration of n points in the unit cube $[0,1]^3$ such that the volume of any tetrahedron spanned by any four of these points is at least $\Omega(\ln n/n^3)$.*

The proof of Theorem 1.2 is based on techniques from combinatorics and algebraic number theory. Some of our arguments are given for the case of arbitrary dimension $k \geq 3$, where appropriate. However, so far we are only able to provide a deterministic polynomial time algorithm for the case $k = 3$.

2 Hypergraphs

In our arguments we will use hypergraphs. It will turn out that the notions *independence number of a hypergraph* and 2-*cycles* are important in our considerations:

Definition 2.1. *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph where each edge $E \in \mathcal{E}$ satisfies $E \subseteq V$. The hypergraph $\mathcal{G}$ is k-*uniform *if every edge $E \in \mathcal{E}$ contains exactly k vertices.*
A subset $I \subseteq V$ is called independent *if I contains no edge $E \in \mathcal{E}$. The largest size of an independent set in $\mathcal{G}$ is called the* independence number $\alpha(\mathcal{G})$.

In a k-uniform hypergraph $\mathcal{G} = (V, \mathcal{E})$, $k \geq 3$, a 2-cycle *is a pair $\{E_1, E_2\}$ of distinct edges $E_1, E_2 \in \mathcal{E}$ with $|E_1 \cap E_2| \geq 2$. A 2-cycle $\{E_1, E_2\}$ in $\mathcal{G}$ is called* $(2, j)$-cycle *if $|E_1 \cap E_2| = j$, where $j = 2, \ldots, k-1$.*

Let $B_k(T) = \{x \in \mathbb{R}^k | \, \|x\| \leq T\} \subset \mathbb{R}^k$ be the k-dimensional ball around the origin with radius T. We will reformulate our combinatorial, geometrical problem as a problem of finding in a suitably defined hypergraph a large independent set. To do so, we will discretize the 3-dimensional search space $[0,1]^3$, namely we will consider only points from the set $B_3(T) \cap \mathbb{Z}^3$, where T will be of suitable size, i.e. polynomial in n. With this discretization, we also have to take care of simplices (tetrahedra) of volume 0, these are degenerated tetrahedra.
For some parameter $B > 0$ and for the given set of grid points in $B_3(T) \cap \mathbb{Z}^3$ we form a 4-uniform hypergraph $\mathcal{G} = \mathcal{G}(B) = (V, \mathcal{E})$ with the vertex set V being this set $B_3(T) \cap \mathbb{Z}^3$ of $\Theta(T^3)$ grid-points. The edges are determined by all subsets of four points from the set $B_3(T) \cap \mathbb{Z}^3$, which form a tetrahedron of volume at most B, where later we will set $B := T^3 \cdot \ln n / n^3$. Then an independent set in this hypergraph $\mathcal{G}(B)$ corresponds to a subset of points in the set $B_3(T) \cap \mathbb{Z}^3$, where no tetrahedron has 'small' volume, i.e. all tetrahedra have volume bigger than B, which after rescaling yields the desired result. In order to show the existence of a large independent set, we will use the following result due to Ajtai, Komlós, Pintz, Spencer and Szemerédi [1], compare [9], stated here in an algorithmic variant proven in [4] see [10]:

Theorem 2.2. [1],[4],[9],[10] *Let $k \geq 3$ be a fixed integer. Let $\mathcal{G} = (V, \mathcal{E})$ be a k-uniform hypergraph on $|V| = n$ vertices and with average degree $t^{k-1} := k \cdot |\mathcal{E}|/|V|$. If for some constant $\gamma > 0$ the hypergraph $\mathcal{G}$ contains at most*

$$n \cdot t^{2k-j-1-\gamma}$$

(2,j)-cycles for $j = 2, \ldots, k-1$, then one can find in $\mathcal{G}$ in polynomial time an independent set of size at least

$$\Omega\left(\frac{n}{t} \cdot (\ln t)^{1/(k-1)}\right) . \tag{1}$$

If the parameter t^{k-1} is an upper bound on the average degree of the hypergraph $\mathcal{G}$, then (1) holds too.

In recent years, several applications of Theorem 2.2 have been found, compare [4]. Here we will give another application of this deep result.
A main part of the proof of Theorem 1.2 consists of counting the degenerated resp. non-degenerated tetrahedra in $\mathbb{Z}^3 \cap B_3(T)$. First we will recall and explain some tools, which we will use in our arguments.

3 Grids in $\mathbb{Z}^k$

We will use some results from linear algebra and number theory, which will be stated in the following.

3.1 Grids

Definition 3.1. *A* grid *L of $\mathbb{Z}^k$ is a subset of $\mathbb{Z}^k$, which is generated by all linear combinations of some linearly independent vectors $q_1, \ldots, q_m \in \mathbb{Z}^k$, where all coefficients are integers, i.e. $L = \mathbb{Z}q_1 + \cdots + \mathbb{Z}q_m$.*

The parameter m *is called the* rank *of the grid* L *and the set* $Q = \{q_1, \ldots, q_m\}$ *is called the* basis *of* L.

Definition 3.2. *Let* $Q = \{q_1, \ldots, q_m\} \subset \mathbb{Z}^k$ *be a set of linear independent vectors.*

i) The $k \times m$ generator matrix *of* Q *(up to the ordering of the vectors) is defined by*

$$G(Q) := (q_1, \ldots, q_m)_{k\times m}.$$

ii) The fundamental parallelopiped F_Q *of* Q *is the following set*

$$F_Q := \Big\{ \sum_{i=1}^{m} \alpha_i \cdot q_i \mid 0 \leq \alpha_i \leq 1,\, i = 1, \ldots, m \Big\} \subseteq \mathbb{R}^k \ .$$

The extreme points *of the fundamental parallelopiped* F_Q *are all the points* $\sum_{i=1}^{m} \alpha_i \cdot q_i$ *with* $\alpha_1, \ldots, \alpha_m \in \{0, 1\}$.

iii) The volume *of the fundamental parallelopiped* $F_Q \subseteq \mathbb{R}^k$ *of* Q *is given by*

$$vol(F_Q) := \Big(\det(G(Q)^\top \cdot G(Q))\Big)^{1/2} ,$$

where $G(q)^\top$ *is the transpose of the generator matrix* $G(Q)$.

The following result can be found in [7].

Lemma 3.3. [7] *Let* Q *and* Q' *be two basis of a grid* L *in* $\mathbb{Z}^k$. *Then the volumes of the corresponding fundamental parallelopipeds are equal, i.e.* $vol(F_Q) = vol(F_{Q'})$. *The parameter* $d(L) := vol(F_Q)$ *is called the* determinant *of the grid* L.

For integers $a_1, \ldots, a_n \in \mathbb{Z}$, which are not all equal to 0, we denote by $\gcd(a_1, \ldots, a_n)$ the *greatest common divisor* of $a_1, \ldots, a_n$. From elementary number theory we recall the well-known Lemma of Bezout:

Lemma 3.4. *Let* $a_1, \ldots, a_k \in \mathbb{Z}$ *be integers, which are not all equal to* 0. *Then there exist integers* $y_1, \ldots, y_k \in \mathbb{Z}$ *such that*

$$a_1 \cdot y_1 + \cdots + a_k \cdot y_k = \gcd(a_1, \ldots, a_k) \ .$$

The next result, see [6] or [8], gives a recursive procedure to find a basis of a grid in $\mathbb{Z}^k$.

Lemma 3.5. [6],[8] *Let* $a = (a_1, \ldots, a_k)^\top \in (\mathbb{Z} \setminus \{0\})^k$ *be a sequence of nonzero integers with* $\gcd(a_1, \ldots, a_k) = 1$. *Then the set* L *of all solutions in* $\mathbb{Z}^k$ *of the equation*

$$a_1 \cdot X_1 + \cdots + a_k \cdot X_k = 0$$

is a grid in $\mathbb{Z}^k$ *with rank* $(L) = k - 1$.
The following recursive procedure yields a basis Q *of the grid* L.
For $k = 1$ *let* $Q = \{0\}$. *For* $k = 2$ *let* $Q = \{(a_2, -a_1)^\top\}$.
For $k \geq 3$ *let* $d = \gcd(a_{k-1}, a_k) = a_{k-1} \cdot y_0 + a_k \cdot y_1$ *with* $y_0, y_1 \in \mathbb{Z}$. *If* $q_1^*, \ldots, q_{k-2}^*$ *is a basis of the set of all solutions of the following equation in* $k - 1$ *variables*

$$a_1 \cdot X_1 + \cdots + a_{k-2} \cdot X_{k-2} + d \cdot Y = 0 \ ,$$

then the vectors $q_1, \ldots, q_{k-1}$ *form a basis of* L *where*

$$\begin{aligned} q_i &= (q_{i,1}^*, \ldots, q_{i,k-2}^*, y_0 \cdot q_{i,k-1}^*, y_1 \cdot q_{i,k-1}^*)^\top \qquad \textit{for } i = 1, \ldots, k-2, \\ q_{k-1} &= \big(0, \ldots, 0, a_k/d, -a_{k-1}/d\big)^\top. \end{aligned}$$

Proof. The proof that $q_1, \ldots, q_{k-1}$ is a basis can be found in [6] or [8]. An algorithm for determining $q_1, \ldots, q_{k-1}$ is based on the Euclidean algorithm. $\square$

We use the standard scalar product $< a, b >:= \sum_{i=1}^{k} a_i \cdot b_i$ for vectors $a = (a_1, \ldots, a_k)^\top \in \mathbb{R}^k$ and $b = (b_1, \ldots, b_k)^\top \in \mathbb{R}^k$. The Euclidean distance $\mathrm{dist}(a, b)$ of the corresponding points is defined by $\mathrm{dist}(a, b) := (\sum_{i=1}^{k} (a_i - b_i)^2)^{1/2}$. The length of a vector $a \in \mathbb{R}^k$ is defined by $\|a\| := \sqrt{< a, a >}$. For a point $p \in \mathbb{R}^k$ and a real linear subspace $V \subseteq R^k$ let $\mathrm{dist}(p, V) := \min\{\mathrm{dist}(p, v) \mid v \in V\}$. For vectors $q_1, \ldots, q_m \in \mathbb{R}^k$ let $\mathrm{span}(q_1, \ldots, q_m)$ be the linear space over the reals, generated by $q_1, \ldots, q_m$.
The following results can be found in [12]:

Lemma 3.6. **[12]** *Let V be a $(k-1)$-dimensional linear subspace of $\mathbb{R}^k$ and let $a \in \mathbb{R}^k \setminus \{0\}$ be a nonzero vector which is orthogonal to V. The distance $dist(p, V)$ of every point $p \in \mathbb{R}^k$ to the subspace V is given by*

$$dist(p, V) = \frac{| < p^\top, a > |}{\|a\|} .$$

Lemma 3.7. **[12]** *Let $q_1, \ldots, q_m \in \mathbb{R}^k$ be linearly independent vectors. Then, with $U := span(q_1, \ldots, q_{m-1})$, the volume of the fundamental parallelopiped $F_{\{q_1,\ldots,q_m\}}$ satisfies*

$$vol(F_{\{q_1,\ldots,q_m\}}) = dist(q_m, U) \cdot vol(F_{\{q_1,\ldots,q_{m-1}\}}) .$$

Lemma 3.8. **[7]** *Let U and L be grids in $\mathbb{Z}^k$ with $U \subseteq L$ and rank $(L) =$ rank $(U) = m$. Then the following holds:*

i) There exists a positive integer $\lambda \in \mathbb{N} \setminus \{0\}$ such that $\lambda \cdot L = \{\lambda \cdot x \mid x \in L\} \subseteq U$.

ii) For every basis $b_1, \ldots, b_m$ of L there is a basis $a_1, \ldots, a_m$ of U of the following form

$$\begin{aligned} a_1 &= v_{1,1} \cdot b_1 \\ a_2 &= v_{2,1} \cdot b_1 + v_{2,2} \cdot b_2 \\ &\vdots \\ a_m &= v_{m,1} \cdot b_1 + \cdots + v_{m,m} \cdot b_m \end{aligned} \tag{2}$$

with $v_{j,i} \in \mathbb{Z}$ and $v_{i,i} \neq 0$ for $1 \leq i, j \leq m$.

iii) For each basis $a_1, \ldots, a_m$ of U there is a basis $b_1, \ldots, b_m$ of L, such that (2) is fulfilled.

Lemma 3.8 can be made constructive in polynomial time, for example by using for (ii) a variant of the LLL-algorithm, see [8],
In our arguments we will only use part (iii) of Lemma 3.8. However, for the proof of part (iii) parts (i) and (ii) are used.

Proof. Our arguments are similar to those in [7]. To keep this paper selfcontained we include the proof of Lemma 3.8.

(i) Let $A = \{a_1, \ldots, a_m\}$ be a basis of U and let $B = \{b_1, \ldots, b_m\}$ be a basis of L. Then there exists a matrix $C \in \mathbb{Z}^{m \times m}$ such that the generator-matrices of A and B satisfy $G(A) = G(B) \cdot C$. Since the matrices $G(A)$ and $G(B)$ both have rank equal to m, the matrix C is regular, hence the inverse C^{-1} of C exists and all entries of C^{-1} are rational numbers. Thus, we can find a positive integer $\lambda \in \mathbb{N}\backslash\{0\}$ such that the matrix $V := \lambda \cdot C^{-1}$ has only integers as entries. With

$$\lambda \cdot G(B) = \lambda \cdot G(A) \cdot C^{-1} = G(A) \cdot V,$$

we have $\lambda \cdot b_i = \sum_j v_{j,i} \cdot a_j \in U$.

(ii) The arguments are similar to those in [7], Theorem I, part A, p. 11-13. Let $b_1, \ldots, b_m \in \mathbb{Z}^k$ be a basis of L. To obtain the desired basis $a_1, \ldots, a_m \in U$, let

$$a_i := v_{i,1} \cdot b_1 + \cdots + v_{i,i} \cdot b_i$$

with $a_i \in U$ and $v_{i,1}, \ldots, v_{i,i} \in \mathbb{Z}$ and $v_{i,i} \neq 0$, where $|v_{i,i}| \neq 0$ is as small as possible among all such choices. Such vectors a_i exist, since by part (i) we have $\lambda \cdot b_i \in U$ for $i = 1, \ldots, m$. We show next that $a_1, \ldots, a_m$ is a basis of U. We have

$$w_1 \cdot a_1 + \cdots + w_m \cdot a_m \in U \tag{3}$$

for every choice of $w_1, \ldots, w_m \in \mathbb{Z}$. Suppose for contradiction, that there is a vector $c \in U$ which is not of the form (3), then since $c \in L$ there are integers $t_1, \ldots, t_k \in \mathbb{Z}$ with $t_k \neq 0$ such that $c = t_1 \cdot b_1 + \cdots + t_k \cdot b_k$, where k is as least as possible among all such choices of c. Since $v_{k,k} \neq 0$, we can choose an integer $s \in \mathbb{Z}$ with

$$|t_k - s \cdot v_{k,k}| < |v_{k,k}| \,. \tag{4}$$

However, since $a_k \in U$, we have $c - s \cdot a_k \in U$ with

$$c - s \cdot a_k = (t_1 - s \cdot v_{1,1}) \cdot b_1 + \cdots + (t_k - s \cdot v_{k,k}) \cdot b_k.$$

But now we have $t_k - s \cdot v_{k,k} \neq 0$ by minimality of k, which contradicts the minimality of $|v_{k,k}|$, thus every vector in U is of the form (3) and $a_1, \ldots, a_m$ is a basis of U.

(iii) The arguments are similar to those in [7], Theorem I, part B, p. 11-13.
Consider the subgrid $L^* := \lambda \cdot L \subseteq U$ of L from part (i). Let $a_1, \ldots, a_m$ be a basis of U. By part (ii) we can find a basis $\lambda \cdot b_1, \ldots, \lambda \cdot b_m$ of L^* where

$$\begin{aligned} \lambda \cdot b_1 &= w_{1,1} \cdot a_1 \\ \lambda \cdot b_2 &= w_{2,1} \cdot a_1 + w_{2,2} \cdot a_2 \\ &\vdots \\ \lambda \cdot b_m &= w_{m,1} \cdot a_1 + \cdots + w_{m,m} \cdot a_m \end{aligned} \tag{5}$$

with integers $w_{j,i}$ and $w_{i,i} \neq 0$ for $1 \leq i, j \leq m$. Solving the system (5) of equations for $a_1, \ldots, a_m$, we obtain a system like in (2) with rational coefficients $v_{j,i}$. Since $b_1, \ldots, b_m$ is a basis of L and $a_1, \ldots, a_m \in L$, the numbers $v_{j,i}$ are indeed integers. $\square$

Crucial in our arguments is the following result:

Lemma 3.9. *Let $k \in \mathbb{N}$ be fixed. Let L be a grid in $\mathbb{Z}^k$ with rank $(L) = m$ and let $a_1, \ldots, a_m \in L$ be linearly independent. Then there exists a basis $b_1, \ldots, b_m$ of L with $\|b_i\| = O(\max_j \|a_j\|)$ for $i = 1, \ldots, m$.*

Proof. The arguments are similar to those in [7], Lemma 8, p. 135-136. For completeness we include the proof.
By Lemma 3.8 (iii) we can find a basis $c_1, \ldots, c_m \in \mathbb{Z}^k$ of L such that

$$\begin{aligned} a_1 &= v_{1,1} \cdot c_1 \\ a_2 &= v_{2,1} \cdot c_1 + v_{2,2} \cdot c_2 \\ &\vdots \\ a_m &= v_{m,1} \cdot c_1 + \cdots + v_{m,m} \cdot c_m \end{aligned} \tag{6}$$

with $v_{j,i} \in \mathbb{Z}$ and $v_{i,i} \neq 0$ for $1 \leq i, j \leq m$. With this we will construct the desired basis $b_1, \ldots, b_m \in \mathbb{Z}^k$ of L with $\|b_j\| = O(\max_i \|a_i\|)$ for $j = 1, \ldots, m$.
For $j = 1, \ldots, m$ we proceed as follows. If $v_{j,j} \in \{+1, -1\}$, then we set $b_j := v_{j,j} \cdot a_j$, hence $\|b_j\| = \|a_j\|$. Otherwise, for $|v_{j,j}| \geq 2$, we solve the system of equations (6) successively for $c_1, \ldots, c_m$ and we obtain

$$c_j = 1/v_{j,j} \cdot a_j + l_{j,j-1} \cdot a_{j-1} + \cdots + l_{j,1} \cdot a_1$$

for $j = 1, \ldots, m$ with rational numbers $l_{j,j-1}, \ldots, l_{j,1} \in \mathbb{Q}$. For $i < j$ choose integers $t_{j,i} \in \mathbb{Z}$ such that

$$|t_{j,i} + l_{j,i}| \leq \frac{1}{2} .$$

Then, we set $k_{j,i} := t_{j,i} + l_{j,i}$ for $i < j$ and $k_{j,j} := 1/v_{j,j}$ and fix b_j by

$$\begin{aligned} b_j &:= k_{j,j} \cdot a_j + k_{j,j-1} \cdot a_{j-1} + \cdots + k_{j,1} \cdot a_1 \\ &= c_j + t_{j,j-1} \cdot a_{j-1} + \cdots + t_{j,1} \cdot a_1 , \end{aligned}$$

hence $b_j \in L$ and $b_1, \ldots, b_m$ is a basis of L. By construction we have $|k_{j,i}| \leq 1/2$ for $i \leq j$, hence

$$\|b_j\| \leq \|k_{j,j} \cdot a_j\| + \|k_{j,j-1} \cdot a_{j-1}\| + \cdots + \|k_{j,1} \cdot a_1\| \leq k/2 \cdot \max_i \|a_i\| .$$

□

3.2 Maximal Grids in $\mathbb{Z}^k$

In our arguments we will use the notion of *maximal* grids.

Definition 3.10. *A grid L in $\mathbb{Z}^k$ is called* m-maximal*, if rank $(L) = m$ and for every grid L' in $\mathbb{Z}^k$ with rank $(L') = m$ and $L \subseteq L'$ it is $L = L'$.*

A nonzero integer-valued vector $a = (a_1, \ldots, a_k)^\top \in \mathbb{Z}^k \setminus \{0^k\}$ is called *primitive*, if $\gcd(a_1, \ldots, a_k) = 1$ and $a_j > 0$ for $j = \min\{i \mid a_i \neq 0\}$.
For a subset $A = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}^k$ of vectors let $A^\perp = \{a \in \mathbb{R}^k \mid\ < a, x_1 >= \cdots < a, x_m >= 0\}$ be the *dual of A*.
Mainly we will deal here with $(k-1)$-maximal grids in $\mathbb{Z}^k$.

Lemma 3.11. *Let $a = (a_1, \ldots, a_k)^\top \in \mathbb{Z}^k \setminus \{0^k\}$ be an integer-valued vector, where not all entries are equal to 0. Then the set $L_a = (\mathbb{R} \cdot a)^\perp \cap \mathbb{Z}^k$ of all solutions of the equation $a_1 \cdot X_1 + \cdots + a_k \cdot X_k = 0$ over $\mathbb{Z}^k$ is a $(k-1)$-maximal grid in $\mathbb{Z}^k$.*

Proof. By linearity we can assume that $\gcd(a_1, \ldots, a_k) = 1$, where w.l.o.g. $a_i \neq 0$ for $i = 1, \ldots, r$ and $a_i = 0$ for $i > r$. By Lemma 3.5 the set of all solutions of the equation $a_1 \cdot X_1 + \cdots + a_r \cdot X_r = 0$ over $\mathbb{Z}^r$ is a grid L^* in $\mathbb{Z}^r$ with rank $(L^*) = r - 1$. Then the set of all solutions of the equation $a_1 \cdot X_1 + \cdots + a_k \cdot X_k = 0$ over $\mathbb{Z}^k$ is the grid $L = L^* \times \mathbb{Z}^{k-r}$ in $\mathbb{Z}^k$ with rank $(L) = k - 1$.
To see the maximality of the grid L, let L' be a grid in $\mathbb{Z}^k$ with rank $(L') = k - 1$ and $L \subseteq L'$. The set $V \subseteq \mathbb{R}^k$ of all solutions of the equation $a_1 \cdot X_1 + \cdots + a_k \cdot X_k = 0$ over $\mathbb{R}^k$ is a $(k-1)$-dimensional linear subspace of $\mathbb{R}^k$. Then we have $L' \subseteq \mathrm{span}(L') = \mathrm{span}(L) = V$, hence each vector in L' is a solution of the equation $a_1 \cdot X_1 + \cdots + a_k \cdot X_k = 0$ and thus $L' \subseteq L$. Therefore, L is a $(k-1)$-maximal grid in $\mathbb{Z}^k$. □

Lemma 3.12. *For every grid L in $\mathbb{Z}^k$ with rank $(L) = k-1$ there is exactly one primitive vector $a_L = (a_1, \ldots, a_k)^\top \in \mathbb{Z}^k$ with $a_L \perp L$, i.e. $< a_L, x >= 0$ for each $x \in L$.*

Proof. Let $Q \subseteq \mathbb{Z}^k$ be a basis of L and let $G(Q)$ be its generator matrix. The system of linear equations

$$G(Q)^\top \cdot (X_1, \ldots, X_k)^\top \quad = \quad 0 \tag{7}$$

has a nontrivial solution $(a_1, \ldots, a_k) \in \mathbb{Z}^k \setminus \{0^k\}$ and every solution X satisfies $X \perp L$. Dividing each entry of $(a_1, \ldots, a_k)$ by $\gcd(a_1, \ldots, a_k)$ and possibly multiplying the resulting vector by -1 we obtain a primitive vector $a_L = (a'_1, \ldots, a'_k)^\top \in \mathbb{Z}^k \setminus \{0^k\}$, which is a solution of (7). Since the rank of the matrix $G(Q)$ is equal to $k-1$, the space of solutions of the system (7) in $\mathbb{R}^k$ has dimension 1, and therefore this vector a_L is unique. □

Corollary 3.13. *(i) For every grid L' in $\mathbb{Z}^k$ with rank $(L) = k-1$ there is exactly one $(k-1)$-maximal grid L in $\mathbb{Z}^k$ with $L' \subseteq L$.*

(ii) There is a bijective mapping between the set of all $(k-1)$-maximal grids L in $\mathbb{Z}^k$ and the set of all primitive vectors a_L in $\mathbb{Z}^k$, i.e. $a_L \in \mathbb{Z}^k$ is the unique primitive normal vector of the grid L.

Definition 3.14. *Let L be a $(k-1)$-maximal grid in $\mathbb{Z}^k$ with primitive normal vector $a_L \in \mathbb{Z}^k$. A* residue class *of L is a set of the following form $L' = x + L$ with $x \in \mathbb{Z}^k$.*

Lemma 3.15. *Let L be a $(k-1)$-maximal grid in $\mathbb{Z}^k$ with primitive normal vector $a_L \in \mathbb{Z}^k$. Then there exists a vector $v \in \mathbb{Z}^k \setminus L$, such that $\mathbb{Z}^k$ can be partitioned into the* residue classes *$s \cdot v + L$, $s \in \mathbb{Z}$, i.e.,*

$$\mathbb{Z}^k = \biguplus_{s \in \mathbb{Z}} (s \cdot v + L) .$$

Moreover, for each point $x \in L$ it is

$$dist(s \cdot v + x, span(L)) = \frac{|s|}{\|a_L\|} .$$

Proof. For every vector $x \in L$ we have $< x, a_L >= 0$ and for each vector $v \in \mathbb{Z}^k \setminus L$ it is $< v, a_L > \in \mathbb{Z} \setminus \{0\}$. As a_L is primitive, the greatest common divisor of its entries is equal to 1, and by Lemma 3.4 there exists a vector $v \in \mathbb{Z}^k \setminus L$ such that $< v, a_L >= 1$.
Using $a_L \perp L$ and Lemma 3.6 we infer for each integer $s \in \mathbb{Z}$ and each vector $x \in L$:

$$\mathrm{dist}(s \cdot v + x, \mathrm{span}(L)) = \frac{| < s \cdot v + x, a_L > |}{\|a_L\|} = \frac{| < s \cdot v, a_L > + < x, a_L > |}{\|a_L\|} = \frac{|s|}{\|a_L\|} ,$$

hence distinct residue classes of L have a distance which is a multiple of $1/\|a_L\|$.
Now let $p \in \mathbb{Z}^k$ be an arbitrary point. We will show that p is contained in some residue class of L. With $s :=< p, a_L >\in \mathbb{Z}$ we have

$$< p - s \cdot v, a_L >=< p, a_L > -s\cdot < v, a_L >= s - s = 0 ,$$

thus $p - s \cdot v \in L$, hence $p \in s \cdot v + L$.
To see that $(s\cdot v+L)\cap(t\cdot v+L) = \emptyset$ for $s \neq t$, we assume to the contrary that $s\cdot v+x = t\cdot v+y$ for some $x, y \in L$. Then we have $(s - t) \cdot v = y - x$, and since $v \notin L$ but $y - x \in L$, we conclude that $s = t$, which is a contradiction. □

Theorem 3.16. *Let L be a $(k-1)$-maximal grid in $\mathbb{Z}^k$ with primitive normal vector $a_L = (a_1, \dots, a_k)^\top \in \mathbb{Z}^k$ and with basis Q. Then the determinant $d(L)$ of L, i.e. the volume of the fundamental parallelopiped determined by Q, satisfies*

$$d(L) = vol(F_Q) = \|a_L\|.$$

Proof. We assume w.l.o.g. that $a_i \neq 0$ for $i = 1, \dots, r$ und $a_i = 0$ for $i = r+1, \dots, k$. Set $a^* := (a_1, \dots, a_r)$ and let $d_i := \gcd(a_i, \dots, a_r)$ for $i = 1, \dots, r$. By Lemma 3.11 the set of all solutions of the equation $a_1 \cdot X_1 + \cdots + a_r \cdot X_r = 0$ over $\mathbb{Z}^r$ is a $(r-1)$-maximal grid L^* in $\mathbb{Z}^r$. By Lemma 3.5 we obtain a basis $Q^* = \{q_1^*, \dots, q_{r-1}^*\} \subset \mathbb{Z}^r$ of L^* with $q_i^* = (q_{i,1}^*, \dots, q_{i,r}^*)$ for $i = 1, \dots, r-1$. By construction of the basis Q^* from Lemma 3.5 the $r \times (r-1)$ generator matrix $G(Q^*)$ of Q^* has the following form

$$\begin{pmatrix} \frac{d_2}{d_1} & 0 & \cdots & 0 \\ * & \frac{d_3}{d_2} & \ddots & \vdots \\ \vdots & \ddots & \frac{d_{r-1}}{d_{r-2}} & 0 \\ \vdots & & \ddots & \frac{d_r}{d_{r-1}} \\ * & \cdots & \cdots & * \end{pmatrix}_{r\times(r-1)} ,$$

since $\frac{\gcd(a_i,\dots,a_r)}{\gcd(a_{i-1},\gcd(a_i,\dots,a_r))} = \frac{d_i}{d_{i-1}}$. Let e_r^* be the r-th unit vector in $\mathbb{Z}^r$ which has entry 1 at position r and entries 0 elsewhere. Consider the determinant of the $r \times r$ matrix $M = (q_1^*, \dots, q_{r-1}^*, e_r^*)$. With $d_1 = 1$ and $d_r = a_r$ we infer

$$|\det(q_1^*, \dots, q_{r-1}^*, e_r^*)| \;=\; 1 \cdot \prod_{i=1}^{r-1} \frac{d_{i+1}}{d_i} = \frac{d_r}{d_1} = a_r. \tag{8}$$

By Lemma 3.6 the distance of the unit vector e_r^* to the span of L^* satisfies

$$\operatorname{dist}(e_r^*, \operatorname{span}(L^*)) = \frac{|< e_r^*, a^* >|}{\|a^*\|} = \frac{|a_r|}{\|a^*\|} = \frac{|a_r|}{\|a_L\|} > 0 \tag{9}$$

and hence, (8) and (9) and Lemmas 3.7 and 3.3 imply for the volume of the parallelopiped F_{Q^*} (determinant of the grid L^*):

$$d(L^*) = \operatorname{vol}(F_{\{q_1^*,\dots,q_{r-1}^*\}}) = \frac{1}{\operatorname{dist}(e_r^*, \operatorname{span}(q_1^*, \dots, q_{r-1}^*))} \cdot \operatorname{vol}(F_{\{q_1^*,\dots,q_{r-1}^*,e_r^*\}}) =$$

$$= \frac{\|a_L\|}{|a_r|} \cdot \left[\det((q_1^*, \dots, q_{r-1}^*, e_r^*)^\top \cdot (q_1^*, \dots, q_{r-1}^*, e_r^*))\right]^{1/2} = \frac{\|a_L\|}{|a_r|} \cdot (a_r^2)^{1/2} = \|a_L\|.$$

The grid L is the set of all solutions of the equation $a_1 \cdot X_1 + \cdots + a_k \cdot X_k = 0$ over $\mathbb{Z}^k$. Therefore, L can be written as $L = L^* \times \mathbb{Z}^{k-r}$ and with $q_i := (q^*_{i,1}, \ldots, q^*_{i,r}, 0, \ldots, 0)^\top \in \mathbb{Z}^k$, $i = 1, \ldots, r-1$, the vectors $q_1, \ldots, q_{r-1}, e_{r+1}, \ldots, e_k$ form a basis of L. Applying Lemma 3.7 several times and using Lemma 3.3, we obtain

$$d(L) = d(L^*) \cdot 1^{k-r} = \|a_L\|,$$

since the unit vectors $e_{r+1}, \ldots, e_k \in \mathbb{Z}^k$ have length 1, are pairwise orthogonal, and are orthogonal to the subspace spanned by $= L^* \times \{0\}^{k-r}$. □

3.3 Simplices and Maximal Grids in $\mathbb{Z}^k$

Definition 3.17. *For a subset $S = \{p_0, \ldots, p_k\} \subset \mathbb{R}^k$ of $(k+1)$ points, the set*

$$S^* = \left\{ p_0 + \sum_{i=1}^{k} \lambda_i \cdot (p_i - p_0) \mid \sum_{i=1}^{k} \lambda_i \le 1; \lambda_1, \ldots, \lambda_k \in [0,1] \right\}$$

is called a simplex. *For short, we identify S and S^* and call each of them* simplex *and specify by calling the points $p_0, \ldots, p_k$* extreme points *of the simplex.*

(i) The rank *of the simplex S is defined by*

$$\mathit{rank}\ (S) = \dim(\mathit{span}(\{p_1 - p_0, \ldots, p_k - p_0\}))\ .$$

(ii) The simplex S is non-degenerated, *if rank $(S) = k$. If rank $(S) < k$, we call S a* degenerated simplex. *The simplex $S = \{p_0, \ldots, p_k\} \subset \mathbb{R}^k$ is called a* triangle *for $k = 2$, and for $k = 3$ it is called a* tetrahedron.

(iii) The volume *of the simplex $S = \{p_0, \ldots, p_k\} \subset \mathbb{R}^k$ is defined by*

$$\mathit{vol}(S) = \frac{1}{k} \cdot h \cdot \mathit{vol}(S')\ ,$$

where h is the distance of p_k to the affine space generated by $p_0, \ldots, p_{k-1}$ and $S' = \{p_0, \ldots, p_{k-1}\}$.

Recall that $B_k(T) = \{x \in \mathbb{R}^k \mid \|x\| \le T\} \subset \mathbb{R}^k$ is the k-dimensional ball around the origin with radius $T \in \mathbb{R}_0^+$.

Lemma 3.18. *Let $S \subseteq B_k(T) \cap \mathbb{Z}^k$ be a set of points with rank $(S) \le k-1$. Then there exists a $(k-1)$-maximal grid L of $\mathbb{Z}^k$ such that S is contained in some residue class $v + L$ of L for some $v \in \mathbb{Z}^k$, and L has a basis $q_1, \ldots, q_{k-1} \subset \mathbb{Z}^k$ with $\max\limits_{i=1,\ldots,k-1} \|q_i\| = O(T)$.*

Proof. Let $S = \{p_0, \ldots, p_m\} \subseteq B_k(T) \cap \mathbb{Z}^k$ with $r = \text{rank } S$. The vectors $p_1 - p_0, \ldots, p_m - p_0$ span a grid L' in $\mathbb{Z}^k$ with rank $(L') = r$, and have length $\|p_i - p_0\| \le 2 \cdot T$ for $i = 1, \ldots, m$. We take $(k-1-r)$ unit vectors from $\mathbb{Z}^k \setminus L'$, add them to L', and we obtain a grid L'' of $\mathbb{Z}^k$ with rank $(L'') = k-1$ and $L' \subseteq L''$. By Lemma 3.12 the grid L'' uniquely determines a $(k-1)$-maximal grid L of $\mathbb{Z}^k$ with $L' \subseteq L'' \subseteq L$. By Lemma 3.9 we can find a basis $q_1, \ldots, q_{k-1}$ of L with $\|q_i\| = O(T)$ for $i = 1, \ldots, k-1$. Then we have $S \subseteq p_0 + L$. □

Recall that in our notation every residue class L' of a grid L in $\mathbb{Z}^k$ is of the form $L' = x + L$ for some $x \in \mathbb{Z}^k$.

Theorem 3.19. *Let $k \in \mathbb{N}$ be fixed. Let L be a $(k-1)$-maximal grid of $\mathbb{Z}^k$ with primitive normal vector $a_L \in \mathbb{Z}^k$ and let $Q = \{q_1, \ldots, q_{k-1}\}$ be a basis of L with $\max_i \|q_i\| = O(T)$. Then the following holds:*

i) The primitive normal vector a_L satisfies $\|a_L\| = O(T^{k-1})$.

ii) There are at most $O(T \cdot \|a_L\|)$ different residue classes L' of L with $L' \cap B_k(T) \neq \emptyset$.

iii) For every residue class L' of L it is $|L' \cap B_k(T)| = O\big(T^{k-1}/\|a_L\|\big)$.

Proof. (i): By Theorem 3.16 we have $\|a_L\| = d(L) = \mathrm{vol}(F_Q)$ and by using the assumption $\max_i \|q_i\| = O(T)$ we know that $F_Q \subseteq B_k(c \cdot T)$ for some constant $c > 0$. Since $\dim(F_Q) = k-1$, the volume of the parallelopiped F_Q is bounded from above by $O(T^{k-1})$. We conclude $\|a_L\| = \mathrm{vol}(F_Q) = O(T^{k-1})$.

(ii): By Lemma 3.15 the distances of different residue classes of L are multiples of $1/\|a_L\|$. The Euclidean distance between any two points in $B_k(T)$ is at most $2 \cdot T$, hence at most $O(T \cdot \|a_L\|)$ distinct residue classes of L have a nonempty intersection with $B_k(T)$.

(iii): The volume of a $(k-1)$-dimensional space S intersected with $B_k(T)$ is bounded from above by $O(T^{k-1})$. Since $F_Q \subseteq B_k(c \cdot T)$ and $\mathrm{vol}(F_Q) = \|a_L\|$, we can cover the set $S \cap B_k(T)$ by at most $O\big(T^{k-1}/\|a_L\|\big)$ distinct translates of the parallelopiped F_Q. As L is maximal, the interior of F_Q (only the extreme points are excluded) contains no points from L, which finishes the proof of the theorem. □

3.4 Representations by Sums of Squares

We will use the following results from elementary number theory. For integers $k, d \in \mathbb{N}$ let $r_k(d)$ be the number of vectors $(x_1, \ldots, x_k)^\top \in \mathbb{Z}^k$ with $x_1^2 + \cdots + x_k^2 = d$.

Lemma 3.20. *For fixed integers $k \in \mathbb{N}$ and all integers $n \in \mathbb{N}$ it is:*

$$\sum_{d=1}^{n} r_k(d) = \Theta(n^{k/2}) \ . \tag{10}$$

Proof. The sum $\sum_{d=1}^{n} r_k(d)$ counts the number of grid points in $\mathbb{Z}^k$ in the k-dimensional ball $B_k(\sqrt{n})$ around the origin with radius $\sqrt{n}$. If we put around each of these grid points k-dimensional unit cubes $[0,1]^k$, with centers being the grid points, then all the points of these unit cubes are contained in a k-dimensional ball around the origin with radius $\sqrt{n} + \sqrt{k}/2$, since the diagonal of every k-dimensional unit cube $[0,1]^k$ has length equal to $\sqrt{k}$. Moreover, the unit cubes cover the k-dimensional ball around the origin with radius equal to $\sqrt{n} - \sqrt{k}/2$. Since the number of unit cubes is equal to $\sum_{d=1}^{n} r_k(d)$ we infer

$$\frac{\pi^{k/2}}{\Gamma(k/2+1)} \cdot \left(\sqrt{n} - \frac{\sqrt{k}}{2}\right)^k \leq \sum_{d=1}^{n} r_k(d) \leq \frac{\pi^{k/2}}{\Gamma(k/2+1)} \cdot \left(\sqrt{n} + \frac{\sqrt{k}}{2}\right)^k ,$$

thus $\sum_{d=1}^{n} r_k(d) = \Theta(n^{k/2})$. □

Corollary 3.21. *For constants $k, r \in \mathbb{N}$ and for all positive integers $n \in \mathbb{N}$ it is*

$$\sum_{d=1}^{n} \frac{r_k(d)}{d^r} = \begin{cases} O\left(n^{k/2-r}\right) & \text{if } k/2 - r > 0 \\ O(\ln n) & \text{if } k/2 - r = 0 \\ O(1) & \text{if } k/2 - r < 0. \end{cases}$$

Proof. We assume w.l.o.g. that n is a power of e, i.e. $n = e^l$. Set $n_i = e^i$ for $i = 0, \ldots, l$. With (10) we infer for some constant $c_k > 0$:

$$\sum_{d=1}^{n} \frac{r_k(d)}{d^r} \leq \sum_{i=0}^{l} \sum_{d=n_i}^{n_{i+1}} \frac{r_k(d)}{d^r} \leq \sum_{i=0}^{l} \sum_{d=n_i}^{n_{i+1}} \frac{r_k(d)}{n_i^r} = \sum_{i=0}^{l} e^{-i \cdot r} \cdot \sum_{d=n_i}^{n_{i+1}} r_k(d) \leq$$

$$\leq \sum_{i=0}^{l} e^{-i \cdot r} \cdot \sum_{d=1}^{n_{i+1}} r_k(d) \leq \sum_{i=0}^{l} e^{-i \cdot r} \cdot c_k \cdot (n_{i+1})^{k/2} = c_k \cdot e^{k/2} \cdot \sum_{i=0}^{l} e^{i \cdot (k/2 - r)} .$$

The sum $\sum_{i=0}^{l} e^{i \cdot (k/2-r)}$ is bounded from above as follows: (i) for $k/2 - r > 0$ by $O\left(e^{l \cdot (k/2-r)}\right) = O\left(n^{k/2-r}\right)$, and (ii) for $k/2 - r = 0$ by $O(l) = O(\ln n)$ and (iii) for $k/2 - r < 0$ by $O(1)$. □

For a $(k-1)$-maximal grid L in $\mathbb{Z}^k$ with primitive normal vector $a_L \in \mathbb{Z}^k$ and a positive integer d, we denote by $r_k(d; a_L)$ the number of grid points P in L such that the square of the Euclidean distance between P and the origin O is equal to d, i.e. $(\mathrm{dist}(O, P))^2 = d$. In our arguments we will use the following variants of Lemma 3.20 and Corollary 3.21:

Lemma 3.22. *Let $k \in \mathbb{N}$ be fixed. Let L be a $(k-1)$-maximal grid in $\mathbb{Z}^k$ with primitive normal vector $a_L \in \mathbb{Z}^k$. Then, for all positive integers $n \in \mathbb{N}$ it is:*

$$\sum_{d=1}^{n} r_k(d; a_L) \quad = \quad O\left(\frac{n^{(k-1)/2}}{\|a_L\|}\right) . \tag{11}$$

Proof. The sum $\sum_{d=1}^{n} r_k(d; a_L)$ is equal to the number of grid points in L in the k-dimensional ball $B_k(\sqrt{n})$ with radius $\sqrt{n}$ around the origin. By Theorem 3.19 (iii) this sum $\sum_{d=1}^{n} r_k(d; a_L)$ is at most $O((\sqrt{n})^{k-1}/\|a_L\|)$, from which inequality (11) follows. □

With (11) the proof of the following is now straightforward with the the same arguments as used in the proof of Corollary 3.21.

Corollary 3.23. *Let $k, r \in \mathbb{N}$ be constants. Let L be a $(k-1)$-maximal grid in $\mathbb{Z}^k$ with primitive normal vector $a_L \in \mathbb{Z}^k$. For all positive integers $n \in \mathbb{N}$ it is*

$$\sum_{d=1}^{n} \frac{r_k(d; a_L)}{d^r} = \begin{cases} O\left(\frac{1}{\|a_L\|} \cdot n^{(k-1)/2-r}\right) & \text{if } (k-1)/2 - r > 0 \\ O\left(\frac{1}{\|a_L\|} \cdot \ln n\right) & \text{if } (k-1)/2 - r = 0 \\ O\left(\frac{1}{\|a_L\|}\right) & \text{if } (k-1)/2 - r < 0. \end{cases}$$

4 Degenerated Simplices in $B_k(T) \cap \mathbb{Z}^k$

Theorem 4.1. *Let $k \in \mathbb{N}$ be fixed. The number $D_k(T)$ of degenerated simplices in $B_k(T) \cap \mathbb{Z}^k$ satisfies*

$$D_k(T) = O\big(T^{k^2} \cdot \ln T\big) .$$

All these degenerated simplices in $B_k(T) \cap \mathbb{Z}^k$ can be constructed in time polynomial in T.

Proof. For fixed $k \in \mathbb{N}$, by inspecting every $(k+1)$-element subset $S = \{p_0, \ldots, p_k\} \subset B_k(T) \cap \mathbb{Z}^k$ we can determine all degenerated simplices in $B_k(T) \cap \mathbb{Z}^k$ in polynomial time $O(T^{k+1})$. To check whether $\mathrm{vol}(S) = 0$, one simply computes in time $O(1)$ the determinant of the matrix with columns $p_1 - p_0, \ldots, p_k - p_0$.

By Lemma 3.18 each degenerated $(k+1)$-element subset of points in $B_k(T) \cap \mathbb{Z}^k$ is contained in a residue class L' for some $(k-1)$-maximal grid L in $\mathbb{Z}^k$, where L has a basis $q_1, \ldots, q_{k-1} \in \mathbb{Z}^k$ with $\|q_i\| = O(T)$ for $i = 1, \ldots, k-1$. By Theorem 3.19 (i) it suffices to consider all primitive normal vectors $a_L \in \mathbb{Z}^k$ of length $\|a_L\| = O(T^{k-1})$ and the corresponding residue classes.
Having fixed a $(k-1)$-maximal grid L in $\mathbb{Z}^k$, determined by its primitive normal vector $a_L \in \mathbb{Z}^k$, by Theorem 3.19 (ii) there are at most $O(T \cdot \|a_L\|)$ residue classes L' of the grid L, which intersect the set $B_k(T) \cap \mathbb{Z}^3$ in a nonempty set.
By Theorem 3.19 (iii) each set $L' \cap B_k(T)$ contains at most $O(T^{k-1}/\|a_L\|)$ points. From each set $L' \cap B_k(T)$ we can select $(k+1)$ points in at most $O(\binom{T^{k-1}/\|a_L\|}{k+1})$ ways to obtain a degenerated simplex. This implies the following upper bound on the number $D_k(T)$ of degenerated simplices in $B_k(T) \cap \mathbb{Z}^k$:

$$D_k(T) = O\left(\sum_{a \in \mathbb{Z}^k,\, \|a\| = O(T^{k-1})} T \cdot \|a\| \cdot \binom{T^{k-1}/\|a\|}{k+1} \right) =$$

$$= O\left(T^{k^2} \cdot \sum_{a \in \mathbb{Z}^k,\, \|a\| = O(T^{k-1})} \frac{1}{\|a\|^k} \right) = O\left(T^{k^2} \cdot \sum_{d=1}^{O(T^{2k-2})} \frac{r_k(d)}{d^{k/2}} \right) = O(T^{k^2} \cdot \ln T) \ ,$$

since by Corollary 3.21 we have $\sum_{d=1}^{n} r_k(d)/d^{k/2} = O(\ln n)$, which finishes the proof. $\square$

5 Non-degenerated Tetrahedra in $B_3(T) \cap \mathbb{Z}^3$

From now on we consider only the case of dimension $k = 3$. We will determine for positive reals B the number $N_3(T; B)$ of non-degenerated tetrahedra $S = \{p_0, \ldots, p_3\}$ in $B_3(T) \cap \mathbb{Z}^3$ with volume $\mathrm{vol}(S) \leq B$. Recall that the *volume* of the tetrahedron S is defined by

$$\mathrm{vol}(S) = \frac{1}{3} \cdot h \cdot G \ ,$$

where h is the distance between p_3 and the affine real space generated by p_0, p_1, p_2 and G is the area determined by the triangle $\{p_0, p_1, p_2\}$.
We will show in this chapter the following result:

Theorem 5.1. *The number $N_3(T; B)$ of non-degenerated tetrahedra $S \subseteq B_3(T) \cap \mathbb{Z}^3$ with vol$(S) \leq B$ is bounded from above as follows*

$$N_3(T; B) = O\left(B \cdot T^9\right) \ . \tag{12}$$

The set of all these non-degenerated tetrahedra in $B_3(T) \cap \mathbb{Z}^3$ can be constructed in time polynomial in T.

Proof. By inspecting every 4-element subset $S = \{p_0, \ldots, p_3\} \subset B_3(T) \cap \mathbb{Z}^3$ we can determine all tetrahedra in $B_3(T) \cap \mathbb{Z}^3$ with volume at most B in time $O(T^{12})$. To check whether $\mathrm{vol}(S) \leq B$, one simply computes in time $O(1)$ the absolute value of the determinant of the matrix with columns $p_0, \ldots, p_3$, augmented by an all 1 row, and multiplies this value by $1/6$.
To prove (12) we first consider the number of non-degenerated triangles S in $B_3(T) \cap L$ for a 2-maximal grid L in $\mathbb{Z}^3$, where we distinguish whether $\mathrm{area}(S) \geq v$ or $\mathrm{area}(S) \leq v$ for some real value $v > 0$.

Lemma 5.2. *Let L be a 2-maximal grid in $\mathbb{Z}^3$ with primitive normal vector $a_L \in \mathbb{Z}^3$. Let $v, B > 0$ be real numbers. For every residue class L' of L the number of non-degenerated tetrahedra $S = \{p_0, p_1, p_2, p_3\}$ in $B_3(T) \cap \mathbb{Z}^3$ with $S \setminus \{p_3\} \subseteq L'$ and area$(S \setminus \{p_3\}) \geq v$ and vol$(S) \leq B$ is at most*

$$O\left(\frac{B \cdot T^8}{v \cdot \|a_L\|^3}\right).$$

Proof. Let L' be a residue class of a 2-maximal grid L in $\mathbb{Z}^3$ with primitive normal vector $a_L \in \mathbb{Z}^3$. By Lemma 3.18 we can assume that L has a basis $q_1, q_2 \in \mathbb{Z}^3$, with $\|q_1\|, \|q_2\| = O(T)$. By Theorem 3.19 (iii) the set $L' \cap B_3(T)$ contains at most $O(T^2/\|a_L\|)$ points. In particular, for every real $v > 0$, we can choose from the set $L' \cap B_3(T)$ three extreme points of a non-degenerated triangle S' with area$(S') \geq v$ in at most $O(\binom{T^2/\|a_L\|}{3})$ ways. Since the desired tetrahedra should have volume at most B, the corresponding fourth point has distance at most $O(B/v)$ from the real affine space of L'. By Lemma 3.15 the distance between distinct residue classes of L is a multiple of $1/\|a_L\|$, and since for every residue class L'' of L the set $L'' \cap B_3(T)$ contains at most $O(T^2/\|a_L\|)$ points, the fourth point can be chosen in at most

$$O\left(\frac{B}{v} \cdot \|a_L\| \cdot \frac{T^2}{\|a_L\|}\right) = O\left(\frac{B \cdot T^2}{v}\right)$$

ways. Altogether we obtain for the number of desired simplices

$$O\left(\binom{T^2/\|a_L\|}{3} \cdot \frac{B \cdot T^2}{v}\right) = O\left(\frac{B \cdot T^8}{v \cdot \|a_L\|^3}\right).$$

□

Next we will consider those non-degenerated triangles in $B_3(T) \cap \mathbb{Z}^3$ with area at most v. For this case we will use the following lemma:

Lemma 5.3. *Let L be a 2-maximal grid in $\mathbb{Z}^3$ with primitive normal vector $a_L \in \mathbb{Z}^3$. For every residue class L' of L, and two fixed distinct points P and Q on L', the number of non-degenerated triangles S in $L' \cap B_3(T)$ with extreme points P and Q and with area$(S) \leq v$ is at most*

$$O\left(\frac{v \cdot T}{dist(P,Q) \cdot \|a_L\|}\right).$$

Proof. By an affine mapping $f\colon L \longrightarrow \mathbb{Z}^2$ with $P' := f(P)$ for $P \in L$, we transform the 2-maximal grid L in $\mathbb{Z}^3$ with primitive normal vector $a_L \in \mathbb{Z}^3$ (or any residue class L' of L) into the standard 2-dimensional rectangular grid $\mathbb{Z}^2$ with basis $(0,1)^\top$ and $(1,0)^\top$. For a basis $a = (a_1, a_2, a_3)^\top \in \mathbb{Z}^3$, $b = (b_1, b_2, b_3)^\top \in \mathbb{Z}^3$ of the grid L, let $f(a) := (0,1)^\top$ and $f(b) := (1,0)^\top$. Points $P, Q, R \in L' \cap B_3(T)$ become the grid points $P', Q', R' \in E \cap \mathbb{Z}^2$ in an ellipsoid E. If area$(P,Q,R) = v$, then area$(P',Q',R') = v/\|a_L\|$, as can be seen easily. We can assume that $L' = L$ and that $P = (0,0,0)$ and $Q = \lambda \cdot a + \mu \cdot b$ for some $\lambda, \mu \in \mathbb{Z}$ are the two given points. Via the mapping $f\colon L \longrightarrow \mathbb{Z}^2$ we obtain the points $f(P) = P' = (0,0)$ and $f(Q) = Q' = (\lambda, \mu)$ which are contained in the ellipsoid E.
Let $g = \gcd(\lambda, \mu)$ and set $\lambda' := \lambda/g$ and $\mu' := \mu/g$. The line L_1 (residue class) through the points P' and Q' in $\mathbb{Z}^2$ has w.l.o.g. the primitive normal vector $a_N := (\mu', -\lambda')^\top$.
To estimate the number of points $R \in L' \cap B_3(T)$ such that area$(P,Q,R) \leq v$, we compute the number of points $R' \in E \cap \mathbb{Z}^2$ such that area$(P',Q',R') \leq v/\|a_L\|$. The distance of

R' to the line L_1 is at most $2 \cdot v/(\|a_L\| \cdot \mathrm{dist}(P', Q'))$. By Lemma 3.15 residue classes L_1' of L_1 have distance a multiple of $1/\|a_N\|$, thus we consider at most

$$O\left(\frac{v \cdot \|a_N\|}{\|a_L\| \cdot \mathrm{dist}(P', Q')}\right)$$

lines L_1'. Since the distance between any two points in $B_3(T)$ is at most $2 \cdot T$, the line L_1 intersects the ellipsoid E in two points with distance D, where

$$D = O\left(\frac{T \cdot \mathrm{dist}(P', Q')}{\mathrm{dist}(P, Q)}\right).$$

By Theorem 3.16 the distance between points in $L_1' \cap \mathbb{Z}^2$ is a multiple of $\|a_N\|$. Every line L_1' parallel to L_1 intersects the ellipsoid E in two points whose distance is at most D, hence $|L_1' \cap E \cap \mathbb{Z}^2| = O(D/\|a_N\|)$. Thus, we have the following upper bound on the number of triangles in $L' \cap B_3(T)$ with area at most v and with fixed extreme points P and Q:

$$O\left(\frac{v \cdot \|a_N\|}{\|a_L\| \cdot \mathrm{dist}(P', Q')} \cdot \frac{D}{\|a_N\|}\right) = O\left(\frac{v \cdot T}{\mathrm{dist}(P, Q) \cdot \|a_L\|}\right).$$

□

Corollary 5.4. *Let L be a 2-maximal grid in $\mathbb{Z}^3$ with primitive normal vector $a_L \in \mathbb{Z}^3$. For every residue class L' of L the number of non-degenerated triangles S in the set $L' \cap B_3(T)$ with area$(S) \leq v$ is at most*

$$O\left(\frac{v \cdot T^4}{\|a_L\|^3}\right).$$

Proof. Let w.l.o.g. $L = L'$. By Lemma 3.18 we can assume that the grid L has a basis $q_1, q_2 \in \mathbb{Z}^3$ with $\|q_1\|, \|q_2\| = O(T)$. Fix a point $P \in L \cap B_3(T)$, by Theorem 3.19 (iii) there are at most $O(T^2/\|a_L\|)$ possibilities for this. Then for every integer $d \in \mathbb{N}$ there are at most $r_3(d, a_L)$ points Q in $L \cap B_3(T)$ such that $(\mathrm{dist}(P, Q))^2 = d$. With $d = O(T^2)$ and by Lemma 5.3 and Corollary 3.23 the number of non-degenerated triangles S in the set $L \cap B_3(T)$ with area$(S) \leq v$ is at most

$$O\left(\frac{T^2}{\|a_L\|} \cdot \sum_{d=1}^{O(T^2)} \frac{v \cdot T}{d^{1/2} \cdot \|a_L\|} \cdot r_3(d, a_L)\right) = O\left(\frac{v \cdot T^3}{\|a_L\|^2} \cdot \sum_{d=1}^{O(T^2)} \frac{r_3(d, a_L)}{d^{1/2}}\right) = O\left(\frac{v \cdot T^4}{\|a_L\|^3}\right).$$

□

Lemma 5.5. *Let L be a 2-maximal grid L in $\mathbb{Z}^3$ with primitive normal vector $a_L \in \mathbb{Z}^3$. Then for every non-degenerated triangle S in L it is*

$$\text{area}(S) \geq \frac{1}{2} \cdot \|a_L\|.$$

Proof. The minimum area of a non-degenerated triangle in L is half of the volume of the fundamental parallelopiped F_Q, i.e. $\mathrm{vol}(F_Q) = \|a_L\|$. □

Lemma 5.6. *Let L be a 2-maximal grid L in $\mathbb{Z}^3$ with primitive normal vector $a_L \in \mathbb{Z}^3$. Let L' be any residue class of L. Let $v > 0$ be a real number. The number of non-degenerated tetrahedra $S = \{p_0, \ldots, p_3\}$ in $B_3(T) \cap \mathbb{Z}^3$ with $S \setminus \{p_3\} \subseteq L'$ and with area$(S \setminus \{p_3\}) \leq v$ and with vol$(S) \leq B$ is at most*

$$O\left(\frac{B \cdot v \cdot T^6}{\|a_L\|^4}\right) .$$

Proof. By Lemma 3.18 we assume that the grid L has a basis $q_1, q_2 \in \mathbb{Z}^3$, with $\|q_1\|, \|q_2\| = O(T)$. Let $S' = \{p_0, p_1, p_2\}$ be a non-degenerated triangle in $L' \cap B_3(T)$ with area$(S') \leq v$. By Corollary 5.4 there at most $O(v \cdot T^4/\|a_L\|^3)$ of them. By Lemma 5.5 we have area$(S') \geq \|a_L\|/2$. To select the point $p_3 \in B_3(T) \cap \mathbb{Z}^3$, the requirement vol$(S' \cup \{p_3\}) \leq B$ has to be satisfied, thus the distance between the point p_3 and the real space generated by S' is at most $O(B/\|a_L\|)$. By Lemma 3.15 the distance between distinct residue classes of L is a multiple of $1/\|a_L\|$. By Theorem 3.19 (iii) for every residue class L' of L the set $L' \cap B_3(T)$ contains at most $O(T^2/\|a_L\|)$ points. Hence, the point p_3 can be chosen in at most

$$O\left(\frac{B}{\|a_L\|} \cdot \frac{T^2}{\|a_L\|} \cdot \|a_L\|\right) = O\left(\frac{B \cdot T^2}{\|a_L\|}\right)$$

ways. Thus, the number of tetrahedra $S = \{p_0, \ldots, p_3\}$ in $B_3(T) \cap \mathbb{Z}^3$ with vol$(S) \leq B$ and $S \setminus \{p_3\} \subseteq L'$ and area$(S \setminus \{p_3\}) \leq v$ is at most

$$O\left(\frac{v \cdot T^4}{\|a_L\|^3} \cdot \frac{B \cdot T^2}{\|a_L\|}\right) = O\left(\frac{B \cdot v \cdot T^6}{\|a_L\|^4}\right) .$$

□

Now we will finish the proof of Theorem 5.1. From Lemmas 5.2 and 5.6 we infer that for every real $v > 0$ the number of tetrahedra $S = \{p_0, \ldots, p_3\}$ in $B_3(T) \cap \mathbb{Z}^3$ with $S \setminus \{p_3\} \subseteq L$ and with vol$(S) \leq B$ is at most

$$O\left(\frac{B \cdot v \cdot T^6}{\|a_L\|^4} + \frac{B \cdot T^8}{v \cdot \|a_L\|^3}\right) . \tag{13}$$

We have

$$\frac{B \cdot v \cdot T^6}{\|a_L\|^4} = \frac{B \cdot T^8}{v \cdot \|a_L\|^3} \qquad \text{if} \qquad v = T \cdot \|a_L\|^{1/2} .$$

For a given vector $a \in \mathbb{Z}^k$ we set $v(a) := T \cdot \|a\|^{1/2}$. Then, (13) becomes:

$$O\left(\frac{B \cdot v \cdot T^6}{\|a_L\|^4} + \frac{B \cdot T^8}{v \cdot \|a_L\|^3}\right) = O\left(\frac{B \cdot T^7}{\|a_L\|^{7/2}}\right) . \tag{14}$$

By Lemma 3.18 we can assume that the grid L has a basis $q_1, q_2 \in \mathbb{Z}^3$ with $\|q_1\|, \|q_2\| = O(T)$, hence $\|a_L\| = O(T^2)$ by Theorem 3.19 (i). For a fixed primitive normal vector $a_L \in \mathbb{Z}^3$ there are by Theorem 3.19 (ii) at most $O(T \cdot \|a_L\|)$ distinct residue classes of the grid L, which intersect $B_3(T)$ in a nonempty set. Thus, summing over all possible grids L we have with (14):

$$N_3(T; B) = O\left(\sum_{\substack{a \in \mathbb{Z}^3 \\ \|a\| = O(T^2)}} T \cdot \|a\| \cdot \frac{B \cdot T^7}{\|a\|^{7/2}}\right) = O\left(B \cdot T^8 \cdot \sum_{\substack{a \in \mathbb{Z}^3 \\ \|a\| = O(T^2)}} \frac{1}{\|a\|^{5/2}}\right) =$$

$$= O\left(B \cdot T^8 \cdot \sum_{d=1}^{O(T^4)} \frac{r_3(d)}{d^{5/4}}\right) = O(B \cdot T^9) ,$$

where the last equation follows with Corollary 3.21, i.e. $\sum_{d=1}^{n} r_3(d)/d^{5/4} = O(n^{1/4})$. $\square$

We summarize the considerations leading to (14) in a separate lemma:

Lemma 5.7. *Let L be a 2-maximal grid in $\mathbb{Z}^3$ with primitive normal vector $a_L \in \mathbb{Z}^3$. For every residue class L' of L the number of non-degenerated tetrahedra $S = \{p_0, p_1, p_2, p_3\} \subseteq B_3(T) \cap \mathbb{Z}^3$ with $S \setminus \{p_3\} \subseteq L'$ and $vol(S) \leq B$ is at most $O(B \cdot T^7/\|a_L\|^{7/2})$.*

6 2-Cycles

For some value $B > 0$, which will be fixed below, and for the given set of points in the set $B_3(T) \cap \mathbb{Z}^3$ we form a 4-uniform hypergraph $\mathcal{G} = \mathcal{G}(B) = (V, \mathcal{E})$ with the vertex set V being this set $B_3(T) \cap \mathbb{Z}^3$ of $\Theta(T^3)$ grid-points. The edges are determined by all four-element subsets of V, which form a tetrahedron of volume at most B including the degenerated tetrahedra. Then an independent set in this hypergraph $\mathcal{G} = \mathcal{G}(B)$ corresponds to a set of points in $B_3(T) \cap \mathbb{Z}^3$, where all tetrahedra have volume bigger than B. In order to apply Theorem 2.2 we will show that the assumptions there are satisfied, i.e. we will give upper bounds on the number of 2-cycles. Set

$$B := \frac{\ln n}{n^3} \cdot T^3 ,$$

where $T = n^{1+\varepsilon}$ for some fixed $\epsilon > 0$, thus $B = \Omega(n^{3\varepsilon} \cdot \ln n)$.
First we estimate the average degree t^3 of the hypergraph $\mathcal{G} = \mathcal{G}(B) = (V, \mathcal{E})$. By Theorems 4.1 and 5.1 we can bound the number of edges in $\mathcal{G}(B)$ by

$$|\mathcal{E}| = O(T^9 \cdot \ln T + B \cdot T^9) = O(B \cdot T^9) ,$$

hence, with $|V| = \Theta(T^3)$, the average degree t^3 of $\mathcal{G} = \mathcal{G}(B)$ satisfies

$$t^3 = \frac{4 \cdot |\mathcal{E}|}{|V|} = O\left(\frac{B \cdot T^9}{T^3}\right) = \left(B \cdot T^6\right) . \tag{15}$$

Next we will give upper bounds on the number of 2-cycles in our hypergraph. We will distinguish two types of 2-cycles, namely $(2,2)$-cycles and $(2,3)$-cycles. In the following we will always assume by Lemma 3.18 that L has a basis $q_1, q_2 \in \mathbb{Z}^3$, with $\|q_1\|, \|q_2\| = O(T)$, hence by Theorem 3.19 (i) we have $\|a_L\| = O(T^2)$.

6.1 $(2,2)$-cycles

Let us consider first the number $s_{2,2}(\mathcal{G})$ of $(2,2)$-cycles in our hypergraph $\mathcal{G} = \mathcal{G}(B)$, that is, the number of pairs of tetrahedra in $B_3(T) \cap \mathbb{Z}^3$, which have exactly two extreme points in common, and both tetrahedra have volume at most B.
We distinguish three cases: (a) both tetrahedra are degenerated, or (b) one tetrahedron is degenerated and the other one is non-degenerated or (c) both tetrahedra are non-degenerated. The corresponding numbers of $(2,2)$-cycles are denoted by $s_{2,2}(\mathcal{G}; dd)$, $s_{2,2}(\mathcal{G}; dn)$, $s_{2,2}(\mathcal{G}; nn)$, respectively.

Case (a): Both tetrahedra are degenerated. By Theorem 4.1 there are at most $O\left(T^9 \cdot \ln T\right)$ degenerated tetrahedra in the set $B_3(T) \cap \mathbb{Z}^3$. Fix one of these tetrahedra. The second degenerated tetrahedron is contained in a 2-maximal grid in $\mathbb{Z}^3$ and has two extreme points in common with the first one, say $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$. Fix a primitive normal vector $b_M := (b_1, b_2, b_3) \in \mathbb{Z}^3$ with $\|b_M\| = O(T^2)$, which belongs to a 2-maximal grid M in $\mathbb{Z}^3$, where $P, Q \in M'$ for some residue class M' of the grid M. Indeed, if $P, Q \in M'$, then with $y_i := p_i - q_i$ for $i = 1, 2, 3$ it must hold that

$$b_1 \cdot y_1 + b_2 \cdot y_2 + b_3 \cdot y_3 = 0 \;,$$

where $(y_1, y_2, y_3) \neq (0, 0, 0)$. By Theorem 3.19 (iii) the set $M' \cap B_3(T)$ contains at most $O(T^2/\|b_M\|)$ points, as by Lemma 3.18 we can assume that L has a basis $q_1, q_2 \in \mathbb{Z}^3$ with $\|q_1\|, \|q_2\| = O(T)$. Having fixed already the extreme points P and Q of the second degenerated tetrahedron, two further extreme points can be chosen from M' in at most $O(\binom{T^2/\|b_M\|}{2})$ ways. Summing over all possible residue classes M' of 2-maximal grids M in $\mathbb{Z}^3$ with $P, Q \in M'$ we obtain, using Corollary 3.21, the following upper bound on the number of the two further points of the second tetrahedron, where, neglecting constant factors, we assume that $y_3 \neq 0$:

$$\left(\sum_{\substack{b=(b_1,b_2,b_3)\in\mathbb{Z}^3; \|b\|=O(T^2) \\ b_1y_1+b_2y_2+b_3y_3=0}} \binom{T^2/\|b\|}{2} \right)$$
$$= \; O\left(T^4 \cdot \sum_{\substack{(b_1,b_2)\in\mathbb{Z}^2 \\ |b_1|,|b_2|=O(T^2)}} \frac{1}{b_1^2 + b_2^2 + (b_1 \cdot y_1/y_3 + b_2 \cdot y_2/y_3)^2} \right) =$$
$$= \; O\left(T^4 \cdot \sum_{\substack{(b_1,b_2)\in\mathbb{Z}^2 \\ |b_1|,|b_2|=O(T^2)}} \frac{1}{b_1^2 + b_2^2} \right) = O\left(T^4 \cdot \sum_{d=1}^{O(T^4)} \frac{r_2(d)}{d} \right) = O(T^4 \cdot \ln T) \;. \quad (16)$$

Hence we infer for the number $s_{2,2}(\mathcal{G}; dd)$ of pairs of degenerated tetrahedra in $B_3(T) \cap \mathbb{Z}^3$ which have two extreme points in common, where y_1, y_2, y_3 refer to the chosen points of the first tetrahedron:

$$s_{2,2}(\mathcal{G}; dd) = O\left(T^9 \cdot \ln T \cdot T^4 \cdot \ln T\right) = O\left(T^{13} \cdot (\ln T)^2\right) \;. \quad (17)$$

Case (b): One tetrahedron is degenerated and the other one is non-degenerated with volume at most B. By Theorem 5.1 the number of non-degenerated tetrahedra with volume at most B in $B_3(T) \cap \mathbb{Z}^3$ is at most $O(B \cdot T^9)$. Fix such a tetrahedron and fix two of its extreme points, say $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ where $y_i := p_i - q_i$ for $i = 1, 2, 3$. As in case (a), for a primitive normal vector $b_M = (b_1, b_2, b_3) \in \mathbb{Z}^3$ with $\|b_M\| = O(T^2)$ using Theorem 3.19 (i), we must have for the case that $P, Q \in M'$ for some residue class M' of M that

$$b_1 \cdot y_1 + b_2 \cdot y_2 + b_3 \cdot y_3 = 0 \;.$$

Two further extreme points of the second degenerated tetrahedron can be chosen from M' in at most $O(\binom{T^2/\|b_M\|}{2})$ ways, hence as in case (a) using (16) altogether at most $O(T^4 \cdot \ln T)$ possibilities. We infer for the number of pairs $s_{2,2}(\mathcal{G}; dn)$ of tetrahedra, where one is non-degenerated with volume at most B and the other one is degenerated, the following upper bound:

$$s_{2,2}(\mathcal{G}; dn) = O\left(B \cdot T^9 \cdot T^4 \cdot \ln T\right) = O\left(B \cdot T^{13} \cdot \ln T\right) \;. \quad (18)$$

Case (c): Both simplices have volume at most B, are non-degenerated and have two extreme points in common.
To count this number $s_{2,2}(\mathcal{G}; nn)$ of $(2,2)$-cycles, we choose a primitive normal vector $a_L \in \mathbb{Z}^3$ of a 2-maximal grid in $\mathbb{Z}^3$ with $\|a_L\| = O(T^2)$. Then we fix a point $P \in \mathbb{Z}^3 \cap B_3(T)$. There is exactly one residue class L' of L with $P \in L'$. For fixed integers $d > 0$ consider a second point $Q \in L'$ with $(\mathrm{dist}(P,Q))^2 = d$, there are at most $r_3(d; a_L)$ of these points. The points P and Q are the two common extreme points of the two tetrahedra.
By Lemma 5.3 there are at most $O(v \cdot T/(\|a_L\| \cdot d^{1/2}))$ points $R \in L' \cap B_3(T)$ such that $\mathrm{area}(P,Q,R) \leq v$. Then the fourth point from $B_3(T) \cap \mathbb{Z}^3$ of a tetrahedron with volume at most B can be chosen in at most $O(B \cdot T^2/\|a_L\|)$ ways. If we assume that the third point $R \in L' \cap B_3(T)$ satisfies $\mathrm{area}(P,Q,R) > v$, then there are at most $O(T^2/\|a_L\|)$ choices for the point R and the fourth point from $B_3(T) \cap \mathbb{Z}^3$ can be chosen in at most $O(B \cdot T^2/v)$ ways, thus we have at most

$$O\left(\frac{B \cdot v \cdot T^3}{\|a_L\|^2 \cdot d^{1/2}} + \frac{B \cdot T^4}{v \cdot \|a_L\|}\right) \tag{19}$$

possibilities for the third and fourth extreme point of the first tetrahedron. With $v(a_L) := T^{1/2} \cdot \|a_L\|^{1/2} \cdot d^{1/4}$ this upper bound (19) becomes

$$O\left(\frac{B \cdot T^{7/2}}{\|a_L\|^{3/2} \cdot d^{1/4}}\right) . \tag{20}$$

Concerning the second tetrahedron in $B_3(T) \cap \mathbb{Z}^3$, which also has the extreme points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ we proceed similarly as above. Consider all 2-maximal grids M in $\mathbb{Z}^3$ with primitive normal vector $b_M = (b_1, b_2, b_3) \in \mathbb{Z}^3$ with $\|b_M\| = O(T^2)$ such that $P - Q \in M$. Let $y_i := p_i - q_i$ for $i = 1, 2, 3$. For the case $P, Q \in M'$ for some residue class M' of M we have

$$b_1 \cdot y_1 + b_2 \cdot y_2 + b_3 \cdot y_3 = 0 . \tag{21}$$

This would leave at most $O(T^4)$ possibilities for the choice of b_M. However, since the third point $R \in M$ of the non-degenerated tetrahedron satisfies $R \in B_3(T) \cap \mathbb{Z}^3$, there are at most $O(T^3)$ choices for the primitive normal vector $b_M \in \mathbb{Z}^3$. Let $C \subseteq \mathbb{Z}^3$ be the set of all possible choices for b_M.
Having fixed P and Q, the number of possibilities to extend these two points to the second tetrahedron in $B_3(T) \cap \mathbb{Z}^3$ is by (20) at most

$$O\left(\sum_{b \in C} \frac{B \cdot T^{7/2}}{\|b\|^{3/2} \cdot d^{1/4}}\right) = O\left(\frac{B \cdot T^{7/2}}{d^{1/4}} \cdot \sum_{b \in C} \frac{1}{\|b\|^{3/2}}\right) = O\left(\frac{B \cdot T^{17/4}}{d^{1/4}}\right) .$$

The last equality can be seen as follows. Since we already fixed the two points P and Q, there are at most $O(T^3)$ possibilities to choose a residue class M' (of a 2-maximal grid M in $\mathbb{Z}^3$) such that $P, Q \in M'$ and where $|M' \cap B_3(T)| \geq 3$. Assume that $y_3 \neq 0$. Then by (21) we have: $(b_1, b_2, b_3), (b_1, b_2, b_3') \in C$ implies that $b_3 = b_3'$. Using also that the function $f(x) = 1/x$ is monoton decreasing, we have by Corollary 3.21

$$\sum_{b=(b_1,b_2,b_3) \in C} \frac{1}{\|b\|^{3/2}} \quad \leq \quad \sum_{b=(b_1,b_2,b_3) \in C} \frac{1}{(b_1^2 + b_2^2)^{3/4}} = O\left(\sum_{d=1}^{O(T^3)} \frac{r_2(d)}{d^{3/4}}\right) = O(T^{3/4}) .$$

Thus, we infer:

$$s_{2,2}(\mathcal{G};nn) = O\left(\sum_{\substack{a\in\mathbb{Z}^3\\ \|a\|=O(T^2)}} T^3 \cdot \sum_{d=1}^{O(T^2)} r_3(d;a)\cdot \frac{B\cdot T^{7/2}}{\|a\|^{3/2}\cdot d^{1/4}}\cdot \frac{B\cdot T^{17/4}}{d^{1/4}}\right) =$$

$$\begin{aligned}
&= O\left(B^2\cdot T^{43/4}\cdot \sum_{\substack{a\in\mathbb{Z}^3\\ \|a\|=O(T^2)}} \frac{1}{\|a\|^{3/2}}\cdot \sum_{d=1}^{O(T^2)} \frac{r_3(d;a)}{d^{1/2}}\right) = \\
&= O\left(B^2\cdot T^{47/4}\cdot \sum_{\substack{a\in\mathbb{Z}^3\\ \|a\|=O(T^2)}} \frac{1}{\|a\|^{5/2}}\right) = O\left(B^2\cdot T^{47/4}\cdot \sum_{d=1}^{O(T^4)} \frac{r_3(d)}{d^{5/4}}\right) = \\
&= O(B^2\cdot T^{51/4})\ . \qquad (22)
\end{aligned}$$

To satisfy the assumptions of Theorem 2.2, we must have for some suitable constant $\gamma > 0$ that

$$\begin{aligned}
s_{2,2}(\mathcal{G}) &= s_{2,2}(\mathcal{G};dd) + s_{2,2}(\mathcal{G};dn) + s_{2,2}(\mathcal{G};nn) = \\
&= O\left(T^3\cdot t^{5-\gamma}\right) = O\left(T^{13-2\gamma}\cdot B^{5/3-\gamma/3}\right)\ , \qquad (23)
\end{aligned}$$

where $t = O(B^{1/3}\cdot T^2)$ with $B = T^3\cdot \ln n/n^3$ and $T = n^{1+\varepsilon}$ for some constant $\varepsilon > 0$. Considering (17), (18) and (22) we have $T^{13}\cdot(\ln T)^2 = O(B\cdot T^{13}\cdot \ln T)$ and $B^2\cdot T^{51/4} = O(B\cdot T^{13}\cdot \ln T)$ for $0 < \varepsilon < 1/11$. Thus it suffices to consider only case (b). In this case (b) we infer with (18) for $0 < \gamma < 2\varepsilon/(2+3\varepsilon)$:

$$\frac{B\cdot T^{13}\cdot \ln T}{T^{13-2\gamma}\cdot B^{5/3-\gamma/3}} = \frac{T^{2\gamma}\cdot \ln T}{B^{2/3-\gamma/3}} = O\left(\frac{(\ln n)^{1/3+\gamma/3}}{n^{2\varepsilon-3\varepsilon\gamma-2\gamma}}\right) = o(1)\ . \qquad (24)$$

Thus by (24) the upper bound (23) holds for $0 < \varepsilon < 1/11$ and $0 < \gamma < 2\varepsilon/(2+3\varepsilon)$.

6.2 $(2,3)$-cycles

Let us now consider the number $s_{2,3}(\mathcal{G})$ of $(2,3)$-cycles in our hypergraph $\mathcal{G} = \mathcal{G}(B)$, that is, the number of pairs of tetrahedra in $B_3(T)\cap\mathbb{Z}^3$, both with volume at most B, having exactly three extreme points in common. As in the case of $(2,2)$-cycles we distinguish three cases: (a) both tetrahedra are degenerated, or (b) one tetrahedron is degenerated and the other one is non-degenerated or (c) both tetrahedra are non-degenerated. The corresponding numbers of $(2,3)$-cycles are denoted by $s_{2,3}(\mathcal{G};dd)$, $s_{2,3}(\mathcal{G};nd)$, $s_{2,3}(\mathcal{G};nn)$, respectively.

Case (a): Both tetrahedra are degenerated and have three extreme points in common. Thus the two tetrahedra are contained in a 2-maximal grid L in $\mathbb{Z}^3$ or a residue class L' of it with $L'\cap B_3(T)\neq\emptyset$, determined by some primitive normal vector $a_L\in\mathbb{Z}^3$ with $\|a_L\| = O(T^2)$ by Theorem 3.19 (i). By Theorem 3.19 (iii) the set $L'\cap B_3(T)$ contains at most $O(T^2/\|a_L\|)$ points. We can choose the five extreme points of the two tetrahedra in at most $O(\binom{T^2/\|a_L\|}{5})$ ways. Again by Theorem 3.19 (ii), there are at most $O(T\cdot\|a_L\|)$

residue classes of L intersecting $B_3(T)$ in a nonempty set, hence

$$s_{2,3}(\mathcal{G}; dd) = O\left(\sum_{\substack{a\in\mathbb{Z}^3\\ \|a\|=O(T^2)}} T\cdot\|a\|\cdot\binom{T^2/\|a\|}{5}\right) =$$

$$= O\left(T^{11}\cdot\sum_{\substack{a\in\mathbb{Z}^3\\ \|a\|=O(T^2)}}\frac{1}{\|a\|^4}\right) = O\left(T^{11}\cdot\sum_{d=1}^{O(T^4)}\frac{r_3(d)}{d^2}\right) = O(T^{11})\ . \tag{25}$$

since by Corollary 3.21 we have $\sum_{d=1}^{n} r_3(d)/d^2 = O(1)$.

Case (b): One tetrahedron is non-degenerated with volume at most B and the other one is degenerated, and they have three extreme points in common. Fix one of the by Theorem 5.1 at most $O(B\cdot T^9)$ non-degenerated tetrahedra in $B_3(T)\cap\mathbb{Z}^3$ with volume at most B. Choose three extreme points of it, say p_0, p_1, p_2, which are common to both tetrahedra. Since the points p_0, p_1, p_2 uniquely determine a residue class L' of a maximal grid L in $\mathbb{Z}^3$, and since the second tetrahedron is degenerated the fourth point p_3' of the second degenerated tetrahedron is contained in L'. Since $|L'\cap B_3(T)| = O(T^2)$ there are at most $O(T^2)$ choices for the point p_3', and we obtain

$$s_{2,3}(\mathcal{G}, dn) \;\;=\;\; O(B\cdot T^9\cdot T^2) = O(B\cdot T^{11})\ . \tag{26}$$

Case (c): Both tetrahedra are non-degenerated, each with volume at most B and they have three extreme points in common. Fix a 2-maximal grid L in $\mathbb{Z}^3$, respectively a residue class L' of it, with primitive normal vector $a_L\in\mathbb{Z}^3$, where $\|a_L\| = O(T^2)$. We count the pairs of non-degenerated tetrahedra $S = \{p_0, p_1, p_2, p_3\}$ and $S' = \{p_0, p_1, p_2, p_3'\}$ in $B_3(T)\cap\mathbb{Z}^3$ with $p_0, p_1, p_2\in L'$ and $\mathrm{vol}(S)\le B$ and $\mathrm{vol}(S')\le B$.
The number of triangles in $L'\cap B_3(T)$ with area at most v is by Lemma 5.4 at most $O(v\cdot T^4/\|a_L\|^3)$. A fourth point from $B_3(T)\cap\mathbb{Z}^3$ of a tetrahedron with volume at most B can be chosen in at most $O(B\cdot T^2/\|a_L\|)$ ways.
On the other hand, the number of triangles in $L'\cap B_3(T)$ with area at least v is at most $O(\binom{T^2/\|a_L\|}{3})$, and a fourth point from $B_3(T)\cap\mathbb{Z}^3$ of a tetrahedron can be chosen in at most $O(B\cdot T^2/v)$ ways.
Altogether, using $v := v(\|a_L\|) := T^{2/3}\cdot\|a_L\|^{2/3}$ the number of such pairs of tetrahedra is at most

$$O\left(\frac{v\cdot T^4}{\|a_L\|^3}\cdot\left(\frac{B\cdot T^2}{\|a_L\|}\right)^2 + \frac{T^6}{\|a_L\|^3}\cdot\left(\frac{B\cdot T^2}{v}\right)^2\right) =$$

$$= O\left(\frac{v\cdot B^2\cdot T^8}{\|a_L\|^5} + \frac{B^2\cdot T^{10}}{v^2\cdot\|a_L\|^3}\right) = O\left(\frac{B^2\cdot T^{26/3}}{\|a_L\|^{13/3}}\right)\ . \tag{27}$$

Summing over all grids L in $\mathbb{Z}^3$ with $\|a_L\| = O(T^2)$ and over all at most $O(T\cdot\|a_L\|)$

residue classes L' of L with $L' \cap B_3(T) \neq \emptyset$, we obtain

$$
\begin{aligned}
s_{2,3}(\mathcal{G}, nn) = O\left(\sum_{\substack{a \in \mathbb{Z}^3 \\ \|a\| = O(T^2)}} T \cdot \|a\| \cdot \frac{B^2 \cdot T^{26/3}}{\|a\|^{13/3}} \right) = \\
= \quad O\left(B^2 \cdot T^{29/3} \cdot \sum_{\substack{a \in \mathbb{Z}^3 \\ \|a\| = O(T^2)}} \frac{1}{\|a\|^{10/3}} \right) = O\left(B^2 \cdot T^{29/3} \cdot \sum_{d=1}^{O(T^4)} \frac{r_3(d)}{d^{5/3}} \right) = \\
= \quad O(B^2 \cdot T^{29/3}) \ , \qquad (28)
\end{aligned}
$$

since by Corollary 3.21 we have $\sum_{d=1}^{n} r_3(d)/d^{5/3} = O(1)$.
To satisfy the assumptions in Theorem 2.2, we must have that for some suitable constant $\gamma > 0$:

$$
s_{2,3}(\mathcal{G}) = s_{2,3}(\mathcal{G}, dd) + s_{2,3}(\mathcal{G}, dn) + s_{2,3}(\mathcal{G}, nn) = O(T^3 \cdot t^{4-\gamma}) \qquad (29)
$$

where $t = O(B^{1/3} \cdot T^2)$. Using the estimates (25), (26) and (28), we need for some constant $\gamma > 0$ the following:

$$
T^{11} + B \cdot T^{11} + B^2 \cdot T^{29/3} = O\left(T^{11-2\gamma} \cdot B^{4/3-\gamma/3} \right) \ ,
$$

where $B := T^3 \cdot \ln n / n^3$ and $T = n^{1+\varepsilon}$ with $\varepsilon > 0$.
Since we have $T^{11} = O(B \cdot T^{11})$ and $B^2 \cdot T^{29/3} = O(B \cdot T^{11})$ for $0 < \varepsilon < 4/5$, it suffices to consider only case (b). For this case (b) with (26) and $0 < \gamma < \varepsilon/(2 + 3\varepsilon)$ we have

$$
s_{2,3}(\mathcal{G}; dn) = O\left(B \cdot T^{11} \right) = O\left(T^{11-2\gamma} \cdot B^{4/3-\gamma/3} \right) \ ,
$$

since

$$
\frac{B \cdot T^{11}}{T^{11-2\gamma} \cdot B^{4/3-\gamma/3}} = \frac{T^{2\gamma}}{B^{1/3-\gamma/3}} = O\left(\frac{1}{(\ln n)^{1/3-\gamma/3} \cdot n^{\varepsilon - 3\gamma\varepsilon - 2\gamma}} \right) = o(1) \ .
$$

For $0 < \varepsilon < 1/11$ and $0 < \gamma < \varepsilon/(2 + 3\varepsilon)$ equations (29) and (23) are satisfied, thus all assumptions of Theorem 2.2 are fulfilled.
We finish the proof of Theorem 1.2 as follows. We apply Theorem 2.2 to our 4-uniform hypergraph $\mathcal{G} = \mathcal{G}(B) = (V, \mathcal{E})$ with average degree $t^3 = O(B \cdot T^6)$ by (15), and we find in time polynomial in T and hence in n an independent set in $\mathcal{G}(B)$ of size at least

$$
\begin{aligned}
\Omega\left(\frac{|V|}{t} \cdot (\ln t)^{1/3} \right) = \Omega\left(\frac{T^3}{B^{1/3} \cdot T^2} \cdot \left(\ln(B^{1/3} \cdot T^2) \right)^{1/3} \right) = \\
= \quad \Omega\left(\frac{T}{(\ln n)^{1/3} \cdot n^{\varepsilon}} \cdot (\ln n)^{1/3} \right) = \Omega(n).
\end{aligned}
$$

Thus we have found in polynomial time $\Omega(n)$ points in $B_3(T) \cap \mathbb{Z}^3$ such that the volume of every tetrahedron is at least $T^3 \cdot \ln n / n^3$. After rescaling, we obtain $\Omega(n)$ points in the unit cube $[0,1]^3$ such that the volume of every tetrahedron is at least $\Omega(\ln n / n^3)$.

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