

# A COLORED VERSION OF THE ERDŐS-KO-RADO THEOREM FOR VECTOR SPACES

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ABSTRACT. We study a variant, in the context of the Erdős–Ko–Rado Theorem for vector spaces [4], of an extremal problem of Erdős and Rothschild [3], who considered edge-colorings of graphs avoiding monochromatic triangles. For fixed positive integers  $r$ ,  $k$  and  $\ell$  with  $1 \leq \ell < r$ , and a family  $\mathcal{F}$  of linear  $r$ -dimensional subspaces in a linear  $n$ -dimensional vector space  $V_n$  over the fixed finite field  $GF(q)$ , let  $C_{(k,\ell)}(\mathcal{F})$  denote the number of  $k$ -colorings of spaces in  $\mathcal{F}$  for which the intersection of any two spaces in the same color class has dimension at least  $\ell$ . Consider the function  $\chi_{(r,k,\ell,q)}(n) = \max_{\mathcal{F} \subseteq \mathcal{S}_{n,q,r}} C_{(k,\ell)}(\mathcal{F})$ , where the maximum runs over all subfamilies of the family  $\mathcal{S}_{n,q,r}$  of all linear  $r$ -dimensional spaces in a linear  $n$ -dimensional space  $V_n$  over  $GF(q)$ . In this paper, we study the asymptotic behavior of the function  $\chi_{(r,k,\ell,q)}(n)$  and the corresponding extremal families.

## 1. INTRODUCTION AND MAIN RESULTS

A wide array of problems in combinatorics may be described as the quest for the largest element in a given class of discrete objects satisfying some prescribed set of properties. For instance, given a graph  $F$ , the famous Turán problem [14] asks for the maximum number of edges in an  $n$ -vertex graph that does not contain a copy of  $F$  as a subgraph. Erdős and Rothschild [3] have proposed the following twist to this problem: instead of considering graphs with no copy of  $F$ , they were interested in *edge-colorings* of  $n$ -vertex graphs with *no monochromatic copy of  $F$* . The aim is now to find  $n$ -vertex graphs with the largest number of distinct such colorings.

A problem with the same flavor may be stated in the setting of finite vector spaces. Indeed, a natural question asks about the maximum number of elements in a family of linear  $r$ -dimensional subspaces ( $r$ -spaces for short) of a linear  $n$ -dimensional vector space  $V_n$ , for which the intersection of any pair of spaces has dimension at least  $\ell$ . This problem has been solved by Frankl and Wilson [4] and their solution is known as the Erdős-Ko-Rado Theorem for vector spaces.

Here we address the Erdős-Rothschild version of this problem. Consider an  $n$ -space  $V_n$  over  $GF(q)$ , where as usual  $GF(q)$  denotes the finite field with  $q$  elements. Let  $\dim(U)$  be the dimension of a linear vector space  $U$ . Given  $r \leq n$ , the number of  $r$ -spaces in  $V_n$  over  $GF(q)$  is given by the *Gaussian coefficient*  $\binom{n}{r}_q = \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-r+1}-1)}{(q^r-1)(q^{r-1}-1)\dots(q-1)}$ . For a fixed integers  $\ell, r$ , where  $0 < \ell < r$ , a family  $\mathcal{F}$  of  $r$ -spaces in  $V_n$  is called  $\ell$ -*intersecting* if  $\dim(F \cap F') \geq \ell$  for all  $F, F' \in \mathcal{F}$ .

**Definition 1.1.** *Let  $\mathcal{F}$  be a family of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$ . A  $(k, \ell)$ -coloring of  $\mathcal{F}$  is a function  $\Delta: \mathcal{F} \rightarrow \{1, \dots, k\}$  associating a color with each  $r$ -space in  $\mathcal{F}$  so that any two  $r$ -spaces  $F_1, F_2 \in \mathcal{F}$  with the same color are  $\ell$ -intersecting. The number of  $(k, \ell)$ -colorings of  $\mathcal{F}$  is denoted by  $C_{(k,\ell)}(\mathcal{F})$ .*

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Define  $\mathcal{S}_{n,q,r}$  to be the family of all  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$ . Given positive integers  $n, q, r, \ell, k$ , where  $n > r > \ell$  and  $q$  is a prime power, we consider the function

$$\chi_{(r,k,\ell,q)}(n) = \max_{\mathcal{F} \subseteq \mathcal{S}_{n,q,r}} \{ C_{(k,\ell)}(\mathcal{F}) \},$$

that is,  $\chi_{(r,k,\ell,q)}(n)$  is the maximum number of  $(k, \ell)$ -colorings over all families  $\mathcal{F}$  of  $r$ -spaces in an  $n$ -space over  $GF(q)$ . A family  $\mathcal{F}$  for which  $\chi_{(r,k,\ell,q)}(n) = C_{(k,\ell)}(\mathcal{F})$  is called  $(k, \ell)$ -*extremal*.

The problem of determining  $\chi_{(r,k,\ell,q)}(n)$  and the corresponding  $(k, \ell)$ -extremal families of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$  is related with the Erdős-Ko-Rado Theorem for vector spaces. This theorem is due to Frankl and Wilson [4], and was complemented by Tanaka [13], who showed that, in the case  $n = 2r$ , there are exactly two non-isomorphic maximum families. It is convenient to work with the following notation.

**Definition 1.2.** *Let  $n > r$  be positive integers and let  $V_n$  be a linear  $n$ -dimensional space over  $GF(q)$ , where  $q$  is a prime power. For a fixed linear space  $L$  in  $V_n$ , let  $\mathcal{E}_1(L, r)$  be the family of all  $r$ -spaces in  $V_n$  that contain  $L$ . For a fixed linear space  $S$  in  $V_n$ , let  $\mathcal{E}_2(S, r)$  be the family of all  $r$ -spaces in  $V_n$  that are contained in  $S$ .*

**Theorem 1.3** (Erdős-Ko-Rado Theorem for Vector Spaces). *Let  $q$  be a prime power, let  $\ell$  and  $r$  be fixed positive integers with  $\ell < r$  and let  $n \geq 2r - \ell + 1$ . Let  $\mathcal{F}$  be an  $\ell$ -intersecting family  $\mathcal{F}$  of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$ . Then, we have*

$$|\mathcal{F}| \leq \max \left\{ \binom{n-\ell}{r-\ell}_q, \binom{2r-\ell}{r}_q \right\}. \quad (1)$$

If  $n > 2r$ , equality in (1) only holds for a unique family  $\mathcal{F}$  (up to isomorphism), which is given by  $\mathcal{E}_1(L, r)$  for a fixed  $\ell$ -space  $L$  in  $V_n$ . For  $2r - \ell < n < 2r$ , equality in (1) is achieved only by  $\mathcal{F} = \mathcal{E}_2(S, r)$  for some fixed  $(2r - \ell)$ -space  $S$  in  $V_n$ . In the case  $n = 2r$ , there are exactly two non-isomorphic extremal families  $\mathcal{F}$ : either  $\mathcal{F}$  is equal to the family  $\mathcal{E}_1(L, r)$  for a fixed  $\ell$ -space  $L$  in  $V_n$  or it is equal to  $\mathcal{E}_2(S, r)$  for some fixed  $(2r - \ell)$ -space  $S$  in  $V_n$ .

Given  $n, q, r, \ell$ , let  $\mathcal{E}_{n,q,r,\ell}$  denote the set of extremal families given in Theorem 1.3. Note that any family  $\mathcal{F} \in \mathcal{E}_{n,q,r,\ell}$  leads to a natural lower bound on  $\chi_{(r,k,\ell,q)}(n)$ , namely

$$\chi_{(r,k,\ell,q)}(n) \geq k^{|\mathcal{F}|}, \quad (2)$$

as  $\mathcal{F}$  is  $\ell$ -intersecting and therefore all  $r$ -spaces in  $\mathcal{F}$  may be colored arbitrarily. When the number of colors available is  $k \in \{2, 3\}$ , we show that (2) holds with equality.

**Theorem 1.4.** *Let  $q$  be a prime power and fix positive integers  $n > r > \ell$ . For a family  $\mathcal{F}$  of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$ , the following facts hold.*

(i) *For each  $n \geq 2r - \ell + 1$ , we have*

$$C_{(2,\ell)}(\mathcal{F}) \leq 2^{\max\left\{\binom{n-\ell}{r-\ell}_q, \binom{2r-\ell}{r}_q\right\}}. \quad (3)$$

(ii) *There exists a positive integer  $n_0$  such that, for  $n \geq n_0$ , we have*

$$C_{(3,\ell)}(\mathcal{F}) \leq 3^{\binom{n-\ell}{r-\ell}_q}. \quad (4)$$

Moreover, a family  $\mathcal{F}$  achieves equality in (3) or (4) if and only if  $\mathcal{F} \in \mathcal{E}_{n,q,r,\ell}$ .

However, extremal  $\ell$ -intersecting families and  $(k, \ell)$ -extremal families do not coincide for  $k \geq 4$  colors. This dichotomy between the cases  $k \in \{2, 3\}$  and  $k \geq 4$  has been observed for several instances of forbidden graphs and hypergraphs [1, 9, 10, 15], as well as in the case of set systems [6]. In this paper, we fully characterize  $(4, \ell)$ -extremal families when the dimension  $n$  of the vector space  $V_n$  is large enough.

**Definition 1.5.** Let  $n > r > \ell$  be positive integers and let  $q$  be a prime power. Let  $\mathcal{F}$  be a family of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$ . A subset  $\mathcal{C}$  of  $\ell$ -spaces of  $V_n$  is called an  $\ell$ -cover for  $\mathcal{F}$  if, for each  $r$ -space  $F \in \mathcal{F}$ , there exists an  $\ell$ -space  $L \in \mathcal{C}$  such that  $L \subseteq F$ . A minimum  $\ell$ -cover for  $\mathcal{F}$  is an  $\ell$ -cover for  $\mathcal{F}$  of minimum cardinality. The family  $\mathcal{F}$  is called  $(\mathcal{C}, r)$ -complete if it contains every  $r$ -space in  $V_n$  that contains some  $\ell$ -space in the  $\ell$ -cover  $\mathcal{C}$ .

In particular, the family  $\mathcal{E}_1(L, r)$  from Definition 1.2 has minimum  $\ell$ -cover of size one, as it is covered by the set  $\mathcal{C} = \{L\}$ , and this family is clearly  $(\mathcal{C}, r)$ -complete.

**Definition 1.6.** Given positive integers  $\ell < r < n$  and  $c$ , and a prime power  $q$ , let  $\mathcal{E}_{n,q,r,\ell;c}$  be the class of all  $(\mathcal{C}, r)$ -complete families of  $r$ -spaces in  $V_n$ , where  $\mathcal{C}$  is a set of  $\ell$ -spaces in  $V_n$  satisfying  $|\mathcal{C}| = c$ . Given  $a \in \{0, \dots, \ell - 1\}$ , we say that an element of  $\mathcal{E}_{n,q,r,\ell;c}$  lies in  $\mathcal{E}_{n,q,r,\ell;c,a}$  if, additionally, every pair of distinct  $\ell$ -spaces  $F_1, F_2 \in \mathcal{C}$  satisfies  $\dim(F_1 \cap F_2) = a$ .

With this definition, note that, for  $n > 2r$ , we have  $\mathcal{E}_{n,q,r,\ell;1} = \mathcal{E}_{n,q,r,\ell}$ , as all the extremal families with respect to Theorem 1.3 are of the form  $\mathcal{E}_1(L, r)$  for some  $\ell$ -space  $L$  in  $V_n$ .

**Theorem 1.7.** Let  $r > \ell$  be positive integers and let  $q$  be a prime power. There exists  $n_0 > 0$  such that, if  $n \geq n_0$  and  $\mathcal{F}$  is a family of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$ , we have

- (i)  $C_{(4,\ell)}(\mathcal{F}) \geq 6 \cdot 4^{\binom{n-\ell}{r-\ell}_q}$  if and only if  $\mathcal{F} \in \mathcal{E}_{n,q,r,\ell;2}$ , i.e.,  $\mathcal{F}$  is  $(\mathcal{C}, r)$ -complete with  $\ell$ -cover of size two.
- (ii)  $C_{(4,\ell)}(\mathcal{F}) = \chi_{(r,4,\ell,q)}(n)$  if and only if  $\mathcal{F} \in \mathcal{E}_{n,q,r,\ell;2,\ell-1}$ , i.e.,  $\mathcal{F}$  has an  $\ell$ -cover of size two and the two cover elements intersect in a space of dimension  $(\ell - 1)$ .

The lower bound  $6 \cdot 4^{\binom{n-\ell}{r-\ell}_q}$  in Theorem 1.7 arises as follows. For an  $\ell$ -cover  $\mathcal{C} = \{t_1, t_2\}$  of size two in  $V_n$  consider the  $(\mathcal{C}, r)$ -complete family  $\mathcal{F}$  of  $r$ -spaces in  $V_n$ . We may generate  $(4, \ell)$ -colorings of  $\mathcal{F}$  by assigning two of the four colors to the cover element  $t_1$ , and the other two colors to  $t_2$ , for which there are six possibilities. We color all  $r$ -spaces containing  $t_1$  arbitrarily with the first two colors, and all  $r$ -spaces containing  $t_2$  with the remaining colors. Those  $r$ -spaces in  $V_n$  which contain  $t_1 \cup t_2$  can be colored with any of the four colors. Up to some minor overcounting, this gives  $6 \cdot 4^{\binom{n-\ell}{r-\ell}_q}$  distinct colorings. Clearly there are other  $(4, \ell)$ -colorings of  $\mathcal{F}$  to compensate for this overcounting<sup>1</sup>.

Although Theorem 1.7 deals only with the case  $k = 4$ , several structural results used for proving it hold for larger values of  $k \geq 5$ , as described in Section 5.

The results in this paper may be analyzed in the context of ‘stability’ as in Simonovits’s Stability Theorem [11] for graphs. Roughly speaking, the problem of maximizing a function  $f$  over a class  $\mathcal{A}$  of combinatorial objects is said to be *stable* if every object that is very close to maximizing  $f$  is almost equal to the object that maximizes  $f$ . In our framework, this can be formalized as follows. Here, for two families  $\mathcal{A}$  and  $\mathcal{B}$ , we write  $\mathcal{A} \triangle \mathcal{B}$  for their symmetric difference  $(\mathcal{A} \setminus \mathcal{B}) \cup (\mathcal{B} \setminus \mathcal{A})$ . Moreover, we say that two families  $\mathcal{F}$  and  $\mathcal{F}'$  of  $r$ -spaces in an  $n$ -space  $V_n$  are isomorphic if there is a one-to-one linear operator  $\phi: V_n \rightarrow V_n$  such that  $F \in \mathcal{F}$  if and only if  $F' = \{\phi(v) : v \in F\} \in \mathcal{F}'$  for each  $F \in \mathcal{F}$ .

**Definition 1.8.** Let  $r > \ell > 0$ ,  $k$  and  $q$  be fixed. The problem  $\mathcal{P}_{n,q,r,k,\ell}$  of determining  $\chi_{(r,k,\ell,q)}(n)$  is stable if, for every  $\varepsilon > 0$ , there exist  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that the following is satisfied. Let  $\mathcal{F}^*$  be a  $(k, \ell)$ -extremal family of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$ , where  $n \geq n_0$ , and let  $\mathcal{F}$  be a family of  $r$ -spaces in  $V_n$  satisfying  $C_{(k,\ell)}(\mathcal{F}) > (1 - \delta) \cdot \chi_{(r,k,\ell,q)}(n)$ . Then  $|E(\mathcal{F}) \triangle E(\mathcal{F}')| < \varepsilon \cdot |E(\mathcal{F}')|$  for some family of  $r$ -spaces  $\mathcal{F}'$  that is isomorphic to  $\mathcal{F}^*$ .

<sup>1</sup>Full proofs of this and other results in this paper may be found at

The proof of Theorem 1.4 implies that, if  $k \in \{2, 3\}$ ,  $\mathcal{P}_{n,q,r,k,\ell}$  is stable for every  $r$ ,  $\ell$  and  $q$ . However, the proof of Theorem 1.7 leads to the conclusion that  $\mathcal{P}_{n,q,r,4,\ell}$  is unstable unless  $\ell = 1$ , since the set of  $(4, \ell)$ -extremal families is given by  $\mathcal{E}_{n,q,r,\ell;2,\ell-1}$ , even though

$$\lim_{n \rightarrow \infty} \frac{C_{(4,\ell)}(\mathcal{F}')}{\chi_{(r,4,\ell,q)}(n)} = 1$$

for every  $\mathcal{F}' \in \mathcal{E}_{n,q,r,\ell;2}$ . In other words,  $C_{(k,\ell)}(\mathcal{F}')$  has the same first-order asymptotic behavior as long as  $\mathcal{F}'$  is  $(\mathcal{C}, r)$ -complete with minimum  $\ell$ -cover  $\mathcal{C}$  of size two, regardless of the dimension of the intersection of the cover spaces. This provides an instance of a natural problem that is provably unstable, but which has a unique optimal solution up to isomorphism. We are not aware of many problems that share this property, but we may cite [2] in the context of multigraphs and [8] in the context of uniform hypergraphs.

## 2. THE CASE OF TWO COLORS

The main objective of this section is to prove Theorem 1.4(i). Recall that we need to show that, for all values of  $n$ , the  $(2, \ell)$ -extremal families of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$  coincide with the extremal families with respect to the Erdős-Ko-Rado Theorem for vector spaces.

*Proof of Theorem 1.4(i).* Let  $\mathcal{F}$  be a family of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$  and consider a  $(2, \ell)$ -coloring of  $\mathcal{F}$ . Let  $\mathcal{E} \subseteq \mathcal{F}$  be a maximal  $\ell$ -intersecting subfamily of  $\mathcal{F}$ . Due to the maximality of  $\mathcal{E}$ , for each  $r$ -space  $R \in \mathcal{F} \setminus \mathcal{E}$ , there is an  $r$ -space  $E \in \mathcal{E}$  such that  $\dim(R \cap E) < \ell$ , so that  $E$  and  $R$  have to be colored differently in any  $(2, \ell)$ -coloring of  $\mathcal{F}$ . This implies that the number of  $(2, \ell)$ -colorings of  $\mathcal{F}$  is at most

$$2^{|\mathcal{E}|} \leq 2^{\max\left\{\binom{n-\ell}{r-\ell}_q, \binom{2r-\ell}{r}_q\right\}}. \quad (5)$$

Theorem 1.3 implies that there is always an  $\ell$ -intersecting family  $\mathcal{F}$  that achieves this upper bound, and hence  $\mathcal{F}$  is not extremal if  $|\mathcal{E}| < \max\left\{\binom{n-\ell}{r-\ell}_q, \binom{2r-\ell}{r}_q\right\}$ . Thus we assume that in (5) equality holds. To prove our result, we still need to show that  $\mathcal{F} \setminus \mathcal{E} = \emptyset$  whenever

$$C_{2,\ell}(\mathcal{F}) = 2^{|\mathcal{E}|} = 2^{\max\left\{\binom{n-\ell}{r-\ell}_q, \binom{2r-\ell}{r}_q\right\}}.$$

For a contradiction, we suppose that there exists an  $r$ -space  $R$  in  $\mathcal{F} \setminus \mathcal{E}$  that is not  $\ell$ -intersecting with some  $r$ -space in  $\mathcal{E}$ . We will show that there are at least  $q^2$  distinct  $r$ -spaces  $W_1, \dots, W_{q^2}$  in  $\mathcal{E}$  that are not  $\ell$ -intersecting with  $R$  (see Lemma 2.1 below). Hence, for every  $(2, \ell)$ -coloring of  $\mathcal{F}$ , all  $r$ -spaces  $W_1, \dots, W_{q^2}$  must be colored the same, which implies that we have at most  $2^{|\mathcal{E}|-q^2+1} < 2^{|\mathcal{E}|}$  distinct  $(2, \ell)$ -colorings of  $\mathcal{F}$ . Hence, for any extremal family  $\mathcal{F}$ , we must indeed have  $\mathcal{F} \setminus \mathcal{E} = \emptyset$ . The characterization of the  $(2, \ell)$ -extremal families for all values of  $n$  and  $r$  follows directly from Theorem 1.3.  $\square$

To conclude the proof of Theorem 1.4 it remains to prove the following lemma.

**Lemma 2.1.** *Let  $n, r, \ell$  be positive integers with  $n \geq r > \ell$ , and let  $q$  be a prime power. Let  $V_n$  be an  $n$ -space over  $GF(q)$  and consider an  $\ell$ -intersecting family  $\mathcal{E}$  of  $r$ -spaces in  $V_n$  with maximum cardinality. If  $F$  is an  $r$ -space in  $V_n$  that is not  $\ell$ -intersecting with some element in  $\mathcal{E}$ , then there exist at least  $q^2$  distinct  $r$ -spaces in  $\mathcal{E}$  with this property.*

To prove Lemma 2.1, we use the following well-known facts. (A proof of this fact is an immediate consequence of Lemma 1.1.2 in [5], for example.)

**Lemma 2.2.** *Let  $g, h, r, n$  be nonnegative integers. Let  $V_n$  be an  $n$ -space over  $GF(q)$ , containing linear spaces  $G$  and  $H$  with dimension  $g$  and  $h$ , respectively. Assume that  $\dim(G \cap H) = 0$ .*

(i) The number of  $r$ -spaces in  $V_n$  that contain the  $g$ -space  $G$  is equal to

$$\binom{n-g}{r-g}_q.$$

(ii) The number of  $(n-g)$ -spaces  $S$  in  $V_n$  with  $\dim(S \cap G) = 0$  is equal to

$$q^{(n-g)g}.$$

(iii) The number of  $r$ -spaces  $R$  in  $V_n$  with  $G \subseteq R$  and  $\dim(H \cap R) = 0$  is equal to

$$q^{(r-g)h} \cdot \binom{n-(g+h)}{r-g}_q.$$

*Proof of Lemma 2.1.* If  $n \leq 2r - \ell$ , then any two  $r$ -spaces in  $V_n$  are  $\ell$ -intersecting, hence we may suppose that  $n > 2r - \ell$ . We consider two cases according to the extremal family.

First assume that the extremal family under consideration is the family  $\mathcal{E}_1(L, r)$  consisting of all  $r$ -spaces in  $V_n$  that contain a fixed  $\ell$ -space  $L$  in  $V_n$ , hence  $|\mathcal{E}_1(L, r)| = \binom{n-\ell}{r-\ell}_q$  and, by Theorem 1.3,  $n \geq 2r$ . By assumption there exists an  $r$ -space  $F \in \mathcal{F}$  and an  $r$ -space  $E_1 \in \mathcal{E}$  such that  $\dim(E_1 \cap F) = s \leq \ell - 1$ , hence  $\dim(F \cap L) = a < \ell$ . Let  $\mathcal{D}(1)$  be the family of all  $r$ -spaces  $R$  in  $V_n$  that contain  $L$ , but do not contain any vector in  $F \setminus L$ . Clearly, each  $r$ -space  $R \in \mathcal{D}(1) \subseteq \mathcal{E}_1$  satisfies  $\dim(R \cap F) < \ell$ . We show that  $|\mathcal{D}(1)| \geq q^2$ . Let  $H$  be an  $h = (r - a)$ -dimensional subspace of  $F$  with  $\dim(H \cap L) = 0$  and consider the family of all  $r$ -spaces  $R$  of  $V_n$  that contain  $L$  and satisfy  $\dim(R \cap H) = 0$ . Clearly, every such  $r$ -space lies in  $\mathcal{D}(1)$ ; moreover, we may use Lemma 2.2(iii) with given  $H$  and  $G = L$  to obtain

$$|\mathcal{D}(1)| \geq q^{(r-\ell)h} \cdot \binom{n-(h+\ell)}{r-\ell}_q = q^{(r-\ell)(r-a)} \cdot \binom{n-(r+\ell-a)}{r-\ell}_q.$$

Since  $r > \ell$  and  $\ell > a$ , we infer that  $q^{(r-\ell)(r-a)} \geq q^2$ . We also have

$$n - (r + \ell - a) \geq r - \ell \iff n \geq 2r - a.$$

In particular, the inequality  $\binom{n-(r+\ell-a)}{r-\ell}_q \geq 1$  follows from  $n \geq 2r$ , so that  $|\mathcal{D}(1)| \geq q^2$ , and we are done in this case.

Next suppose that the extremal family under consideration is the family  $\mathcal{E}_2(S, r)$  consisting of all  $r$ -spaces in some fixed  $(2r - \ell)$ -space  $S$  in  $V_n$ , hence  $|\mathcal{E}_2(S, r)| = \binom{2r-\ell}{r}_q$  and, by Theorem 1.3,  $2r - \ell < n \leq 2r$ . Let  $F$  be an  $r$ -space in  $V_n$  that is not  $\ell$ -intersecting with at least one member of  $\mathcal{E}_2(S, r)$ . Clearly, in this case we must have  $\dim(F \cap S) = a < r$ . If  $a < \ell$ , then  $F$  is not  $\ell$ -intersecting with any element of  $\mathcal{E}_2(S, r)$ , and the result follows because  $\binom{2r-\ell}{r}_q \geq \binom{r+1}{r}_q = (q^{r+1} - 1)/(q - 1) \geq q^2$  for  $r \geq 2$ .

Therefore, we may assume that  $\ell \leq a \leq r - 1$ . Consider the family  $\mathcal{D}(2)$  of all  $r$ -spaces in  $S$  whose intersection with the space  $S \cap F$  has dimension at most  $(\ell - 1)$ . Any  $r$ -space  $R \in \mathcal{D}(2)$  satisfies  $\dim(R \cap F) < \ell$ , hence it suffices to show that  $|\mathcal{D}(2)| \geq q^2$ . For every nonnegative integer  $g \leq \ell - 1$  and every  $g$ -subspace  $G$  of  $F \cap S$ , there are  $q^{(r-g)(a-g)} \cdot \binom{2r-\ell-a}{r-g}_q$  distinct  $r$ -spaces  $R$  in  $S$  such that  $F \cap R = G$  (we apply Lemma 2.2(iii) for  $G$  and an associated  $(a - g)$ -space  $H$  contained in  $F \cap S$  whose intersection with  $G$  is the null vector). Since there are  $\binom{a}{g}_q$  distinct  $g$ -spaces in  $F \cap S$ , we infer that

$$\begin{aligned} |\mathcal{D}(2)| &\geq \sum_{g=0}^{\ell-1} \binom{a}{g}_q \cdot q^{(r-g)(a-g)} \cdot \binom{2r-\ell-a}{r-g}_q \\ &\geq \binom{a}{\ell-1}_q \cdot q^{(r-\ell+1)(a-\ell+1)} \cdot \binom{2r-\ell-a}{r-\ell+1}_q. \end{aligned} \tag{6}$$

Now, since  $a \leq r - 1$ , we have  $\binom{2r-\ell-a}{r-\ell+1}_q \geq \binom{r-\ell+1}{r-\ell+1}_q \geq 1$ . Moreover, with  $a \geq \ell$  and  $r > \ell$ , we have  $\binom{a}{\ell-1}_q \cdot q^{(r-\ell+1)(a-\ell+1)} \geq \binom{\ell}{\ell-1}_q \cdot q^2 \geq q^2$  for  $\ell \geq 1$ . Using these bounds in (6), we obtain  $|\mathcal{D}(2)| \geq q^2$ , which finishes the proof of Lemma 2.1.  $\square$

### 3. COLORING WITH THREE AND FOUR COLORS

In this section, we study the function  $\chi_{(r,k,\ell,q)}(n)$  when the number of colors is  $k \in \{3, 4\}$ . We start with a simple result stating that  $(k, \ell)$ -colorable families of  $r$ -spaces cannot have a large minimum  $\ell$ -cover.

**Lemma 3.1.** *Let  $\ell < r < n$  and  $k$  be positive integers. Let  $\mathcal{F}$  be a  $(k, \ell)$ -colorable family of  $r$ -spaces in an  $n$ -space over  $GF(q)$ . Then  $\mathcal{F}$  has an  $\ell$ -cover  $\mathcal{C} = \{t_1, \dots, t_c\}$  of cardinality  $c \leq k \binom{r}{\ell}_q$  such that  $\dim(\langle t_1 \cup \dots \cup t_c \rangle) \leq kr$ .*

*Proof.* The  $(k, \ell)$ -colorability of  $\mathcal{F}$  implies that there cannot be more than  $k$  distinct  $r$ -spaces in  $\mathcal{F}$  that pairwise intersect in a space of dimension less than  $\ell$ . Hence, there exists a subfamily  $\mathcal{F}' \subseteq \mathcal{F}$  with at most  $k$  distinct  $r$ -spaces such that every  $r$ -space in  $\mathcal{F}$  is  $\ell$ -intersecting with some  $r$ -space in  $\mathcal{F}'$ , and the set  $\mathcal{C} = \{t \in \binom{F}{\ell}_q; F \in \mathcal{F}'\}$  is an  $\ell$ -cover for  $\mathcal{F}$  with  $|\mathcal{C}| \leq k \binom{r}{\ell}_q$  and  $\dim(\langle \bigcup_{t \in \mathcal{C}} t \rangle) \leq kr$ .  $\square$

Let  $n > r > \ell$  and  $k$  be positive integers and consider a family  $\mathcal{F} \in \mathcal{S}_{n,q,r}$  with minimum  $\ell$ -cover  $\mathcal{C} = \{t_1, \dots, t_c\}$ . Let  $V_{\mathcal{C}} := \langle t_1 \cup \dots \cup t_c \rangle$  be the linear space generated by the  $\ell$ -cover  $\mathcal{C}$ , where  $\dim(V_{\mathcal{C}}) \leq kr$  by Lemma 3.1. We split the family  $\mathcal{F}$  as  $\mathcal{F} = \mathcal{F}' \cup \mathcal{G}$ , where  $F \in \mathcal{F}$  is assigned to  $\mathcal{F}'$  if  $\dim(F \cap V_{\mathcal{C}}) = \ell$ , and  $F \in \mathcal{G}$  if  $\dim(F \cap V_{\mathcal{C}}) \geq \ell + 1$ . The number of  $(k, \ell)$ -colorings of  $\mathcal{F}$  is clearly bounded by

$$C_{(k,\ell)}(\mathcal{F}) \leq C_{(k,\ell)}(\mathcal{F}') \cdot C_{(k,\ell)}(\mathcal{G}).$$

We may bound the size of  $\mathcal{G}$  by

$$|\mathcal{G}| \leq \binom{\dim(V_{\mathcal{C}})}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q \leq \binom{kr}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q, \quad (7)$$

so that

$$C_{(k,\ell)}(\mathcal{G}) \leq k^{|\mathcal{G}|} \leq k^{\binom{kr}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q}. \quad (8)$$

The contribution of the  $(k, \ell)$ -colorings of  $\mathcal{G}$  will be negligible, since, for  $n$  sufficiently large, the upper bound (7) is much smaller than the largest possible size of  $\mathcal{F}'$ , namely  $\binom{kr}{\ell}_q \cdot \binom{n-kr}{r-\ell}_q$ .

We focus on  $\mathcal{F}'$ . Fix a  $(k, \ell)$ -coloring  $\Delta$  of  $\mathcal{F}'$ . For  $i = 1, \dots, c$ , let  $\mathcal{F}'_i \subseteq \mathcal{F}'$  be the family of all  $r$ -spaces in  $\mathcal{F}'$  containing the cover element  $t_i \in \mathcal{C}$ . For each  $\ell$ -space  $t_i \in \mathcal{C}$ , and each color  $\sigma \in \{1, \dots, k\}$  we say that  $\sigma$  is *substantial* for  $t_i$  with respect to the coloring  $\Delta$  if the number of  $r$ -spaces in  $\mathcal{F}'_i$  with color  $\sigma$  is larger than

$$L = \binom{r-\ell+1}{1}_q \binom{n-\ell-1}{r-\ell-1}_q. \quad (9)$$

We say that color  $\sigma$  is *influential* with respect to the coloring  $\Delta$  if it is substantial for some cover element  $t_i \in \mathcal{C}$ . If color  $\sigma$  is not influential with respect to  $\Delta$ , then it is called *non-influential*.

**Lemma 3.2.** *Let  $\Delta$  be a  $(k, \ell)$ -coloring of  $\mathcal{F}$ . Let color  $\sigma$  be substantial for the cover space  $t_i \in \mathcal{C}$ . Then, for  $n$  sufficiently large, each  $r$ -space  $F$  in  $\mathcal{F}$  with color  $\sigma$  satisfies  $t_i \subseteq F$ .*

*Proof.* Let  $\mathcal{F}'_{i,\sigma}$  be the subfamily of all  $r$ -spaces in  $\mathcal{F}'_i$  that have color  $\sigma$  and contain the cover element  $t_i$ . For a contradiction assume that the  $r$ -space  $F \in \mathcal{F}$  is colored with  $\sigma$ , but  $t_i \not\subseteq F$ . Fix a cover element  $t_{i'}$  satisfying  $t_{i'} \subseteq F$ . Then, by Lemma 2.2(i), the number of  $r$ -spaces in  $\mathcal{F}'_{i,\sigma}$  that are  $\ell$ -intersecting with  $F$  is at most

$$T = \binom{r - \dim(t_i \cap t_{i'})}{\ell - \dim(t_i \cap t_{i'})}_q \binom{n - 2\ell + \dim(t_i \cap t_{i'})}{r - 2\ell + \dim(t_i \cap t_{i'})}_q.$$

Taking the maximum over all possible sizes of  $\dim(t_i \cap t_{i'})$  and choosing  $n$  sufficiently large, we infer that

$$T \leq \max_{0 \leq m \leq \ell-1} \binom{r-m}{\ell-m}_q \binom{n-2\ell+m}{r-2\ell+m}_q = \binom{r-\ell+1}{1}_q \binom{n-\ell-1}{r-\ell-1}_q = L,$$

which contradicts our assumption that color  $\sigma$  is substantial for the cover element  $t_i \in \mathcal{C}$ .  $\square$

**Lemma 3.3.** *Let  $\mathcal{F}$  be a  $(k, \ell)$ -colorable family of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$  with minimum  $\ell$ -cover  $\mathcal{C} = \{t_1, \dots, t_c\}$  of size  $c$ . Consider the class  $\mathcal{F}^*(\sigma)$  of all subfamilies  $\mathcal{F}(\sigma, \Delta) \subseteq \mathcal{F}$  with the property that  $\mathcal{F}(\sigma, \Delta)$  is the family of all  $r$ -spaces colored  $\sigma$  with respect to a  $(k, \ell)$ -coloring  $\Delta$  for which  $\sigma$  is non-influential. Then, we have*

$$|\mathcal{F}^*(\sigma)| \leq 2^c \cdot \binom{n}{r-\ell}_q^{cL} \leq 2^{k \binom{r}{\ell}_q} \cdot \binom{n}{r-\ell}_q^{k \binom{r}{\ell}_q \binom{r-\ell+1}{1}_q \binom{n-\ell-1}{r-\ell-1}_q}.$$

*Proof.* Let  $\mathcal{C} = \{t_1, \dots, t_c\}$  be a minimum  $\ell$ -cover for the family  $\mathcal{F}$ . For  $i = 1, \dots, c$ , there are at most  $\binom{n}{r-\ell}_q$  distinct  $r$ -spaces in  $\mathcal{F}$  that contain the cover element  $t_i$ . For every  $(k, \ell)$ -coloring  $\Delta$  of  $\mathcal{F}$  for which  $\sigma$  is non-influential, the color  $\sigma$  may appear at most  $L$  times among the  $r$ -spaces covered by each cover element  $t_i$ , hence

$$|\mathcal{F}^*(\sigma)| \leq \sum_{(a_1, \dots, a_c)} \left( \prod_{i=1}^c \binom{\binom{n}{r-\ell}_q}{a_i} \right),$$

where the choice of each  $a_i$  in the sum is within the set  $\{0, 1, \dots, L\}$ . Recall that  $c \leq k \binom{r}{\ell}_q$  by Lemma 3.1, and note that  $\binom{n}{m} \leq n^m$  and that  $\sum_{j=0}^L x^j \leq 2x^L$  for every  $x \geq 2$ . We infer that, for  $n > r - \ell$ ,

$$\begin{aligned} |\mathcal{F}^*(\sigma)| &\leq \sum_{(a_1, \dots, a_c)} \binom{n}{r-\ell}_q^{\sum_{i=1}^c a_i} = \left( \sum_{j=0}^L \binom{n}{r-\ell}_q^j \right)^c \\ &\leq 2^c \cdot \binom{n}{r-\ell}_q^{cL} \leq 2^{k \binom{r}{\ell}_q} \cdot \binom{n}{r-\ell}_q^{k \binom{r}{\ell}_q \binom{r-\ell+1}{1}_q \binom{n-\ell-1}{r-\ell-1}_q}. \end{aligned}$$

$\square$

We will now use these results to prove Theorem 1.4(ii), which deals with  $(3, \ell)$ -colorings, and Theorem 1.7(i), which involves  $(4, \ell)$ -colorings.

*Proof of Theorem 1.4(ii).* Let  $\mathcal{F} \in \mathcal{S}_{n,q,r}$  with minimum  $\ell$ -cover  $\mathcal{C} = \{t_1, \dots, t_c\}$ . If  $c = 1$ , Theorem 1.3 tells us that for  $n \geq 2r$  (and hence for large  $n$ )

$$C_{(3,\ell)}(\mathcal{F}) \leq 3^{\binom{n-\ell}{r-\ell}_q}. \quad (10)$$

Moreover for  $n > 2r$  this upper bound is achieved if and only if  $\mathcal{F}$  is  $(\mathcal{C}, r)$ -complete.

We now suppose that  $c \geq 2$ . Recall that  $\mathcal{F}$  may be partitioned as  $\mathcal{F}' \cup \mathcal{G}$  so that all the  $r$ -spaces in  $\mathcal{F}'$  contain a single  $\ell$ -space in  $\mathcal{C}$  and

$$C_{(3,\ell)}(\mathcal{F}) \leq C_{(3,\ell)}(\mathcal{F}') \cdot C_{(3,\ell)}(\mathcal{G}) \leq 3^{\binom{3r}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q} \cdot C_{(3,\ell)}(\mathcal{F}').$$

First consider colorings of  $\mathcal{F}'$  such that all three colors are influential. In light of Lemma 3.2, there are no such colorings unless  $c \leq 3$ . If  $c = 3$ , the three colors must be assigned to distinct cover elements and every  $r$ -space containing a cover element  $t_i$  must be colored with the color assigned to it, which gives at most six colorings of  $\mathcal{F}'$ . If  $c = 2$ , again by Lemma 3.2, we assign the three colors to the cover elements  $t_1$  and  $t_2$  so that each of them receives at least one color (there are six ways of doing this) and we color the  $r$ -spaces containing a cover element  $t_i$  with one of the colors assigned to it, which gives at most

$$6 \cdot 2^{\binom{n-\ell}{r-\ell}_q}$$

$(3, \ell)$ -colorings of  $\mathcal{F}'$ . In particular, with (8) the number of  $(3, \ell)$ -colorings of  $\mathcal{F}$  for which all colors are influential is bounded above by

$$6 \cdot 3^{\binom{3r}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q} \cdot 2^{\binom{n-\ell}{r-\ell}_q} = 6 \left( 3^{\frac{q^{r-\ell-1}-1}{q^{n-\ell-1}-1} \binom{3r}{\ell+1}_q} \cdot 2 \right)^{\binom{n-\ell}{r-\ell}_q},$$

which is much smaller than  $3^{\binom{n-\ell}{r-\ell}_q}$  for large  $n$ .

We now consider colorings for which  $a \leq 2$  colors are influential (and at most three colors are non-influential). Once  $a$  is fixed, we can choose the  $a$  influential colors in  $\binom{3}{a} \leq 3$  ways, and there are at most  $c^2$  possibilities to assign these to the cover elements. This yields at most  $3c^2 \cdot 2^{\binom{n-\ell}{r-\ell}_q}$  ways of coloring  $r$ -spaces with influential colors. With Lemma 3.3, the non-influential colors generate at most

$$\left( 2^{3\binom{r}{\ell}_q} \cdot \binom{n}{r-\ell}_q^{3\binom{r}{\ell}_q \binom{r-\ell+1}{1}_q \binom{n-\ell-1}{r-\ell-1}_q} \right)^3$$

partial colorings of  $\mathcal{F}$ . Combining the number of partial colorings generated by influential and non-influential colors, and taking into account that there are three choices for the value of  $a$ , we obtain the following upper bound on the number of all  $(3, \ell)$ -colorings of  $\mathcal{F}$ :

$$\begin{aligned} & 9c^2 \cdot 2^{\binom{n-\ell}{r-\ell}_q} \cdot \left( 2^{3\binom{r}{\ell}_q} \cdot \binom{n}{r-\ell}_q^{3\binom{r}{\ell}_q \binom{r-\ell+1}{1}_q \binom{n-\ell-1}{r-\ell-1}_q} \right)^3 \cdot 3^{\binom{3r}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q} \\ & \leq 9c^2 \cdot 2^{\binom{n-\ell}{r-\ell}_q} \cdot 2^{9\binom{r}{\ell}_q} \cdot \binom{n}{r-\ell}_q^{9\binom{r}{\ell}_q \binom{r-\ell+1}{1}_q \binom{n-\ell-1}{r-\ell-1}_q} \cdot 3^{\binom{3r}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q}. \end{aligned} \quad (11)$$

We claim that, for  $n$  sufficiently large, (11) is much less than (10). To see why this is true, we use the following inequalities for  $0 \leq r \leq n$ :

$$q^{(n-r)r} \leq \binom{n}{r}_q \leq 2^r \cdot q^{(n-r)r}. \quad (12)$$

This implies that there is a constant  $D > 0$  such that

$$\binom{n}{r-\ell}_q^{\binom{n-\ell-1}{r-\ell-1}_q} \leq \left( 2^{r-\ell} \cdot q^{(n-r+\ell)(r-\ell)} \right)^{2^{(r-\ell-1)} q^{(n-r)(r-\ell-1)}} \leq q^{Dnq^{(n-r)(r-\ell-1)}},$$

so that, for a positive constant  $D'$ , the quantity in (11) is bounded above by

$$\begin{aligned} D'q^{Dnq^{(n-r)(r-\ell-1)}} \left( \frac{q^{r-\ell-1}}{3^{q^{n-\ell-1}}} \binom{3r}{\ell+1}_q \cdot 2 \right)^{\binom{n-\ell}{r-\ell}_q} &\leq \left( \frac{6}{5} \right)^{\binom{n-\ell}{r-\ell}_q} \left( 3 \frac{q^{r-\ell-1}}{q^{n-\ell-1}} \binom{3r}{\ell+1}_q \cdot 2 \right)^{\binom{n-\ell}{r-\ell}_q} \\ &\leq \left( \frac{5}{2} \right)^{\binom{n-\ell}{r-\ell}_q}, \end{aligned}$$

which is much smaller than  $3^{\binom{n-\ell}{r-\ell}_q}$  for large  $n$ . For this we used the lower bound in (12):

$$\frac{q^{Dnq^{(n-r)(r-\ell-1)}}}{\left( \frac{6}{5} \right)^{\binom{n-\ell}{r-\ell}_q}} \leq \frac{\left( q^{q^{(n-r)(r-\ell-1)}} \right)^{Dn}}{\left( \left( \frac{6}{5} \right)^{q^{(n-r)(r-\ell-1)}} \right)^{q^{n-r}}} \rightarrow 0 \quad \text{for } n \rightarrow \infty. \quad (13)$$

□

We shall now consider the case of  $k = 4$  colors. To prove Theorem 1.7(i), we use the following strategy. We first show that, if a family allows a large number of distinct  $(4, \ell)$ -colorings, then most colorings are such that all colors are influential. We then argue that extremality is achieved for families whose minimum  $\ell$ -cover  $\mathcal{C}$  has size two, and which are  $(\mathcal{C}, r)$ -complete.

**Lemma 3.4.** *Let  $\mathcal{F}$  be a family of  $r$ -spaces over  $GF(q)$ . Let  $\Lambda$  be the class of all  $(4, \ell)$ -colorings of  $\mathcal{F}$  for which at least one of the colors is non-influential. Then there exist  $0 < \gamma < 1$  and  $n_0$  such that, for  $n \geq n_0$ , we have*

$$|\Lambda| \leq 4^{(1-\gamma)\binom{n-\ell}{r-\ell}_q}.$$

*Proof.* For a fixed family  $\mathcal{F}$  of  $r$ -spaces over  $GF(q)$ , consider the family  $\Lambda$  of  $(4, \ell)$ -colorings of  $\mathcal{F}$  for which at least one of the colors is non-influential. Let  $\mathcal{C} = \{t_1, \dots, t_c\}$  be a minimum  $\ell$ -cover of  $\mathcal{F}$  and, as before, let  $\mathcal{F}'$  denote the subfamily of  $\mathcal{F}$  containing all  $r$ -spaces whose intersection with  $V_{\mathcal{C}} = \langle t_1 \cup \dots \cup t_c \rangle$  has dimension  $\ell$ . The family  $\mathcal{G} = \mathcal{F} \setminus \mathcal{F}'$  is given by all  $F \in \mathcal{F}$  with  $\dim(F \cap V_{\mathcal{C}}) \geq \ell + 1$ . Let  $\Lambda'$  contain the restrictions of the colorings in  $\Lambda$  to  $\mathcal{F}'$ . We have

$$|\Lambda| \leq 4^{|\mathcal{G}|} \cdot |\Lambda'| \leq 4^{\binom{4r}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q} \cdot |\Lambda'|.$$

We bound the number of  $(4, \ell)$ -colorings of  $\Lambda'$ . By Lemma 3.3 the number of possible partial colorings of  $\mathcal{F}'$  with color  $\sigma$  so that  $\sigma$  is not influential is at most

$$2^{4\binom{r}{\ell}_q} \cdot \binom{n}{r-\ell}_q^{4\binom{r}{\ell}_q \binom{r-\ell+1}{1}_q \binom{n-\ell-1}{r-\ell-1}_q}.$$

Let  $1 \leq i \leq 4$  denote the number of non-influential colors used in a partial coloring of  $\Lambda'$ . Then there are at most

$$\sum_{i=1}^4 \binom{4}{i} 2^{4i\binom{r}{\ell}_q} \cdot \binom{n}{r-\ell}_q^{4i\binom{r}{\ell}_q \binom{r-\ell+1}{1}_q \binom{n-\ell-1}{r-\ell-1}_q} \leq 15 \cdot 2^{16\binom{r}{\ell}_q} \cdot \binom{n}{r-\ell}_q^{16\binom{r}{\ell}_q \binom{r-\ell+1}{1}_q \binom{n-\ell-1}{r-\ell-1}_q} \quad (14)$$

distinct partial colorings induced by non-influential colors.

Now suppose that we have  $j \leq 3$  influential colors. By Lemma 3.2, all the  $r$ -spaces colored with each influential color must be covered by the same cover element, so that we may assume that influential colors are assigned to cover elements. If all colors are assigned to the same cover element, there are at most  $j^{\binom{n-\ell}{r-\ell}_q}$  ways to color the  $r$ -spaces in  $\mathcal{F}'$  covered by it. Analogously, if we assign  $j_1 \geq 1$  colors to the  $r$ -spaces that contain some cover space  $t_1$  and

$j_2 \geq 1$  colors to the  $r$ -spaces that contain a second cover space  $t_2$ , then, once the colors are fixed, the number of possible colorings of the  $r$ -spaces in  $\mathcal{F}'$  covered by  $t_1$  or  $t_2$  is at most

$$j_1^{\binom{n-\ell}{r-\ell}_q} \cdot j_2^{\binom{n-\ell}{r-\ell}_q}. \quad (15)$$

Note that (15) is smaller than  $3^{\binom{n-\ell}{r-\ell}_q}$  whenever  $j_1 + j_2 \leq 3$ . Finally, if we assign  $j_i \geq 1$  colors to three cover spaces  $t_i$ ,  $i = 1, 2, 3$ , then  $j_1 = j_2 = j_3 = 1$ , so that, once the colors are fixed, there is a single way of using them to color the  $r$ -spaces in  $\mathcal{F}'$  covered by  $t_1$ ,  $t_2$  or  $t_3$ , which gives another six colorings.

Summarizing the above (and considering that the influential colors may be assigned in at most  $c^3$  ways, as at least one of the colors is non-influential), we infer that, for some  $\gamma > 0$ ,

$$|\Lambda'| \leq 15c^3 \cdot 2^{16\binom{r}{\ell}_q} \cdot \binom{n}{r-\ell}_q^{16\binom{r}{\ell}_q \binom{r-\ell+1}{1}_q \binom{n-\ell-1}{r-\ell-1}_q} \cdot 3^{\binom{n-\ell}{r-\ell}_q} \leq 4^{(1-2\gamma)\binom{n-\ell}{r-\ell}_q}$$

for  $n$  large enough. Clearly, the value of  $\gamma$  may be chosen so that it also satisfies  $4^{|\mathcal{G}|} \leq 4^{\gamma\binom{n-\ell}{r-\ell}_q}$  for  $n$  large, which implies our result, i.e.,  $|\Lambda| \leq 4^{(1-\gamma)\binom{n-\ell}{r-\ell}_q}$ .  $\square$

**Lemma 3.5.** *Let  $\mathcal{F}$  be a family of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$  with minimum  $\ell$ -cover  $\mathcal{C} = \{t_1, \dots, t_c\}$ , where  $n$  is large. Let  $\mathcal{F}'$  and  $\mathcal{F}'_i$  be defined as above. If  $C_{(4,\ell)}(\mathcal{F}) \geq 4^{\binom{n-\ell}{r-\ell}_q}$ , then for each cover element  $t_i$  we have  $|\mathcal{F}'_i| > 4L$ ,  $i = 1, \dots, c$ .*

*Proof.* Let  $\mathcal{F}$  be a family of  $r$ -spaces in  $V_n$  with minimum  $\ell$ -cover  $\mathcal{C} = \{t_1, \dots, t_c\}$ , and suppose that  $|\mathcal{F}'_c| \leq 4L$ . By (8) we know that

$$C_{(4,\ell)}(\mathcal{F}) \leq 4^{\binom{4r}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q} \cdot C_{(4,\ell)}(\mathcal{F}'),$$

and from now on we focus on an upper bound on  $C_{(4,\ell)}(\mathcal{F}')$ .

Let  $\Lambda_1$  be the family of all  $(4, \ell)$ -colorings of  $\mathcal{F}'$  for which at least one color is influential for the cover element  $t_c$ , and let  $\Lambda_2$  contain all remaining  $(4, \ell)$ -colorings, i.e.,  $C_{(4,\ell)}(\mathcal{F}') = |\Lambda_1| + |\Lambda_2|$ . First consider colorings in  $\Lambda_1$ . Let  $\hat{\mathcal{F}}'$  denote the family  $\mathcal{F}' \setminus \{F \in \mathcal{F}' : t_c \subseteq F\}$ , that is, we omit from  $\mathcal{F}'$  all  $r$ -spaces which contain the cover space  $t_c$ . Any coloring  $\Delta$  of  $\mathcal{F}'$  induces a coloring  $\hat{\Delta}$  of  $\hat{\mathcal{F}}'$ . Let  $\hat{\Lambda}_1$  be the set of all such  $(4, \ell)$ -colorings  $\hat{\Delta}$ . If color  $\sigma$  is substantial for  $t_c$  with respect to  $\Delta$ , then it cannot be substantial for any other cover space  $t_i$ ,  $i \neq c$ , hence, due to Lemma 3.2, color  $\sigma$  cannot be influential with respect to  $\hat{\Delta}$ . Therefore, by Lemma 3.4 there is  $0 < \gamma < 1$  such that

$$|\hat{\Lambda}_1| \leq 4^{(1-\gamma)\binom{n-\ell}{r-\ell}_q}$$

for  $n$  sufficiently large. Each coloring of  $\hat{\Lambda}_1$  corresponds to at most  $4^{4L}$  colorings of  $\Lambda_1$  as  $|\mathcal{F}'_c| \leq 4L$  by assumption. Since  $L \leq \frac{\gamma}{8} \binom{n-\ell}{r-\ell}_q$  for  $n$  sufficiently large, we infer that

$$|\Lambda_1| \leq 4^{4L} \cdot |\hat{\Lambda}_1| \leq 4^{\frac{\gamma}{2} \binom{n-\ell}{r-\ell}_q} 4^{(1-\gamma)\binom{n-\ell}{r-\ell}_q} = 4^{(1-\frac{\gamma}{2})\binom{n-\ell}{r-\ell}_q}. \quad (16)$$

Next consider the class  $\Lambda_2$ . For a fixed coloring  $\Delta \in \Lambda_2$ , let  $\sigma$  be the color of some  $r$ -space whose single subspace in the cover is the element  $t_c$ . Since by assumption  $\sigma$  is not substantial for  $t_c$ , by Lemma 3.2 it cannot be influential with respect to the coloring  $\Delta$ . Lemma 3.4 applied to the family  $\mathcal{F}'$  tells us that

$$|\Lambda_2| \leq 4^{(1-\gamma)\binom{n-\ell}{r-\ell}_q}$$

with the same value  $\gamma$  as in (16). For  $n$  large, combining (16) and (17), we obtain

$$C_{(4,\ell)}(\mathcal{F}') = |\Lambda_1| + |\Lambda_2| \leq 4^{(1-\frac{\gamma}{2})\binom{n-\ell}{r-\ell}_q} + 4^{(1-\gamma)\binom{n-\ell}{r-\ell}_q} \leq 4^{(1-\frac{\gamma}{3})\binom{n-\ell}{r-\ell}_q}, \quad (17)$$

hence with (8) we conclude that

$$C_{(4,\ell)}(\mathcal{F}) \leq 4^{|\mathcal{G}|} \cdot C_{(4,\ell)}(\mathcal{F}') \leq 4^{\binom{4r}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q} 4^{(1-\frac{\gamma}{3})\binom{n-\ell}{r-\ell}_q} \leq 4^{(1-\frac{\gamma}{4})\binom{n-\ell}{r-\ell}_q} < 4^{\binom{n-\ell}{r-\ell}_q}, \quad (18)$$

as required.  $\square$

Next we show that in a  $(4, \ell)$ -extremal family the size of a minimum  $\ell$ -cover is at most two.

**Lemma 3.6.** *Let  $\mathcal{F}$  be a family of  $r$ -spaces in an  $n$ -space over  $GF(q)$ , where  $\mathcal{F}$  has minimum  $\ell$ -cover  $\mathcal{C}$ . For  $n$  sufficiently large, if  $C_{(4,\ell)}(\mathcal{F}) \geq 4^{\binom{n-\ell}{r-\ell}_q}$ , then  $|\mathcal{C}| \leq 2$ .*

*Proof.* Let  $\mathcal{F}$  be a family of  $r$ -spaces in  $V_n$  with minimum  $\ell$ -cover  $\mathcal{C} = \{t_1, \dots, t_c\}$ , where  $c \geq 3$ . By Lemma 3.5 we may assume that  $|\mathcal{F}'_i| > 4L$ ,  $i = 1, \dots, c$ . By the pigeonhole principle, for each cover space  $t_i$  there is a color  $\sigma_i$  that is substantial for  $t_i$ . Each color is substantial for at most one cover space, hence  $c \leq 4$ .

If  $c = 4$ , Lemma 3.2 tells us that in any  $(4, \ell)$ -coloring each family  $\mathcal{F}'_i$  is monochromatic for  $i = 1, \dots, c$ , so that, for  $n$  large, we have

$$C_{(4,\ell)}(\mathcal{F}) \leq 4^{\binom{4r}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q} \cdot 4! \ll 4^{\binom{n-\ell}{r-\ell}_q}.$$

If  $c = 3$ , three colors are needed for the three families  $\mathcal{F}'_1, \mathcal{F}'_2$  and  $\mathcal{F}'_3$  while the fourth color is either non-influential or is used for coloring more than  $L$  distinct  $r$ -spaces in some family  $\mathcal{F}'_i$ . In the first case we may apply Lemma 3.4 with some  $0 < \gamma < 1$  and in the second case we use (8) to bound the number of colorings directly, so that, for  $n$  sufficiently large, we have

$$C_{(4,\ell)}(\mathcal{F}) \leq 4^{(1-\gamma)\binom{n-\ell}{r-\ell}_q} + 4^{\binom{4r}{\ell+1}_q \binom{n-\ell-1}{r-\ell-1}_q} 2^{\binom{n-\ell}{r-\ell}_q} \cdot \binom{4}{2} \cdot 3 \ll 4^{\binom{n-\ell}{r-\ell}_q},$$

establishing our result.  $\square$

We are now ready to prove Theorem 1.7(i). To accomplish this task, we consider a particular set of families of  $r$ -spaces and a particular class of colourings.

**Definition 3.7.** *Let  $n > r > \ell$  be positive integers, let  $q$  be a prime power, and fix a family  $\mathcal{C}$  of  $\ell$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$ . The  $(\mathcal{C}, r)$ -complete family of  $r$ -spaces with  $\ell$ -cover  $\mathcal{C}$ , which we denote  $\mathcal{F}_{(n,q,r,\mathcal{C})}$ , is the family of all  $r$ -spaces in  $V_n$  that contain an  $\ell$ -space in  $\mathcal{C}$ .*

**Definition 3.8.** *Let  $\mathcal{F}$  be a family of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$  with  $\ell$ -cover  $\mathcal{C} = \{t_1, \dots, t_c\}$ . A star coloring of  $\mathcal{F}$  is a  $(k, \ell)$ -coloring such that, for every color  $\sigma$ , all the  $r$ -spaces in  $\mathcal{F}$  with color  $\sigma$  contain some fixed element  $t_i = t_i(\sigma)$  of the  $\ell$ -cover  $\mathcal{C}$ . A  $(k, \ell)$ -coloring of  $\mathcal{F}$  that is not a star coloring is called a non-star coloring.*

*Proof of Theorem 1.7(i).* By Lemma 3.6, we know that we may restrict our attention to families with minimum  $\ell$ -cover  $\mathcal{C}$  satisfying  $|\mathcal{C}| \leq 2$ . For an  $\ell$ -cover  $\mathcal{C}$ , let  $\mathcal{F}_{(n,q,r,\mathcal{C})}$  be the  $(\mathcal{C}, r)$ -complete family of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$  with  $\ell$ -cover  $\mathcal{C}$ . Clearly, if  $|\mathcal{C}| = 1$ , we have  $C_{(4,\ell)}(\mathcal{F}_{(n,q,r,\mathcal{C})}) = 4^{\binom{n-\ell}{r-\ell}_q}$ . Thus our result follows if we show that, for an arbitrary set  $\mathcal{C} = \{t_1, t_2\}$  of  $\ell$ -spaces in  $V_n$ , where  $n$  is sufficiently large, we have

$$C_{(4,\ell)}(\mathcal{F}_{(n,q,r,\mathcal{C})}) \geq 6 \cdot 4^{\binom{n-\ell}{r-\ell}_q}, \quad (19)$$

and if we show that  $C_{(4,\ell)}(\mathcal{F}) < 6 \cdot 4^{\binom{n-\ell}{r-\ell}_q}$  whenever  $\mathcal{F}$  is not  $(\mathcal{C}, r)$ -complete.

Let  $\dim(t_1 \cap t_2) = y \leq \ell - 1$ . We shall first consider star colorings of the  $(\mathcal{C}, r)$ -complete family  $\mathcal{F}_{(n,q,r,\mathcal{C})}$ . To construct the colorings, we may first assign colors to the cover elements and then use them to color the  $r$ -spaces containing each cover element. The  $(\mathcal{C}, r)$ -completeness of  $\mathcal{F}_{(n,q,r,\mathcal{C})}$  implies that, for large  $n$ , there must be at least one substantial color for each cover

element, so that we may assign two colors to each cover space, or three colors to one cover space and one color to the other.

Let  $\mathcal{F}_1^* = \{U \in \mathcal{F}; t_1 \subset U, t_2 \not\subset U\}$  and  $\mathcal{F}_2^* = \{U \in \mathcal{F}; t_1 \not\subset U, t_2 \subset U\}$  be the sets of  $r$ -spaces containing exactly one of the cover spaces. We begin with the case where two colors are assigned to each cover space, which can be done in  $\binom{4}{2}$  ways. We first assume that  $y \geq 2\ell - r$ , which implies that there exist  $r$ -spaces in  $\mathcal{F}$  containing  $t_1 \cup t_2$ , namely  $\binom{n-2\ell+y}{r-2\ell+y}_q$  of them. The number of 4-colorings of the  $r$ -spaces that contain  $t_1 \cup t_2$  is  $4^{\binom{n-2\ell+y}{r-2\ell+y}_q}$ . We may color the  $r$ -spaces in  $\mathcal{F}_1^*$  in  $2^{\binom{n-\ell}{r-\ell}_q - \binom{n-2\ell+y}{r-2\ell+y}_q}$  ways (the same holds for  $r$ -spaces in  $\mathcal{F}_2^*$ ). Note that, in this argument, the colorings for which both  $\mathcal{F}_1^*$  and  $\mathcal{F}_2^*$  are monochromatic are counted exactly twice, so that we need to subtract the term  $\binom{4}{2} \cdot 2 \cdot 4^{\binom{n-2\ell+y}{r-2\ell+y}_q}$ . This amounts to

$$\binom{4}{2} \cdot 2^2 \binom{\binom{n-\ell}{r-\ell}_q - \binom{n-2\ell+y}{r-2\ell+y}_q}{2} 4^{\binom{n-2\ell+y}{r-2\ell+y}_q} - \binom{4}{2} \cdot 2 \cdot 4^{\binom{n-2\ell+y}{r-2\ell+y}_q} = 6 \cdot 4^{\binom{n-\ell}{r-\ell}_q} - 12 \cdot 4^{\binom{n-2\ell+y}{r-2\ell+y}_q} \quad (20)$$

colorings. In the case when  $y < 2\ell - r$ , that is, in the case when  $\dim(\langle t_1 \cup t_2 \rangle) > r$  and no  $r$ -space contains both cover spaces, the same formula holds observing that  $\binom{n-2\ell+y}{r-2\ell+y}_q = 0$ .

We now consider the star colorings for which three colors are assigned either to  $\mathcal{F}_1^*$  or to  $\mathcal{F}_2^*$ . The number of ways of assigning the colors is  $2\binom{4}{3}$ , and we need to count the number of 3-colorings of the  $r$ -spaces in  $\mathcal{F}_1^*$  (or in  $\mathcal{F}_2^*$ ) for which the three colors are used, as the other colorings have already been counted in the case where two colors were assigned to each cover space. By inclusion-exclusion, we see that there are

$$2\binom{4}{3} \left( 3^{\binom{n-\ell}{r-\ell}_q - \binom{n-2\ell+y}{r-2\ell+y}_q} - 3 \cdot 2^{\binom{n-\ell}{r-\ell}_q - \binom{n-2\ell+y}{r-2\ell+y}_q} + 3 \right) 4^{\binom{n-2\ell+y}{r-2\ell+y}_q} \quad (21)$$

such colorings. Again, this formula holds for all values of  $y$ , but it does not depend on  $y$  if  $y < 2\ell - r$ , since  $\binom{n-2\ell+y}{r-2\ell+y}_q = 0$  in this case.

Combining equations (20) and (21), we deduce that the number  $T(y)$  of star colorings of  $\mathcal{F}$  is given by

$$\begin{aligned} T(y) &= 6 \cdot 4^{\binom{n-\ell}{r-\ell}_q} - 12 \cdot 4^{\binom{n-2\ell+y}{r-2\ell+y}_q} + \\ &\quad + 8 \left( 3^{\binom{n-\ell}{r-\ell}_q - \binom{n-2\ell+y}{r-2\ell+y}_q} - 3 \cdot 2^{\binom{n-\ell}{r-\ell}_q - \binom{n-2\ell+y}{r-2\ell+y}_q} + 3 \right) 4^{\binom{n-2\ell+y}{r-2\ell+y}_q} \\ &= 6 \cdot 4^{\binom{n-\ell}{r-\ell}_q} + \\ &\quad + 8 \cdot 3^{\binom{n-\ell}{r-\ell}_q} \left( \frac{4}{3} \right)^{\binom{n-2\ell+y}{r-2\ell+y}_q} \left( 1 - 3 \left( \frac{2}{3} \right)^{\binom{n-\ell}{r-\ell}_q - \binom{n-2\ell+y}{r-2\ell+y}_q} + \frac{3/2}{3^{\binom{n-\ell}{r-\ell}_q - \binom{n-2\ell+y}{r-2\ell+y}_q}} \right). \end{aligned} \quad (22)$$

We know that  $y \leq \ell - 1$  and that the term within brackets in (22) lies between  $7/8$  and  $1$  for  $n$  sufficiently large, so that

$$6 \cdot 4^{\binom{n-\ell}{r-\ell}_q} + 7 \cdot 3^{\binom{n-\ell}{r-\ell}_q} \left( \frac{4}{3} \right)^{\binom{n-2\ell+y}{r-2\ell+y}_q} \leq T(y) \leq 6 \cdot 4^{\binom{n-\ell}{r-\ell}_q} + 8 \cdot 3^{\binom{n-\ell}{r-\ell}_q} \left( \frac{4}{3} \right)^{\binom{n-2\ell+y}{r-2\ell+y}_q} \quad (23)$$

Hence, for large  $n$ , we have  $T(y-1) \leq T(y)$  for every  $y$ , with equality if and only if  $\binom{n-2\ell+y}{r-2\ell+y}_q = 0$ , that is, if and only if there are no  $r$ -spaces containing  $t_1 \cup t_2$  when  $\dim(t_1 \cap t_2) = y$ . Observe that, for  $y = \ell - 1$ , we have  $r - 2\ell + y = r - \ell - 1 \geq 0$ , so that  $\binom{n-2\ell+y}{r-2\ell+y}_q \geq 1$  in this case. (In particular,  $T(\ell - 1) > T(\ell - 2)$ .) Therefore  $T(y)$  is maximum for  $y = \ell - 1$  and minimum for

$y = 0$  for  $n$  sufficiently large. Since  $T(0) \geq 6 \cdot 4^{\binom{n-\ell}{r-\ell}_q}$ , the number of distinct  $(k, \ell)$ -colorings of  $\mathcal{F}_{(n,q,r,C)}$  is at least  $6 \cdot 4^{\binom{n-\ell}{r-\ell}_q}$ .

To conclude our proof, we argue that, if  $\mathcal{F}$  has minimum  $\ell$ -cover  $\mathcal{C} = \{t_1, t_2\}$ , but is not  $(\mathcal{C}, r)$ -complete, then it has fewer than  $6 \cdot 4^{\binom{n-\ell}{r-\ell}_q}$  distinct  $(k, \ell)$ -colorings.

By Lemma 3.5 we may assume that the number of  $r$ -spaces covered exclusively by each  $t_i$  is bigger than  $4L$ . Let  $R$  be an  $r$ -space containing  $t_i \in \mathcal{C}$  such that  $R \notin \mathcal{F}$ . Let  $\mathcal{F}'_i$  be defined as before. Let  $\Delta$  be a  $(4, \ell)$ -coloring of  $\mathcal{F}$ . At least one color, say  $\sigma$ , appears more than  $L$  times in  $\mathcal{F}'_i$  by the pigeonhole principle. By Lemma 3.2 all  $r$ -spaces in  $\mathcal{F}$  that are colored  $\sigma$  by  $\Delta$  must contain  $t_i$ , so the coloring  $\Delta$  may be extended to a  $(4, \ell)$ -coloring of  $\mathcal{F} \cup \{R\}$  by assigning  $\sigma$  to  $R$ . For the particular coloring  $\Delta$  that colors all  $r$ -spaces in  $\mathcal{F}'_1$  with color 1 and all  $r$ -spaces in  $\mathcal{F}'_2$  with color 2, we have at least three possibilities to color  $R$ , namely with color  $i \in \{1, 2\}$  or with color 3 or 4, hence  $C_{(4,\ell)}(\mathcal{F} \cup \{R\}) > C_{(4,\ell)}(\mathcal{F})$ . Moreover, Lemma 3.4 implies that the great majority of colorings of a  $(4, \ell)$ -extremal family are such that every color is influential. It is easy to see that most colorings assign two colors to each of the cover elements. This implies that, for most of the colorings of  $\mathcal{F}$  considered above, we have at least two different ways of extending these to a coloring of  $\mathcal{F} \cup \{R\}$ . In particular, it is easy to prove that

$$C_{(4,\ell)}(\mathcal{F} \cup \{R\}) > \frac{3}{2} \cdot C_{(4,\ell)}(\mathcal{F}).$$

This implies that  $C_{(4,\ell)}(\mathcal{F}) \leq \frac{2}{3} C_{(4,\ell)}(\mathcal{F}_{(n,q,r,C)})$ . For large  $n$ , our result follows from the inequality

$$C_{(4,\ell)}(\mathcal{F}_{(n,q,r,C)}) \leq 6 \cdot 4^{\binom{n-\ell}{r-\ell}_q} + 8 \cdot 3^{\binom{n-\ell}{r-\ell}_q} \left(\frac{4}{3}\right)^{\binom{n-\ell-1}{r-\ell-1}_q} + 4^{(1-\gamma)\binom{n-\ell}{r-\ell}_q},$$

which is a consequence of the upper bounds on star colorings and non-star colorings given (23) and Lemma 3.4, respectively.  $\square$

#### 4. UNIQUENESS OF THE EXTREMAL CONFIGURATION FOR FOUR COLORS

In this section, we shall prove Theorem 1.7(ii), that is, we shall establish that the  $\ell$ -spaces in the minimum  $\ell$ -cover of a  $(4, \ell)$ -extremal family of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$  intersect in a space of dimension  $\ell - 1$ , i.e.,  $\mathcal{F} \in \mathcal{E}_{n,q,r,\ell,2,\ell-1}$ . To this end, we count the number of  $(4, \ell)$ -colorings of a family  $\mathcal{F}$  of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$  with high accuracy.

*Proof of Theorem 1.7(ii).* We now show that for a cover  $\mathcal{C} = \{t_1, t_2\}$  of  $\ell$ -spaces, if  $\dim(t_1 \cap t_2) = y < \ell - 1$ , then the  $(\mathcal{C}, r)$ -complete family  $\mathcal{F}^* = \mathcal{F}_{(n,q,r,C)}$  is not  $(4, \ell)$ -extremal. To this end, we find an upper bound on the number of non-star colorings of  $\mathcal{F}^*$ , and we show that this number is smaller than the gap between  $T(\ell - 1)$  and  $T(\ell - 2)$ , where  $T(y)$  is the number of star colorings of  $\mathcal{F}^* = \mathcal{F}_{(n,q,r,C)}$  when  $\dim(t_1 \cap t_2) = y$ .

For a non-star coloring  $\Delta$ , there exists at least one pair  $(X, Z)$  of  $r$ -spaces  $X, Z \in \mathcal{F}^*$  of the same color such that  $\dim(X \cap Z) \geq \ell$  with  $t_1 \subset X$  and  $t_2 \subset Z$ , but  $t_1 \cup t_2 \not\subseteq X$  and  $t_1 \cup t_2 \not\subseteq Z$ . Let  $\dim(X \cap t_2) = p_2 + y$  and  $\dim(Z \cap t_1) = p_1 + y$ , where we may assume  $p_1 \leq p_2$  by symmetry, thus

$$p_1 + y \leq \ell - 1 \quad \text{and} \quad p_2 + y \leq \ell - 1. \quad (24)$$

Note that, since  $X$  contains the vector space  $\langle (X \cap Z) \cup t_1 \rangle$ , we have  $r \geq \dim(X \cap Z) + \dim(t_1) - \dim(t_1 \cap Z) \geq \ell + \ell - (p_1 + y) = 2\ell - p_1 - y$ . Similarly we obtain  $r \geq 2\ell - p_2 - y$ .

Let

$$\mathcal{F}_1(Z) = \{U \in \mathcal{F}^* \mid t_1 \subset U, t_2 \not\subset U, \Delta(U) = \Delta(Z)\} \quad (25)$$

$$\mathcal{F}_2(X) = \{U \in \mathcal{F}^* \mid t_2 \subset U, t_1 \not\subset U, \Delta(U) = \Delta(X)\}. \quad (26)$$

**Lemma 4.1.** *There exist constants  $C > 0$  and  $n_0 > 0$  such that, for all integers  $n \geq n_0$ , we have*

$$|\mathcal{F}_1(Z)| \leq Cq^{n(r-2\ell+p_1+y)} \quad \text{and} \quad |\mathcal{F}_2(X)| \leq Cq^{n(r-2\ell+p_2+y)}. \quad (27)$$

*Proof.* Every  $r$ -space  $U$  in  $\mathcal{F}_1(Z)$  contains  $t_1$  and satisfies  $\dim(U \cap Z) \geq \ell$ . The number of  $\ell$ -spaces  $W$  in  $Z$  that contain the  $(p_1 + y)$ -space  $Z \cap t_1$  is at most  $\binom{r-(p_1+y)}{\ell-(p_1+y)}_q$ . The number of  $r$ -spaces  $U$  in  $V_n$  that contain the space  $\langle t_1 \cup W \rangle$  is at most  $\binom{n-(2\ell-p_1-y)}{r-(2\ell-p_1-y)}_q$ . Thus, with (12) there is a constant  $C > 0$  such that

$$|\mathcal{F}_1(Z)| \leq \binom{r-(p_1+y)}{\ell-(p_1+y)}_q \binom{n-(2\ell-p_1-y)}{r-(2\ell-p_1-y)}_q \leq Cq^{n(r-2\ell+p_1+y)}.$$

The second may be derived similarly, namely

$$|\mathcal{F}_2(X)| \leq \binom{r-(p_2+y)}{\ell-(p_2+y)}_q \binom{n-(2\ell-p_2-y)}{r-(2\ell-p_2-y)}_q \leq Cq^{n(r-2\ell+p_2+y)}.$$

□

Next, with the  $r$ -spaces  $X$  and  $Z$  fixed, subfamilies  $\mathcal{G}_1 \subseteq \mathcal{F}_1(Z)$  and  $\mathcal{G}_2 \subseteq \mathcal{F}_2(X)$  may be assigned the same color as  $Z$  and  $X$ , provided that  $\dim(X' \cap Z') \geq \ell$  for all  $r$ -spaces  $X' \in \mathcal{G}_1$  and  $Z' \in \mathcal{G}_2$ .

By (12) the  $r$ -spaces  $X$  and  $Z$  may be chosen in at most  $\binom{n-\ell}{r-\ell}_q^2 \leq 2^{2r-2\ell} \cdot q^{(n-r)(2r-2\ell)} \leq q^{n(2r-2\ell)}$  ways, the subfamilies  $\mathcal{G}_1$  and  $\mathcal{G}_2$  may be fixed in at most  $2^{|\mathcal{F}_1(Z)|+|\mathcal{F}_2(X)|}$  ways and their color may be chosen in four ways. With Lemma 4.1 we deduce that, for  $n$  large (recall that  $p_1 \leq p_2$ ), the total number of choices for  $X$ ,  $Z$ ,  $\mathcal{F}_1(Z)$  and  $\mathcal{F}_2(X)$  is at most

$$\begin{aligned} 4q^{n(2r-2\ell)} \cdot 2^{|\mathcal{F}_1(Z)|+|\mathcal{F}_2(X)|} &\leq 4q^{n(2r-2\ell)} \cdot 2^{Cq^{n(r-2\ell+p_1+y)}+Cq^{n(r-2\ell+p_2+y)}} \\ &\leq 4q^{n(2r-2\ell)} \cdot 2^{2Cq^{n(r-2\ell+p_2+y)}}. \end{aligned} \quad (28)$$

Having fixed the color of  $X$  and  $Z$  and of all  $r$ -spaces in  $\mathcal{G}_1 \cup \mathcal{G}_2$ , we may use this color only for  $r$ -spaces containing  $t_1 \cup t_2$ , which appear if  $r \geq 2\ell - y$ . We may finish the coloring in two ways.

On the one hand, we may use the remaining three colors for a star coloring of the set of uncolored  $r$ -spaces. This can be done in at most

$$2 \binom{3}{2} \cdot 2^{\binom{n-\ell}{r-\ell}_q - \binom{n-2\ell+y}{r-2\ell+y}_q} 4^{\binom{n-2\ell+y}{r-2\ell+y}_q} = 6 \cdot 2^{\binom{n-\ell}{r-\ell}_q + \binom{n-2\ell+y}{r-2\ell+y}_q} \quad (29)$$

ways; hence, with (28) and (29), for  $n$  sufficiently large, the number of such non-star colorings is at most

$$24q^{n(2r-2\ell)} \cdot 2^{2Cq^{n(r-2\ell+p_2+y)}} \cdot 2^{\binom{n-\ell}{r-\ell}_q + \binom{n-2\ell+y}{r-2\ell+y}_q}. \quad (30)$$

On the other hand, assume that there is a second pair of  $r$ -spaces  $(X_1, Z_1)$  colored differently from  $(X, Z)$  where  $X_1, Z_1 \in \mathcal{F}^*$  with  $\dim(X_1 \cap Z_1) \geq \ell$  and  $t_1 \subset X_1, t_2 \subset Z_1$ , but  $t_1 \cup t_2 \not\subseteq X_1$  and  $t_1 \cup t_2 \not\subseteq Z_1$ . Also assume that  $\dim(X_1 \cap t_2) = p'_2 + y$  and  $\dim(Z_1 \cap t_1) = p'_1 + y$ , say  $p'_1 \leq p'_2$ , where  $\Delta(X_1) = \Delta(Z_1)$ . Let the families  $\mathcal{F}_1(Z_1)$  and  $\mathcal{F}_2(X_1)$  be defined as in (25) and (26). As above,  $r$ -spaces in subfamilies  $\mathcal{G}'_1 \subseteq \mathcal{F}_1(Z_1)$  and  $\mathcal{G}'_2 \subseteq \mathcal{F}_2(X_1)$  may be assigned the same color as  $X_1$  and  $Z_1$ , provided that  $\dim(X' \cap Z') \geq \ell$  for all  $r$ -spaces  $X' \in \mathcal{G}'_1$  and  $Z' \in \mathcal{G}'_2$ . The pair  $(X_1, Z_1)$  may be chosen in at most  $\binom{n-r}{r-\ell}_q^2 \leq q^{n(2r-2\ell)}$  ways and there are

three choices for the color. Again by Lemma 4.1 with  $p'_1 \leq p'_2$ , for  $n$  sufficiently large, we have that, for some constant  $C' > 0$ ,

$$\begin{aligned} 3q^{n(2r-2\ell)} \cdot 2^{|\mathcal{F}_1(Z_1)|+|\mathcal{F}_2(X_1)|} &\leq 3q^{n(2r-2\ell)} \cdot 2^{C'q^{n(r-2\ell+p'_1+y)}+C'q^{n(r-2\ell+p'_2+y)}} \\ &\leq 3q^{n(2r-2\ell)} \cdot 2^{2C'q^{n(r-2\ell+p'_2+y)}}. \end{aligned} \quad (31)$$

Combining (28) and (31) gives us, for  $n$  sufficiently large, at most

$$\begin{aligned} &4q^{n(2r-2\ell)} \cdot 2^{2Cq^{n(r-2\ell+p_2+y)}} \cdot 2 \cdot 3 \cdot q^{n(2r-2\ell)} \cdot 2^{2C'q^{n(r-2\ell+p'_2+y)}} \cdot 4^{\binom{n-2\ell+y}{r-2\ell+y}_q} \\ &= 24q^{n(4r-4\ell)} \cdot 2^{2Cq^{n(r-2\ell+p_2+y)}+2C'q^{n(r-2\ell+p'_2+y)}} \cdot 4^{\binom{n-2\ell+y}{r-2\ell+y}_q} \end{aligned} \quad (32)$$

such non-star colorings, since the two colors not used so far can be taken for those remaining  $r$ -spaces not containing  $t_1 \cup t_2$  in at most two ways. Indeed, if we use one of the two remaining colors for another pair  $(X_2, Z_2)$  of  $r$ -spaces that is distinct from the pairs  $(X, Z)$  and  $(X_1, Z_1)$ , where  $t_1 \cup t_2 \not\subseteq X_2$  and  $t_1 \cup t_2 \not\subseteq Z_2$ , then by Lemma 4.1 with (24), for some constant  $C^* > 0$  this color can be used for at most  $C^*q^{n(r-\ell-1)}$  distinct  $r$ -spaces containing  $t_1$  but not  $t_2$ , or  $t_2$  but not  $t_1$ , respectively. However, this leaves at least  $\binom{n-\ell}{r-\ell}_q - (C + C' + C^*)q^{n(r-\ell-1)}$  uncolored  $r$ -spaces containing  $t_1$  or  $t_2$ , but not  $t_1 \cup t_2$ , which cannot be colored properly with a single color.

Hence, combining (30) and (32) with the inequalities  $p_2 + y \leq \ell - 1$  and  $p'_2 + y \leq \ell - 1$  given in (24), we have that, for some constant  $C^* > 0$ , for  $n$  sufficiently large, the total number of non-star colorings of  $F^*$  is bounded above by

$$\begin{aligned} &24q^{n(2r-2\ell)} \cdot 2^{2Cq^{n(r-2\ell+p_2+y)}} \cdot 2^{\binom{n-\ell}{r-\ell}_q + \binom{n-2\ell+y}{r-2\ell+y}_q} \\ &+ 24q^{n(4r-4\ell)} \cdot 2^{2Cq^{n(r-2\ell+p_2+y)}+2C'q^{n(r-2\ell+p'_2+y)}} \cdot 4^{\binom{n-2\ell+y}{r-2\ell+y}_q} \\ &\leq 24 \cdot 2^{C^*q^{n(r-\ell-1)}} \left( q^{n(2r-2\ell)} \cdot 2^{\binom{n-\ell}{r-\ell}_q + \binom{n-\ell-1}{r-\ell-1}_q} + q^{n(4r-4\ell)} \cdot 4^{\binom{n-\ell-1}{r-\ell-1}_q} \right) \\ &= 2^{(1+o(1))\binom{n-\ell}{r-\ell}_q}. \end{aligned} \quad (33)$$

For fixed  $p_1$  and  $p_2$  (respectively  $p'_1$  and  $p'_2$ ) the number of possibilities for choosing the intersections  $X \cap t_2$ , and  $Z \cap t_1$ , and  $X_1 \cap t_2$ , as well as  $Z_1 \cap t_1$ , where  $\dim(X \cap t_2) = p_2 + y$  and  $\dim(Z \cap t_1) = p_1 + y$  as well as  $\dim(X_1 \cap t_2) = p'_2 + y$  and  $\dim(Z_1 \cap t_1) = p'_1 + y$  is bounded from above by a constant, which affects the upper bound (33) by at most a constant factor.

The non-star colorings are not enough to bridge the gap between  $T(\ell - 1)$  and  $T(\ell - 2)$ , as by (23) and with  $\binom{n+1}{k+1}_q = \binom{n}{k}_q + q^{k+1} \cdot \binom{n}{k+1}_q$  we have for  $n$  sufficiently large that

$$\begin{aligned} T(\ell - 1) - T(\ell - 2) &\geq 7 \cdot 3^{\binom{n-\ell}{r-\ell}_q} \cdot \left(\frac{4}{3}\right)^{\binom{n-\ell-1}{r-\ell-1}_q} - 8 \cdot 3^{\binom{n-\ell}{r-\ell}_q} \cdot \left(\frac{4}{3}\right)^{\binom{n-\ell-2}{r-\ell-2}_q} \\ &= 3^{\binom{n-\ell}{r-\ell}_q} \cdot \left(\frac{4}{3}\right)^{\binom{n-\ell-1}{r-\ell-1}_q} \left(7 - \frac{8}{\left(\frac{4}{3}\right)^{q^{r-\ell-1} \cdot \binom{n-\ell-2}{r-\ell-1}_q}}\right). \end{aligned}$$

Since  $7 - \frac{8}{\left(\frac{4}{3}\right)^{q^{r-\ell-1} \cdot \binom{n-\ell-2}{r-\ell-1}_q}} \geq 1$  for  $r > \ell$  and  $n$  sufficiently large, we have

$$T(\ell - 1) - T(\ell - 2) \geq 3^{\binom{n-\ell}{r-\ell}_q} \cdot \left(\frac{4}{3}\right)^{\binom{n-\ell-1}{r-\ell-1}_q},$$

which, for large  $n$ , is much larger than the upper bound in (33).

Therefore, the number of  $(4, \ell)$ -colorings of  $\mathcal{F}^*$  is maximized for  $y = \ell - 1$ , which finishes the proof of Theorem 1.7(ii).  $\square$

## 5. FINAL REMARKS

For every  $k \in \{2, 3, 4\}$  and all integers  $1 \leq \ell < r$ , we have found the  $(k, \ell)$ -extremal families of  $r$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$ , provided that  $n$  is sufficiently large. When  $k = 2$ , our results hold for all values of  $n$ .

We briefly discuss how to extend our arguments to the case of  $k \geq 5$  colors. We are coloring  $r$ -spaces in an  $n$ -space over  $GF(q)$ , where each color class is  $\ell$ -intersecting. Suppose that we have a family  $\mathcal{F}$  of  $r$ -spaces in an  $n$ -space with an  $\ell$ -cover  $\mathcal{C} = \{t_1, \dots, t_c\}$ . A *star coloring* of  $\mathcal{F}$  is a  $(k, \ell)$ -coloring such that, for every color  $\sigma$ , all the  $r$ -spaces in  $\mathcal{F}$  with color  $\sigma$  contain some fixed element  $t_i = t_i(\sigma)$  of the  $\ell$ -cover  $\mathcal{C}$ .

We know from the proof of Theorem 1.7 that the size of the  $\ell$ -cover is bounded by a function of  $r$ ,  $k$  and  $\ell$ , and, we may show that the bulk of the colorings of a  $(k, \ell)$ -extremal configuration consists of star colorings. Moreover, along the lines of the proof of Theorem 1.7, it may be proved that extremality is achieved by  $(\mathcal{C}, r)$ -complete families. Because of this, we may restrict our attention to  $(k, \ell)$ -colorings where we assign each color to a cover element  $t_i$ , and we use the set of colors associated with  $t_i$  to color the  $r$ -spaces containing it. In particular, if  $s_i$  colors go to cover space  $t_i$ ,  $i = 1, \dots, c$ , we generate

$$\prod_{i=1}^c s_i^{\binom{n-\ell}{r-\ell}_q} \quad (34)$$

star colorings of this type (ignoring  $r$ -spaces that contain at least two cover elements). If we maximize this quantity in terms of  $c$  and the integral vector  $\mathbf{s} = (s_1, \dots, s_c)$  subject to the constraint  $s_1 + \dots + s_c = k$ , it is not hard to see that a maximum is achieved when  $c = \lceil k/3 \rceil$  and  $\mathbf{s}$  has as many components equal to 3 as possible, while the remaining components are equal to 2. In other words, the components of  $\mathbf{s}$  are all equal to 3 if  $k \equiv 0 \pmod{3}$ , exactly two components are equal to 2 if  $k \equiv 1 \pmod{3}$ , and exactly one component is equal to 2 if  $k \equiv 2 \pmod{3}$ . For a proof of this fact, see Lemma 2.4 in [6].

In light of this, and based on the results [6], we pose the following conjecture, which generalizes the results obtained in this paper. For integers  $k \geq 2$  and  $1 \leq \ell < r < n$ , let  $\mathcal{C}$  be a set of cardinality  $c$  whose elements are  $\ell$ -spaces in an  $n$ -space  $V_n$  over  $GF(q)$ . The family  $\mathcal{F}_{n,q,r,k,\ell}$  is a  $(\mathcal{C}, r)$ -complete family of  $r$ -spaces in  $V_n$  for which  $|\mathcal{C}| = c = c(k) = \lceil k/3 \rceil$  and all distinct  $\ell$ -spaces in  $\mathcal{C}$  have pairwise intersection of dimension zero.

**Conjecture 5.1.** *Let  $1 \leq \ell < r$  and  $k$  be fixed integers. Then the following holds.*

(i) *If  $k \geq 5$  and  $r < 2\ell - 1$ , then*

$$\chi_{(r,k,\ell,q)}(n) = (1 + o(1)) \cdot C_{(k,\ell)}(\mathcal{F}_{n,q,r,k,\ell}).$$

(ii) *If  $k \geq 5$  and  $r \geq 2\ell - 1$ , then*

$$\chi_{(r,k,\ell,q)}(n) = C_{(k,\ell)}(\mathcal{F}_{n,q,r,k,\ell}).$$

The conclusions of Conjecture 5.1 are known to hold in the case of set systems (see [6]). Moreover, it might turn out that the family  $\mathcal{F}_{n,q,r,k,\ell}$  is *not* extremal in case (i). In fact, we have seen that, for  $k = 4$ , among all the  $(\mathcal{C}, r)$ -complete configurations with minimum  $\ell$ -cover of size two, the maximum number of  $(k, \ell)$ -colorings is achieved for  $\ell$ -spaces whose intersection has the largest possible dimension. If the parallelism between set systems and vector spaces were to hold further, the family  $\mathcal{F}_{n,q,r,k,\ell}$  would never be extremal in case (i), which would then imply that the problem  $\mathcal{P}_{n,q,r,k,\ell}$  is unstable whenever  $k \geq 5$  and  $r < 2\ell - 1$ .

We would also like to mention that the problem considered in this paper, which has been addressed earlier in terms of graphs [1, 3], hypergraphs [9, 10] and set systems [6], appears

naturally in other contexts as well. For instance, it could be viewed in the framework of lattices or partial orders, such as the power set lattice, the linear lattice, or any other, where the elements are colored with a fixed number of colors and we wish to avoid a fixed (monochromatic) substructure. Again, we ask for configurations achieving the largest number of such colorings.

## REFERENCES

1. N. Alon, J. Balogh, P. Keevash, and B. Sudakov, The number of edge colorings with no monochromatic cliques, *J. London Math. Soc.* **70**(2) (2004), 273–288.
2. W. G. Brown and M. Simonovits, Extremal Multigraph and Digraph Problems, in: Paul Erdős and his Mathematics, Volume II, Springer, 2002, 157–204.
3. P. Erdős, Some new applications of probability methods to combinatorial analysis and graph theory, Proc. of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1974), 39–51. *Congressus Numerantium*, No. X, Utilitas Math. (1974), 39–51.
4. P. Frankl and R. M. Wilson, The Erdős-Ko-Rado theorem for vector spaces, *Journal of Combinatorial Theory Series A* **43** (1986), 228–236.
5. C. Godsil, Finite Geometry, 2004, <http://quoll.uwaterloo.ca/mine/Notes/fgeom.pdf>.
6. C. Hoppen, Y. Kohayakawa, and H. Lefmann, Hypergraphs with many Kneser colorings, *European Journal of Combinatorics* **33** (2012), 816–843.
7. C. Hoppen, Y. Kohayakawa, and H. Lefmann, Edge Colorings of Graphs Avoiding Monochromatic Subgraphs With Small Turán Density, Proceedings of the Sixth European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2011, *Electronic Notes in Discrete Mathematics* **38** (2011), 469–474.
8. C. Hoppen, Y. Kohayakawa, and H. Lefmann, An unstable hypergraph problem with a unique optimal solution, In: *Information Theory, Combinatorics, and Search Theory Information Theory, Combinatorics, and Search Theory (Series: Lecture Notes in Computer Science)*, v. 7777 (2013), Aydinian, Harout; Cicalese, Ferdinando; Deppe, Christian. (Org.), Springer Verlag, 432–450.
9. H. Lefmann, Y. Person, V. Rödl, and M. Schacht, On colorings of hypergraphs without monochromatic Fano planes, *Combinatorics, Probability & Computing* **18** (2009), 803–818.
10. H. Lefmann, Y. Person, and M. Schacht, A structural result for hypergraphs with many restricted edge colorings, *Journal of Combinatorics* **1** (2010), 441–475.
11. M. Simonovits, *A Method for Solving Extremal Problems in Graph Theory, Stability Problems, Theory of Graphs* (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, 279–319.
12. H. Tanaka, Classifications of subsets with minimal width and dual width in Grassmann, bilinear forms and dual polar graphs, *Journal of Combinatorial Theory Series A* **113** (2006), 903–910.
13. P. Turán, On an extremal problem in graph theory (in Hungarian), *Matematikai és Fizikai Lapok* **48** (1941), 436–452.
14. R. Yuster, The number of edge colorings with no monochromatic triangle, *J. Graph Theory* **21** (1996), 441–452.

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