

Generalizations of Heilbronn's Triangle Problem

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Abstract

For integers $d, j \geq 2$ and $n \geq j$, distributions of n points in the d -dimensional unit cube $[0, 1]^d$ are investigated, such that the minimum volume of the convex hull determined by j of n points is large. Lower and upper bounds on these minimum volumes are given. For obtaining lower bounds, results on the independence number of non-uniform, linear hypergraphs are used, which might be of interest by their own.

Keywords: Hypergraphs, Independence Number, Large Simplices.

1 Introduction

Originally, Heilbronn's triangle problem asks for the supremum value $\Delta_3(n)$, over all distributions of n points in the unit square $[0, 1]^2$, of the minimum area of a triangle among n points. For prime numbers n , the points $(1/n) \cdot (i \bmod n, i^2 \bmod n)$, $i = 0, \dots, n - 1$, yield $\Delta_3(n) = \Omega(1/n^2)$ as has been observed by Erdős. By using results on the independence number of certain 3-uniform hypergraphs, this lower bound has been improved by Komlós, Pintz and Szemerédi [8] to $\Delta_3(n) = \Omega(\log n/n^2)$. Earlier upper bounds on $\Delta_3(n)$ have been proved by Roth and Schmidt, and the currently best upper bound

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$\Delta_3(n) = O(2^{c\sqrt{\log n}}/n^{8/7})$ is due to Komlós, Pintz and Szemerédi [7]. A variant of this problem, considered by Schmidt [10], asks for integers $2 \leq j \leq n$ for the supremum value $\Delta_j(n)$, over all distributions of n points in $[0, 1]^2$, of the minimum area of the convex hull of j among n points, and for $j = 4$ he obtained $\Delta_4(n) = \Omega(1/n^{3/2})$. For fixed $j \geq 3$, the currently best lower bound $\Delta_j(n) = \Omega((\log n)^{1/(j-2)}/n^{(j-1)/(j-2)})$ is due to this author. However, for fixed $j \geq 4$ only the upper bound $\Delta_j(n) = O(1/n)$ is known.

An extension of Heilbronn’s triangle problem to higher dimensions has been considered by Barequet and Naor [2,3]. For integers $d, j, n \geq 2$, let $\Delta_{j,d}(n)$ be the supremum, over all distributions of n points in the unit cube $[0, 1]^d$, of the minimum $((j-1)$ -dimensional for $j \leq d+1$) volume of a j -point simplex among n points. For fixed $3 \leq j \leq d+1$, the currently best bounds are $\Delta_{j,d}(n) = \Omega((\log n)^{1/(d-j+2)}/n^{(j-1)/(d-j+2)})$ and $\Delta_{j,d}(n) = O(n^{(j-1)/d})$, moreover $\Delta_{j,d}(n) = O(n^{((j-1)/d)+(j-2)/(2d(d-1))})$ for even $j \geq 4$, see [9].

In connection with some range searching problems Chazelle [5] has investigated the function $\Delta_{j,d}(n)$ for values j , which depend on n , and he showed the asymptotically correct order $\Theta(j/n)$ with $\log n \leq j \leq n$ and fixed $d \geq 2$. In view of these results, it might be of interest, whether for a given range of j , say $3 \leq j \leq K$, there is a single configuration of n points in $[0, 1]^d$ which yields good lower bounds on the minimum volume of any j points among the n points, simultaneously for $j = 3, \dots, K$. To do so, we consider an extension of a result of Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] on the independence number of hypergraphs without short cycles. Some of the probabilistic existence arguments, which are used in the proofs, can be derandomized in suitable discrete lattices and yield deterministic polynomial time algorithms.

2 Uncrowded and Linear Hypergraphs

Definition 2.1 For a hypergraph \mathcal{G} the notation $\mathcal{G} = (V, \mathcal{E}_2 \cup \dots \cup \mathcal{E}_k)$ means that V is its vertex-set and \mathcal{E}_i is the set of all i -element edges in \mathcal{G} , $i = 2, \dots, k$. The quantity $i \cdot |\mathcal{E}_i|/|V|$, denoted by t_i^{i-1} , is the *average degree* of a vertex for the i -element edges $E \in \mathcal{E}_i$ in \mathcal{G} . A hypergraph $\mathcal{G} = (V, \mathcal{E}_k)$ is called *k -uniform*, i.e., each edge $E \in \mathcal{E}$ contains exactly k vertices. The *independence number* $\alpha(\mathcal{G})$ of a hypergraph $\mathcal{G} = (V, \mathcal{E})$ is the largest size of a subset $I \subseteq V$ which contains no edges from \mathcal{E} . A *j -cycle* in $\mathcal{G} = (V, \mathcal{E})$ is a sequence E_1, \dots, E_j of distinct edges from \mathcal{E} , such that $E_i \cap E_{i+1} \neq \emptyset$, $i = 1, \dots, j-1$, and $E_j \cap E_1 \neq \emptyset$, and a sequence v_1, \dots, v_j of distinct vertices with $v_{i+1} \in E_i \cap E_{i+1}$, $i = 1, \dots, j-1$, and $v_1 \in E_1 \cap E_j$. An unordered pair $\{E, E'\}$ of distinct edges $E, E' \in \mathcal{E}$ with $|E \cap E'| \geq 2$ is called *2-cycle*. A 2-cycle $\{E, E'\}$ in $\mathcal{G} = (V, \mathcal{E}_2 \cup \dots \cup \mathcal{E}_k)$ with

$E \in \mathcal{E}_i$ and $E' \in \mathcal{E}_j$ is called $(2; (g, i, j))$ -cycle iff $|E \cap E'| = g$, $2 \leq g \leq i \leq j$ but $g < j$. The hypergraph \mathcal{G} is called *linear* if it does not contain any 2-cycles, and it is called *uncrowded* if it does not contain any 2-, 3-, or 4-cycles.

Well known is Turán's theorem for hypergraphs:

Theorem 2.2 *Let $\mathcal{G} = (V, \mathcal{E}_2 \cup \dots \cup \mathcal{E}_k)$ be a hypergraph on $|V| = N$ vertices and with average degree $t_i^{i-1} := i \cdot |\mathcal{E}_i|/N$ for the i -element edges, $i = 2, \dots, k$. Let $t_{i_0} := \max \{t_i \mid 2 \leq i \leq k\} \geq 1/2$. Then, there exists an independent set $I \subseteq V$ in \mathcal{G} with $|I| = \Omega(N/t_{i_0})$.*

The next result by Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] improves on Theorem 2.2 for uncrowded, uniform hypergraphs, compare [4] for a deterministic polynomial time algorithm.

Theorem 2.3 *Let $k \geq 3$ be fixed. Let $\mathcal{G} = (V, \mathcal{E}_k)$ be an uncrowded, k -uniform hypergraph on $|V| = N$ vertices and with average degree $t_k^{k-1} := k \cdot |\mathcal{E}_k|/N$. Then, the independence number $\alpha(\mathcal{G})$ satisfies $\alpha(\mathcal{G}) = \Omega((N/t) \cdot (\log t_k)^{1/(k-1)})$.*

Theorem 2.3 can be extended to non-uniform hypergraphs as follows:

Theorem 2.4 *Let $k \geq 3$ be fixed. Let $\mathcal{G} = (V, \mathcal{E}_3 \cup \dots \cup \mathcal{E}_k)$ be an uncrowded hypergraph on $|V| = N$ vertices, such that for some number $T \geq 1$ for $i = 3, \dots, k$ the average degrees $t_i^{i-1} := i \cdot |\mathcal{E}_i|/N$ for the i -element edges satisfy $t_i^{i-1} \leq T^{i-1} \cdot (\log T)^{(k-i)/(k-1)}$. Then, the independence number $\alpha(\mathcal{G})$ satisfies $\alpha(\mathcal{G}) = \Omega((N/T) \cdot (\log T)^{1/(k-1)})$.*

For a certain range $k < T < N$ of the parameters Theorem 2.4 is asymptotically best possible up to a constant factor, as can be seen by considering random non-uniform hypergraphs $\mathcal{G} = (V, \mathcal{E}_3 \cup \dots \cup \mathcal{E}_k)$ on $|V| = N$ vertices. Theorem 2.4 also holds for linear hypergraphs, compare [6] for the case of uniform hypergraphs:

Theorem 2.5 *Let $k \geq 3$ be fixed. Let $\mathcal{G} = (V, \mathcal{E}_3 \cup \dots \cup \mathcal{E}_k)$ be a linear hypergraph with $|V| = N$, such that for some number $T \geq 1$ for $i = 3, \dots, K$ the average degrees $t_i^{i-1} := i \cdot |\mathcal{E}_i|/N$ for the i -element edges satisfy $t_i^{i-1} \leq T^{i-1} \cdot (\log T)^{(k-i)/(k-1)}$. Then, there exists an independent set $I \subseteq V$ with $|I| = \Omega((N/T) \cdot (\log T)^{1/(k-1)})$.*

With Theorem 2.5 the following has been shown recently by this author:

Theorem 2.6 *Let $d \geq 2$ and $K \geq 3$ be fixed. For integers $n \geq K$ there exist n points in $[0, 1]^2$, such that, simultaneously for $j = 3, \dots, K$, the area of the convex hull of any j among the n points is $\Omega((\log n)^{1/(j-2)}/n^{(j-1)/(j-2)})$.*

3 Volumes of the Convex Hull of j Points

In view of Theorem 2.6 we turn our attention to distributions of n points in higher dimensional unit cubes. A *simplex* given by j points $P_1, \dots, P_j \in [0, 1]^d$, where $j \leq d + 1$, is the set of all points $P_1 + \sum_{i=2}^j \lambda_i \cdot (P_i - P_1)$ with $\lambda_i \geq 0$, $i = 2, \dots, j$, and $\sum_{i=2}^j \lambda_i \leq 1$. The $((j - 1)$ -dimensional) *volume of a simplex* determined by points $P_1, \dots, P_j \in [0, 1]^d$, $2 \leq j \leq d + 1$, is $\text{vol}(P_1, \dots, P_j) := (1/(j - 1)!) \cdot \prod_{i=2}^j \text{dist}(P_i; \langle P_1, \dots, P_{i-1} \rangle)$, where $\text{dist}(P_i; \langle P_1, \dots, P_{i-1} \rangle)$ is the Euclidean distance of the point P_i from the affine space $\langle P_1, \dots, P_{i-1} \rangle$ generated by the points P_1, \dots, P_{i-1} with $\langle P_1 \rangle := P_1$.

By using Theorem 2.2 and, for an algorithmic version, results on the theory of lattices, the following has been shown by this author:

Theorem 3.1 *Let $K, d \geq 3$ be fixed. For every $n \geq K$ there exists a configuration of n points in the unit cube $[0, 1]^d$ such that, simultaneously for $j = 2, \dots, K$, the volume of the convex hull of any j among the n points is $\Omega(1/n^{(j-1)/(1+d-j+1)})$.*

By Theorem 3.1 we obtain a single distribution of n points in $[0, 1]^d$, which is simultaneously good for small sets of points, however for $3 \leq j \leq d + 1$ in view of the best available lower bounds we loose a polylogarithmic factor. The next result takes care of this factor.

Theorem 3.2 *Let $d \geq 2$ and $K \geq 3$ with $K \leq d + 1$ be fixed. For every $n \geq K$ there exists a configuration of n points in the unit cube $[0, 1]^d$, such that, simultaneously for $j = 3, \dots, K$, the $(j - 1)$ -dimensional volume of any j -point simplex among these n points is $\Omega((\log n)^{1/(d-j+2)}/n^{(j-1)/(d-j+2)})$.*

Proof (Sketch) Let $\beta, \gamma_2, \dots, \gamma_K, d_3, \dots, d_K > 0$ be suitable constants, and set $N := n^{1+\beta}$, and $V_j := d_j \cdot (\log n)^{1/(d-j+2)}/n^{(j-1)/(d-j+2)}$, $j = 3, \dots, K$. Uniformly at random and independently of each other select N points $P_1, \dots, P_N \in [0, 1]^d$. Form a hypergraph $\mathcal{G} = (V, \mathcal{E}_2^* \cup \mathcal{E}_3^* \cup \mathcal{E}_3 \cup \dots \cup \mathcal{E}_K^* \cup \mathcal{E}_K)$, where $\mathcal{E}_j^*, \mathcal{E}_j \subseteq [V]^j$, $j = 2, \dots, K$, with vertex-set $V = \{P_1, \dots, P_N\}$ and with two types of j -element edges. For $j = 2, \dots, K$, and points P_{i_1}, \dots, P_{i_j} , $1 \leq i_1 < \dots < i_j \leq N$, let $\{P_{i_1}, \dots, P_{i_j}\} \in \mathcal{E}_j^*$ iff $\text{vol}(P_{i_1}, \dots, P_{i_j}) \leq N^{-\gamma_j}$. Let $E = \{P_{i_1}, \dots, P_{i_j}\} \in \mathcal{E}_j$ iff $\text{vol}(P_{i_1}, \dots, P_{i_j}) \leq V_j$ and $E' \not\subseteq E$ for all edges $E' \in \mathcal{E}_2^* \cup \dots \cup \mathcal{E}_{j-1}^*$.

Let $C_{(g;i,j)}$ be the set of all $(2; (g; i, j))$ -cycles in the hypergraph \mathcal{G} , $2 \leq g \leq i \leq j \leq K$ and $g < j$. Set $\gamma_j := (j - 1)/(d - j + 3/2)$, $j = 2, \dots, K$, and $\beta := 1/(5 \cdot d \cdot K)$. One can show that there exists a hypergraph $\mathcal{G} = (V, \mathcal{E}_2^* \cup \mathcal{E}_3^* \cup \mathcal{E}_3 \cup \dots \cup \mathcal{E}_K^* \cup \mathcal{E}_K)$, which satisfies for constants $c_j > 0$:

$$|\mathcal{E}_j^*| = o(|V|) \quad \text{and} \quad |C_{(g;i,j)}| = o(|V|) \quad \text{and} \quad |\mathcal{E}_j| \leq c_j \cdot N^j \cdot V_j^{d-j+2}.$$

Delete one vertex from each edge $E \in \mathcal{E}_2^* \cup \dots \cup \mathcal{E}_K^*$ and each $(2; (g, i, j))$ -cycle in $C_{(g;i,j)}$, $2 \leq g \leq i \leq j \leq K$ with $g < j$, and obtain an induced, linear subhypergraph $\mathcal{G}^0 = (V^0, \mathcal{E}_3^0 \cup \dots \cup \mathcal{E}_K^0)$ of \mathcal{G} with $\mathcal{E}_j^0 := \mathcal{E}_j \cap [V^0]^j$, $j = 3, \dots, K$, on $|V^0| = (1 - o(1)) \cdot N$ vertices. With $T := c' \cdot (N/n) \cdot (\log n)^{1/(k-1)}$ for a suitable constant $c' > 0$ all assumptions in Theorem 2.5 are fulfilled and we infer for the independence number $\alpha(\mathcal{G}^0) \geq n$. Thus, there exist n points such that the volume of every j -point simplex, $j = 3, \dots, K$, among these n points is $\Omega((\log n)^{1/(d-j+2)} / n^{(j-1)/(d-j+2)})$ as claimed. \square

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