Distributions of Points in d Dimensions and Large k-Point Simplices

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Abstract. We consider a variant of Heilbronn's triangle problem by investigating for fixed dimension $d \geq 2$ and for integers $k \geq 2$ with $k \leq d$ distributions of n points in the d-dimensional unit cube $[0,1]^d$ such that the minimum volume of the simplices, which are determined by $(k+1)$ of these n points, is as large as possible. Denoting by $\Delta_{k,d}(n)$ the supremum of the minimum volume of a $(k+1)$-point simplex among n points over all distributions of n points in $[0,1]^d$, we show that $c_{k,d} \cdot (\log n)^{1/(d-k+1)}/n^{k/(d-k+1)} \leq \Delta_{k,d}(n) \leq c'_{k,d}/n^{k/d}$ for fixed $2 \leq k \leq d$, and, moreover, for odd integers $k \geq 1$ we show the upper bound $\Delta_{k,d}(n) \leq c''_{k,d}/n^{k/d+(k-1)/(2d(d-1))}$, where $c_{k,d}, c'_{k,d}, c''_{k,d} > 0$ are constants.

1 Introduction

For integers $n \geq 3$, Heilbronn's problem asks for the supremum $\Delta_2(n)$ of the minimum area of a triangle formed by three of n points over all distributions of n points in the unit square $[0,1]^2$. For primes n, no three of the points $P_k = (1/n) \cdot (l \bmod n, l^2 \bmod n)$, $l = 0, 1, \ldots, n-1$ are collinear, which gives the lower bound $\Delta_2(n) = \Omega(1/n^2)$, as has been observed by Erdős, see [17]. Komlós, Pintz and Szemerédi [12] improved this to the currently best known lower bound $\Delta_2(n) = \Omega(\log n/n^2)$, and in [7] a deterministic polynomial in n time algorithm was given, which achieves this lower bound on $\Delta_2(n)$. Upper bounds on $\Delta_2(n)$ have been given in a series of papers by Roth [17–20] and Schmidt [22]. The currently best known upper bound has been obtained by Komlós, Pintz and Szemerédi [11], who proved for some constant $c > 0$ that $\Delta_2(n) = O(2^{c\sqrt{\log n}}/n^{8/7})$. If n points are chosen uniformly at random and independently of each other in $[0,1]^2$, the expected value of the minimum area of a triangle formed by three of these n points is $\Theta(1/n^3)$, as has been shown by Jiang, Li and Vitany [10].
A variant of Heilbronn's problem, which has been considered by Barequet in [2], asks, for dimension $d \geq 2$, for the supremum $\Delta_d(n)$ of the minimum volume of a simplex determined by $(d+1)$ of n points in the d-dimensional unit cube $[0,1]^d$, where the supremum is taken over all distributions of n points in $[0,1]^d$. For fixed $d \geq 1$, he showed in [2] the lower bound $\Delta_d(n) = \Omega(1/n^d)$. This has been improved in [13] by a logarithmic factor to $\Delta_d(n) = \Omega(\log n/n^d)$ for fixed $d \geq 2$, and in [16], for the case of dimension $d = 3$ a deterministic polynomial in n time

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algorithm has been given, which achieves the lower bound $\Delta_3(n) = \Omega(\log n/n^3)$. An upper bound of $\Delta_d(n) = O(1/n)$ follows from the pigeonhole-principle. Recently, by considering the angles between lines, which are determined by pairs of points, Brass [8] improved this upper bound to $\Delta_d(n) = O(1/n^{1+1/(2d)})$ for fixed odd integers $d \geq 3$.
Here we consider the following generalization of Heilbronn's problem: given fixed integers d, k with $1 \leq k \leq d$, find for every integer $n \geq k$ a distribution of n points in the d-dimensional unit cube $[0,1]^d$ such that the minimum volume of a $(k+1)$-point simplex arising from these n points is as large as possible. Let $\Delta_{k,d}(n)$ denote the supremum – over all distributions of n points in $[0,1]^d$ – of the minimum volume of a $(k+1)$-point simplex among n points in $[0,1]^d$, i.e., $\Delta_d(n) = \Delta_{d,d}(n)$.
It is easy to see that $\Delta_{1,d} = \Theta(1/n^{1/d})$ for fixed dimension $d \geq 1$: the lower bound follows by considering the points of the standard $n^{1/d} \times \cdots \times n^{1/d}$-grid in $[0,1]^d$, and the upper bound follows by an argument of packing balls in $[0,1]^d$. Lower and upper bounds on $\Delta_{2,d}(n)$ for fixed $d \geq 2$, i.e., areas of triangles in $[0,1]^d$, were given by this author in [14], where it has been shown that $c_{2,d} \cdot (\log n)^{1/(d-1)}/n^{2/(d-1)} \leq \Delta_{2,d}(n) \leq c'_{2,d}/n^{2/d}$ for constants $c_{2,d}, c'_{2,d} > 0$.
Here we prove the following lower and upper bounds on $\Delta_{k,d}(n)$:

Theorem 1. *Let d, k be fixed integers with $2 \leq k \leq d$. Then, for constants $c_{k,d}, c'_{k,d}, c''_{k,d} > 0$, for every integer $n \geq k$ it is*

$$c_{k,d} \cdot \frac{(\log n)^{\frac{1}{d-k+1}}}{n^{\frac{k}{d-k+1}}} \leq \Delta_{k,d}(n) \leq \frac{c'_{k,d}}{n^{\frac{k}{d}}} \qquad \text{for every } k \tag{1}$$

$$\Delta_{k,d}(n) \leq \frac{c''_{k,d}}{n^{\frac{k}{d}+\frac{k-1}{2d(d-1)}}} \qquad \text{for } k \text{ odd.} \tag{2}$$

For $d = 2$ and $k = 2$, the lower bound in (1) is just the result from [12]. For $k = d$, the upper bound in (2) yields the bound from [8] and the lower bound in (1) gives the result from [13]. For $k = 2$ and any fixed dimension $d \geq 2$, the bounds in (1) yield the above mentioned result from [14]. Indeed, our arguments for proving Theorem 1 give a randomized polynomial in n time algorithm, which finds a distribution of n points in $[0,1]^d$ that achieves the lower bound in (1).
Independently from this work, in [4] Barequet and Naor showed – taking careful attention to the involved parameters – the bounds $f(k,d)/n^{\frac{k}{d-k+1}} \leq \Delta_{k,d}(n) \leq (k^{k/d} \cdot d^{k/2})/(k! \cdot n^{\frac{k}{d}})$ for arbitrary integers $1 \leq k \leq d$, where the function $f(k,d)$ only depends on d, k. Note that our bounds on $\Delta_{k,d}(n)$ are better for fixed $2 \leq k \leq d$: the lower bound in (1) by a factor of $\Theta((\log n)^{1/(d-k+1)})$, and, for odd k, the upper bound by a factor of $\Theta(n^{(k-1)/((2d(d-1))})$.
We remark that the on-line situation – the points are positioned one after the other in $[0,1]^d$ and suddenly this process stops – of the variant of Heilbronn's problem for $(d+1)$-point simplices in $[0,1]^d$ has been investigated by Barequet [3], where he proved the existence of distributions of n points in $[0,1]^d$ for the cases $d = 3$ and $d = 4$, such that the volume of every $(d+1)$-simplex

is $\Omega(1/n^{10/3})$ and $\Omega(1/n^{127/24})$, respectively. In extending these results, recently, Barequet and Shaikhet [5, 21] showed by packing arguments for the on-line situation the existence of configurations of n points in $[0,1]^d$, where for fixed $k \leq d$ the volume of each $(k+1)$-point simplex among these n points is $\Omega(1/n^{(d+1)\ln\frac{d-2}{d-k+1}+0.735d-k+2.8881})$ for fixed $d \geq 5$ and $3 \leq k \leq d$. Thus, with respect to the off-line situation as discussed above, where the number n of points is known in advance, there is a large gap between the lower bounds.

2 Notation

We introduce some notation, which is used throughout this paper.
For points P, Q with $P = (p_1, \ldots, p_d) \in [0,1]^d$ and $Q = (q_1, \ldots, q_d) \in [0,1]^d$ let dist $(P, Q) := ((p_1 - q_1)^2 + \ldots + (p_d - q_d)^2)^{1/2}$ denote the *Euclidean distance* between P and Q. A $(k+1)$*-point simplex* is given by $(k+1)$ points $P_1, \ldots, P_{k+1} \in [0,1]^d$ and is defined as the convex hull of $P_1, \ldots, P_{k+1}$, i.e., it is the set of all points $P_1 + \sum_{i=2}^{k+1} \lambda_i \cdot (P_i - P_1)$ with $\lambda_i \geq 0$, $i = 2, \ldots, k+1$, and $\sum_{i=2}^{k+1} \lambda_i \leq 1$. The ($k$-dimensional) *volume of a* $(k+1)$*-point simplex* determined by the points $P_1, \ldots, P_{k+1} \in [0,1]^d$, $1 \leq k \leq d$, is defined by vol $(P_1, \ldots, P_{k+1}) := (1/k!) \cdot \prod_{j=2}^{k+1}$ dist $(P_j; \langle P_1, \ldots, P_{j-1} \rangle)$, where dist $(P_j; \langle P_1, \ldots, P_{j-1} \rangle)$ denotes the Euclidean distance of the point P_j from the affine space $\langle P_1, \ldots, P_{j-1} \rangle$, which is generated by the points $P_1, \ldots, P_{j-1}$ with $\langle P_1 \rangle := P_1$. Hence, if $(k+1)$ points are contained in a $(k-1)$-dimensional space, then vol $(P_1, \ldots, P_{k+1}) = 0$.
In our arguments we transform the geometrical problem into a problem on hypergraphs.
A hypergraph $\mathcal{G} = (V, \mathcal{E})$ with vertex-set V and edge-set $\mathcal{E}$ is called k*-uniform* if $|E| = k$ for each edge $E \in \mathcal{E}$. If the hypergraph $\mathcal{G}$ contains edges of different cardinalities, then $\mathcal{G}$ is called *non-uniform*. For a hypergraph $\mathcal{G}$ we indicate by $\mathcal{G} = (V, \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_k)$ that $\mathcal{E}_i$ is the set of all i-element edges in $\mathcal{G}$, $i = 2, \ldots, k$. A subset $I \subseteq V$ of the vertex-set V is called *independent* if no edge from $\mathcal{E}$ is contained in I. The largest size $|I|$ of an independent set in $\mathcal{G}$ is the *independence number* $\alpha(\mathcal{G})$ of $\mathcal{G}$. A hypergraph $\mathcal{G} = (V, \mathcal{E})$ is called *linear* if $|E \cap E'| \leq 1$ for all distinct edges $E, E' \in \mathcal{E}$.

3 A Lower Bound on $\Delta_{k,d}(n)$

In this section we prove the lower bound (1) in Theorem 1, namely, that for fixed integers d, k with $2 \leq k \leq d$ there are constants $c_{k,d} > 0$ such that for every integer $n \geq k$ it is $\Delta_{k,d}(n) \geq c_{k,d} \cdot (\log n)^{\frac{1}{d-k+1}} / n^{\frac{k}{d-k+1}}$.

Proof. Let d, k be fixed integers with $2 \leq k \leq d$. For arbitrary integers $n \geq k$ and a suitable constant $\beta > 0$, we select uniformly at random and independently of each other $N := n^{1+\beta}$ points $P_1, P_2, \ldots, P_N$ from the d-dimensional unit cube $[0,1]^d$.

For suitable constants $\gamma_j > 0$, $j = 2, \ldots, k$, and a number $V_0 > 0$, which are fixed later in connection with Lemmas 3 and 4, we form a random, non-uniform hypergraph $\mathcal{G} = \mathcal{G}(N^{-\gamma_2}, \ldots, N^{-\gamma_k}, V_0) = (V, \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_{k+1})$ with vertex-set $V = \{1, 2, \ldots, N\}$, where vertex i corresponds to the random point $P_i \in [0,1]^d$, $i = 1, \ldots, N$. For $j = 2, \ldots, k$, let $\{i_1, \ldots, i_j\} \in \mathcal{E}_j$ be a j-element edge if and only if the $(j-1)$-dimensional volume of the simplex determined by the points $P_{i_1}, \ldots, P_{i_j}$ is at most $N^{-\gamma_j}$, i.e., vol $(P_{i_1}, \ldots, P_{i_j}) \leq N^{-\gamma_j}$. Moreover, let $\{i_1, \ldots, i_{k+1}\} \in \mathcal{E}_{k+1}$ be a $(k+1)$-element edge if and only if vol $(P_{i_1}, \ldots, P_{i_{k+1}}) \leq V_0$ and $\{i_1, \ldots, i_{k+1}\}$ does not contain any j-element edges $E \in \mathcal{E}_j$ for $j = 2, \ldots, k$.

Let $I \subseteq V$ be an independent set in this hypergraph $\mathcal{G}$. Then, by definition of the edge-set of $\mathcal{G}$, for distinct vertices $i_1, \ldots, i_{k+1} \in I$ we infer that the volume of the simplex, which is determined by the corresponding points $P_{i_1}, \ldots, P_{i_{k+1}} \in [0,1]^d$, satisfies vol $(P_{i_1}, \ldots, P_{i_{k+1}}) > V_0$. Thus, an independent set $I \subseteq V$ in $\mathcal{G}$ yields $|I|$ many points in $[0,1]^d$ such that the volume of each simplex determined by k of these $|I|$ points is bigger than V_0.

Our aim is to show the existence of a large independent set $I \subseteq V$ in $\mathcal{G}$. For doing so, we use an extension by Duke, Rödl and this author [9] of a result by Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] on the independence number of linear, uniform hypergraphs.

Theorem 2. [1, 9] *Let $k \geq 2$ be a fixed integer. Let $\mathcal{G} = (V, \mathcal{E}_{k+1})$ be a linear, $(k+1)$-uniform hypergraph on $|V| = N$ vertices with average degree $t^k := (k+1) \cdot |\mathcal{E}_{k+1}|/N$.*

Then, for some constant $C^*_{k+1} > 0$ the independence number $\alpha(\mathcal{G})$ of $\mathcal{G}$ satisfies

$$\alpha(\mathcal{G}) \geq C^*_{k+1} \cdot \frac{N}{t} \cdot (\log t)^{\frac{1}{k}}. \tag{3}$$

We remark that for arbitrary $(k+1)$-uniform hypergraphs $\mathcal{G}$ on N vertices with average degree t^k one can prove only the lower bound $\alpha(\mathcal{G}) = \Omega(N/t)$ (Turán bound) and there exist $(k+1)$-uniform hypergraphs with an upper bound on the independence number of $O(N/t)$. However, as Theorem 2 shows, if the hypergraph $\mathcal{G}$ is linear, then there is a bigger lower bound on the independence number $\alpha(\mathcal{G})$.

The difficulty in our arguments is, to find a suitable uniform subhypergraph of the random, non-uniform hypergraph $\mathcal{G}$ to which we can apply Theorem 2. For doing so, we select a suitable induced $(k+1)$-uniform subhypergraph $\mathcal{G}^*$ of $\mathcal{G}$. For $j = 2, \ldots, k$, let $BP_j(\mathcal{G})$ be the set of all 'bad j-pairs of $(k+1)$-point simplices' in $\mathcal{G}$, which are those unordered pairs $\{E, E'\}$ of distinct $(k+1)$-element edges $E, E' \in \mathcal{E}_{k+1}$ in $\mathcal{G}$, which share j vertices, i.e., $|E \cap E'| = j$. We show that in the random, non-uniform hypergraph $\mathcal{G}$ the expected numbers $E(|\mathcal{E}_i|)$ and $E(|BP_j(\mathcal{G})|)$ of i-element edges and bad j-pairs of $(k+1)$-point simplices, respectively, $i, j = 2, \ldots, k$, are not too big, i.e., are of the order $o(N)$. Then we discard one vertex from each i-element edge $E \in \mathcal{E}_i$, $i = 2, \ldots, k$, which yields a $(k+1)$-uniform subhypergraph of $\mathcal{G}$. Moreover, we discard one vertex from each bad j-pair of $(k+1)$-point simplices, $j = 2, \ldots, k$. This yields a

linear, $(k+1)$*-uniform* subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}^*_{k+1})$ of $\mathcal{G}$, and $\mathcal{G}^*$ fulfills the assumptions of Theorem 2 and then we apply it.
To obtain upper bounds on the expected numbers $E(|\mathcal{E}_i|)$ of i-element edges in $\mathcal{G}$, $i = 2, \ldots, k+1$, we estimate for a given number $v > 0$ the probability that i points, which are chosen uniformly at random and independently of each other in $[0,1]^d$, determine a simplex of volume at most v.

Lemma 1. *Let d, k be fixed integers with $1 \leq k \leq d$. For $i = 2, \ldots, k+1$, and random points $P_1, \ldots, P_i \in [0,1]^d$ there are constants $c_i^* > 0$, such that for every number $v > 0$ it is*

$$\textit{Prob}\ (\textit{vol}\ (P_1, \ldots, P_i) \leq v) \leq c_i^* \cdot v^{d-i+2}. \tag{4}$$

Proof. Let $P_1, \ldots, P_i$ be i points, which are chosen uniformly at random and independently of each other in $[0,1]^d$. We assume that these points are numbered such that for $2 \leq g \leq h \leq i$ it is

$$\text{dist}\ (P_g; \langle P_1, \ldots, P_{g-1} \rangle) \geq \text{dist}\ (P_h; \langle P_1, \ldots, P_{g-1} \rangle). \tag{5}$$

The point P_1 may be anywhere in $[0,1]^d$. Given the point $P_1 \in [0,1]^d$, the probability, that the Euclidean distance of the point $P_2 \in [0,1]^d$ from P_1 is in the infinitesimal interval $[r_1, r_1 + \text{ d}r_1]$, is at most the difference of the volumes of the d-dimensional balls with center P_1 and with radii $(r_1 + \text{ d}r_1)$ and r_1, respectively, hence

$$\text{Prob}\ (r_1 \leq \text{ dist}\ (P_1, P_2) \leq r_1 + \text{ d}r_1) \leq d \cdot C_d \cdot r_1^{d-1} \text{ d}r_1,$$

where $C_l := \pi^{l/2}/\Gamma(l/2+1)$ denotes the volume of the l-dimensional unit ball in $\mathbb{R}^l$, $l = 1, \ldots, d$.
Given the points $P_1, P_2 \in [0,1]^d$ with dist $(P_1, P_2) = r_1$, the probability, that the distance dist $(P_3; \langle P_1, P_2 \rangle)$ of the point $P_3 \in [0,1]^d$ from the affine line $\langle P_1, P_2 \rangle$ is in the interval $[r_2, r_2 + \text{ d}r_2]$, is at most the difference of the volumes of two cylinders, which are centered at the affine line $\langle P_1, P_2 \rangle$ and have radii $(r_2 + \text{ d}r_2)$ and r_2, respectively. By assumption (5) and the triangle inequality, both cylinders have height $2 \cdot r_1 = 2 \cdot \text{dist}\ (P_1, P_2)$. Thus we infer

$$\text{Prob}\ (r_2 \leq \text{ dist}\ (P_3; \langle P_1, P_2 \rangle) \leq r_2 + \text{ d}r_2) \leq 2 \cdot r_1 \cdot (d-1) \cdot C_{d-1} \cdot r_2^{d-2} \text{ d}r_2.$$

In general, given $P_1, \ldots, P_g \in [0,1]^d$, $g < i$, with dist $(P_f; \langle P_1, \ldots, P_{f-1} \rangle) = r_{f-1}$, $f = 2, \ldots, g$, by (5) and the triangle inequality the projection of the point P_{g+1} onto the affine space $\langle P_1, \ldots, P_g \rangle$ is contained in a $(g-1)$-dimensional parallelepiped of volume $2^{g-1} \cdot r_1 \cdot \ldots \cdot r_{g-1}$. If dist $(P_{g+1}; \langle P_1, \ldots, P_g \rangle) \leq r$, then the point P_{g+1} is contained in the Cartesian product of a $(g-1)$-dimensional parallelepiped of volume $2^{g-1} \cdot r_1 \cdot \ldots \cdot r_{g-1}$ and a $(d-g+1)$-dimensional ball of radius r. Hence, for $g < i-1$ we obtain

$$\begin{aligned} &\text{Prob}\ (r_g \leq \text{dist}\ (P_{g+1}; \langle P_1, \ldots, P_g \rangle) \leq r_g + \text{ d}r_g) \\ &\leq 2^{g-1} \cdot r_1 \cdot \ldots \cdot r_{g-1} \cdot (d-g+1) \cdot C_{d-g+1} \cdot r_g^{d-g} \text{ d}r_g. \end{aligned} \tag{6}$$

For $g = i-1$, given points $P_1, \ldots, P_{i-1} \in [0,1]^d$ with dist $(P_f; \langle P_1, \ldots, P_{f-1}\rangle) = r_{f-1}$, $f = 2, \ldots, i-1$, to satisfy vol $(P_1, \ldots, P_i) \leq v$, we must have

$$\frac{1}{(i-1)!} \cdot \text{dist } (P_i; \langle P_1, \ldots, P_{i-1}\rangle) \cdot \prod_{f=2}^{i-1} r_{f-1} \leq v,$$

hence

$$\text{dist } (P_i; \langle P_1, \ldots, P_{i-1}\rangle) \leq \frac{(i-1)! \cdot v}{r_1 \cdot \ldots \cdot r_{i-2}}. \tag{7}$$

With (5) the projection of the point P_i onto the affine space $\langle P_1, \ldots, P_{i-1}\rangle$ is contained in a $(i-2)$-dimensional parallelepiped of volume $2^{i-2} \cdot r_1 \cdot \ldots \cdot r_{i-2}$. Thus, by (7) the point P_i is contained in the Cartesian product of an $(i-2)$-dimensional parallelepiped of volume $2^{i-2} \cdot r_1 \cdot \ldots \cdot r_{i-2}$ and a $(d-i+2)$-dimensional ball of radius $(i-1)! \cdot v/(r_1 \cdot \ldots \cdot r_{i-2})$, which happens with probability at most

$$2^{i-2} \cdot r_1 \cdot \ldots \cdot r_{i-2} \cdot C_{d-i+2} \cdot \left(\frac{(i-1)! \cdot v}{r_1 \cdot \ldots \cdot r_{i-2}} \right)^{d-i+2}. \tag{8}$$

Summarizing the estimates (6) and (8), we obtain for constants $c_i^*, c_i^{**} > 0$:

$$\begin{aligned}
&\text{Prob } (\text{vol } (P_1, \ldots, P_i) \leq v) \\
\leq &\int_{r_{i-2}=0}^{\sqrt{d}} \cdots \int_{r_1=0}^{\sqrt{d}} 2^{i-2} \cdot C_{d-i+2} \cdot \frac{((i-1)! \cdot v)^{d-i+2}}{(r_1 \cdot \ldots \cdot r_{i-2})^{d-i+1}} \cdot \\
&\cdot \prod_{g=1}^{i-2} \left(2^{g-1} \cdot r_1 \cdot \ldots \cdot r_{g-1} \cdot (d-g+1) \cdot C_{d-g+1} \cdot r_g^{d-g} \right) \; dr_{i-2} \ldots \; dr_1 \leq \\
\leq &\; c_i^{**} \cdot v^{d-i+2} \cdot \int_{r_{i-2}=0}^{\sqrt{d}} \cdots \int_{r_1=0}^{\sqrt{d}} \left(\prod_{g=1}^{i-2} r_g^{2i-2g-3} \right) \; dr_{i-2} \ldots \; dr_1 \\
\leq &\; c_i^* \cdot v^{d-i+2} \qquad\qquad \text{as } 2 \cdot i - 2 \cdot g - 3 > 0,
\end{aligned}$$

which proves inequality (4). □

Corollary 1. *Let d, k be fixed integers with $2 \leq k \leq d$. For $i = 2, \ldots, k$, there exist constants $c_i, c_{k+1} > 0$, such that*

$$E(|\mathcal{E}_i|) \leq c_i \cdot N^{i-\gamma_i(d-i+2)} \qquad \textit{and} \qquad E(|\mathcal{E}_{k+1}|) \leq c_{k+1} \cdot V_0^{d-k+1} \cdot N^{k+1}. \tag{9}$$

Proof. There are $\binom{N}{i}$ possibilities to choose i out of the N random points $P_1, \ldots, P_N \in [0,1]^d$, and, using the definition of the edge-set of $\mathcal{G}$, by (4) with $v := N^{-\gamma_i}$, $i = 2, \ldots, k$, and $v := V_0$ for $i = k+1$ the inequalities (9) follow. □

Next we give upper bounds on the expected numbers $E(|BP_j(\mathcal{G})|)$ of bad j-pairs of $(k+1)$-point simplices in $\mathcal{G}$, $j = 2, \ldots, k$.

Lemma 2. *Let d, k be fixed integers with $2 \le k \le d$. For $j = 2, \ldots, k$, there exist constants $c_{p,j} > 0$, such that*

$$E(|BP_j(\mathcal{G})|) \le c_{p,j} \cdot V_0^{2(d-k+1)} \cdot N^{2k+2-j+\gamma_j(d-k+1)}. \tag{10}$$

Proof. For $j = 2, \ldots, k$, we show an upper bound of $O(V_0^{2(d-k+1)} \cdot N^{\gamma_j(d-k+1)})$ on the probability that $(2k+2-j)$ points, which are chosen uniformly at random and independently of each other in $[0,1]^d$, form a bad j-pair of $(k+1)$-point simplices.
Note that $\{i_1, \ldots, i_{k+1}\} \in \mathcal{E}_{k+1}$ if and only if vol $(P_{i_1}, \ldots, P_{i_{k+1}}) \le V_0$ and $\{i_1, \ldots, i_{k+1}\}$ does not contain any i-element edges $E \in \mathcal{E}_i$, $i = 2, \ldots, k$. Let the two $(k+1)$-point simplices, which form a bad j-pair, are given by the points $P_1, \ldots, P_{k+1} \in [0,1]^d$ and $P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_{k+1} \in [0,1]^d$, and both sets of points determine an edge in $\mathcal{E}_{k+1}$, hence

$$\text{vol}\,(P_1, \ldots, P_{k+1}) \le V_0 \quad \text{and} \quad \text{vol}\,(P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_{k+1}) \le V_0.$$

By (4) with $v := V_0$ we know that for some constant $c^*_{k+1} > 0$:

$$\text{Prob}\,(\text{vol}\,(P_1, \ldots, P_{k+1}) \le V_0) \le c^*_{k+1} \cdot V_0^{d-k+1}. \tag{11}$$

By construction of the hypergraph $\mathcal{G}$ we have vol $(P_1, \ldots, P_j) > N^{-\gamma_j}$, and we condition on this in the following. Given the points $P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_g \in [0,1]^d$, we infer for $g = j, \ldots, k-1$:

$$\begin{aligned} &\text{Prob}\,(r_g \le \text{dist}\,(Q_{g+1}; \langle P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_g \rangle) \le r_g + \,\mathrm{d}r_g) \\ &\le (\sqrt{d})^{g-1} \cdot (d+1-g) \cdot C_{d+1-g} \cdot r_g^{d-g} \,\mathrm{d}r_g, \end{aligned} \tag{12}$$

since all points Q_{g+1} with dist $(Q_{g+1}; \langle P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_g \rangle) \le r$, are contained in the Cartesian product of a $(g-1)$-dimensional parallelepiped of volume at most $(\sqrt{d})^{g-1}$ and a $(d+1-g)$-dimensional ball of radius r.
Finally, given the points $P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_k$ in the unit cube $[0,1]^d$, such that dist $(Q_f; \langle P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_{f-1} \rangle) = r_{f-1}$, $f = j+1, \ldots, k$, to fulfill vol $(P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_{k+1}) \le V_0$, we must have

$$\frac{(j-1)!}{k!} \cdot \text{dist}\,(Q_{k+1}; \langle P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_k \rangle) \cdot \text{vol}\,(P_1, \ldots, P_j) \cdot \prod_{g=j}^{k-1} r_g \le V_0,$$

thus, with vol $(P_1, \ldots, P_j) > N^{-\gamma_j}$, we conclude

$$\text{dist}\,(Q_{k+1}; \langle P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_k \rangle) < \frac{k!}{(j-1)!} \cdot \frac{V_0 \cdot N^{\gamma_j}}{\prod_{g=j}^{k-1} r_g}.$$

Hence the point Q_{k+1} is contained in the Cartesian product of a $(k-1)$-dimensional parallelepiped of volume $(\sqrt{d})^{k-1}$ and a $(d-k+1)$-dimensional ball of radius $(k! \cdot V_0 \cdot N^{\gamma_j})/((j-1)! \cdot \prod_{g=j}^{k-1} r_g)$, which happens with probability at most

$$(\sqrt{d})^{k-1} \cdot C_{d-k+1} \cdot \left(\frac{k!}{(j-1)!} \cdot \frac{V_0 \cdot N^{\gamma_j}}{\prod_{g=j}^{k-1} r_g} \right)^{d-k+1} . \tag{13}$$

Putting (11)–(13) together, we obtain for constants $c^*_{k+1}, c_{p,j}, c^*_{p,j} > 0$, $j = 2, \ldots, k$, the following upper bound

$$\begin{aligned}
&\text{Prob}\left(\{P_1, \ldots, P_{k+1}\}, \{P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_{k+1}\} \text{ is a bad } j\text{-pair}\right) \\
&\leq c^*_{k+1} \cdot V_0^{d-k+1} \cdot \int_{r_{k-1}=0}^{\sqrt{d}} \cdots \int_{r_j=0}^{\sqrt{d}} d^{\frac{k-1}{2}} \cdot C_{d-k+1} \cdot \frac{(k! \cdot V_0 \cdot N^{\gamma_j})^{d-k+1}}{((j-1)! \cdot \prod_{g=j}^{k-1} r_g)^{d-k+1}} \cdot \\
&\quad \cdot \prod_{g=j}^{k-1} \left(d^{\frac{g-1}{2}} \cdot (d+1-g) \cdot C_{d+1-g} \cdot r_g^{d-g} \right) \, \mathrm{d}r_{k-1} \ldots \, \mathrm{d}r_j \leq \\
&\leq c^*_{p,j} \cdot V_0^{2(d-k+1)} \cdot N^{\gamma_j (d-k+1)} \cdot \int_{r_{k-1}=0}^{\sqrt{d}} \cdots \int_{r_j=0}^{\sqrt{d}} \prod_{g=j}^{k-1} r_g^{k-g-1} \, \mathrm{d}r_{k-1} \ldots \, \mathrm{d}r_j \\
&\leq c_{p,j} \cdot V_0^{2(d-k+1)} \cdot N^{\gamma_j (d-k+1)} \qquad \text{as } k-g-1 \geq 0.
\end{aligned} \tag{14}$$

As there are $\binom{N}{j}$ possibilities to choose j out of the N random points, and less than $\binom{N}{k+1-j}$ choices for $(k+1-j)$ out of $(N-j)$ points, by (14) we infer for constants $c_{p,j} > 0$, $j = 2, \ldots, k$:

$$\begin{aligned}
E(|BP_j(\mathcal{G})|) \leq \binom{N}{j} \cdot \binom{N}{k+1-j}^2 \cdot c_{p,j} \cdot V_0^{2(d-k+1)} \cdot N^{\gamma_j (d-k+1)} \leq \\
\leq c_{p,j} \cdot V_0^{2(d-k+1)} \cdot N^{2k+2-j+\gamma_j(d-k+1)},
\end{aligned}$$

which proves (10) and finishes the proof of Lemma 2. $\square$

By (9) and (10) and Markov's inequality, there exist $N = n^{1+\beta}$ points $P_1, \ldots, P_N$ in $[0,1]^d$ such that the corresponding hypergraph $\mathcal{G} = (V, \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_{k+1})$ on $|V| = N$ vertices, which we consider in the following, satisfies for $i, j = 2, \ldots, k$:

$$|V| = N \tag{15}$$

$$|\mathcal{E}_i| \leq 2 \cdot k \cdot c_i \cdot N^{i-\gamma_i(d-i+2)} \tag{16}$$

$$|\mathcal{E}_{k+1}| \leq 2 \cdot k \cdot c_{k+1} \cdot V_0^{d-k+1} \cdot N^{k+1} \tag{17}$$

$$|BP_j(\mathcal{G})| \leq 2 \cdot k \cdot c_{p,j} \cdot V_0^{2(d-k+1)} \cdot N^{2k+2-j+\gamma_j(d-k+1)}. \tag{18}$$

Set for some suitable constant $c^* > 0$, which will be fixed later in connection with (23) :

$$V_0 := (c^*)^{\frac{k}{d-k+1}} \cdot (\log n)^{\frac{1}{d-k+1}} / n^{\frac{k}{d-k+1}} . \tag{19}$$

Lemma 3. *Let d, k be fixed integers with $2 \le k \le d$. For $j = 2, \ldots, k$, and fixed $\gamma_j > 0$ with $\gamma_j < (2 \cdot k)/((1+\beta) \cdot (d-k+1)) - (2 \cdot k + 1 - j)/(d-k+1)$ it is*

$$|BP_j(\mathcal{G})| = o(|V|) . \tag{20}$$

Proof. Using (15), (18), (19) and $N = n^{1+\beta}$, we infer

$$\begin{aligned}
&|BP_j(\mathcal{G})| = o(|V|) \\
\Longleftarrow\; & V_0^{2(d-k+1)} \cdot N^{2k+2-j+\gamma_j(d-k+1)} = o(N) \\
\Longleftrightarrow\; & \frac{(\log n)^2}{n^{2k}} \cdot N^{2k+1-j+\gamma_j(d-k+1)} = o(1) \\
\Longleftrightarrow\; & (\log n)^2 \cdot n^{(1+\beta)((2k+1-j)+\gamma_j(d-k+1))-2k} = o(1) \\
\Longleftrightarrow\; & \gamma_j < \frac{2 \cdot k}{(1+\beta) \cdot (d-k+1)} - \frac{2 \cdot k + 1 - j}{d-k+1} ,
\end{aligned}$$

as claimed. □

Lemma 4. *Let d, k be fixed integers with $2 \le k \le d$. For $i = 2, \ldots, k$, and fixed γ_i with $\gamma_i > (i-1)/(d-i+2)$ it is*

$$|\mathcal{E}_i| = o(|V|) . \tag{21}$$

Proof. By (15), (16) and using $N = n^{1+\beta}$, we infer

$$\begin{aligned}
&|\mathcal{E}_i| = o(|V|) \\
\Longleftarrow\; & N^{i-\gamma_i(d-i+2)} = o(N) \\
\Longleftrightarrow\; & \gamma_i > \frac{i-1}{d-i+2},
\end{aligned}$$

as desired. □

Now we fix $\gamma_i := (i-1)/(d-i+3/2)$, $i = 2, \ldots, k$, and $\beta := 1/(8 \cdot k \cdot d)$. Certainly, for these choices of γ_i, $i = 2, \ldots, k$, the assumptions of Lemma 4 are satisfied. To see that also the assumptions of Lemma 3 are fulfilled, notice that for $0 < \beta \le 1/(8 \cdot k \cdot d)$ it is $(2 \cdot k)/(1+\beta) \ge 2 \cdot k - 1/(4 \cdot d)$ and that

$$\begin{aligned}
&\frac{i-1}{d-i+3/2} < \frac{2 \cdot k}{(d-k+1) \cdot (1+\beta)} - \frac{2 \cdot k + 1 - i}{d-k+1} \\
\Longleftarrow\; & \frac{i-1}{d-i+3/2} < \frac{i-1-1/(4 \cdot d)}{d-k+1} \\
\Longleftrightarrow\; & i^2 - i \cdot (k+3/2) - i/(4 \cdot d) < -k - 3/4 - 3/(8 \cdot d) .
\end{aligned} \tag{22}$$

For $2 \le i \le k$, the left hand side of (22) achieves its maximum for $i = 2$ or $i = k$. For $i := 2$ and $i := k$ inequality (22) is equivalent to $k > 7/4 - 1/(8 \cdot d)$ and $k - 3/2 > (3 - 2 \cdot k)/(4 \cdot d)$, respectively. Both inequalities hold for $k \ge 2$. Hence, by choice of the constants γ_i and β, $i = 2, \ldots, k$, the assumptions in both Lemmas 3 and 4 are fulfilled.

In the hypergraph $\mathcal{G} = (V, \mathcal{E}_2 \cup \dots \cup \mathcal{E}_{k+1})$ we discard one vertex from each i-element edge and from each bad j-pair of $(k+1)$-point simplices, $i, j = 2, \dots, k$. Let $V^* \subseteq V$ be the set of remaining vertices. On the vertex-set V^* the induced subhypergraph $\mathcal{G}^*$ of $\mathcal{G}$ is $(k+1)$-uniform and linear, hence $\mathcal{G}^* = (V^*, \mathcal{E}^*_{k+1})$ with $\mathcal{E}^*_{k+1} := \mathcal{E}_{k+1} \cap [V^*]^{k+1}$, and fulfills $|V^*| = (1 - o(1)) \cdot |V|$ by (20) and (21). Thus, we have $|V^*| \geq N/2$ and $|\mathcal{E}^*_{k+1}| \leq |\mathcal{E}_{k+1}| \leq 2 \cdot k \cdot c_{k+1} \cdot V_0^{d-k+1} \cdot N^{k+1}$ by (15) and (17), and $\mathcal{G}^*$ has average degree $t^k = (k+1) \cdot |\mathcal{E}^*_{k+1}|/|V^*| \leq 4 \cdot k^2 \cdot c_{k+1} \cdot V_0^{d-k+1} \cdot N^k =: t_1^k$, hence $t_1 = \omega(n^\beta)$ by (19) and with $N = n^{1+\beta}$. Now the assumptions of Theorem 2 are fulfilled by this subhypergraph $\mathcal{G}^*$, i.e., $\mathcal{G}^*$ is $(k+1)$-uniform and linear, and with (3) and (19), by using that the function $f(t) = (\log t)^{1/k}/t$ is decreasing for $t \geq e^{1/k}$, we obtain with $N = n^{1+\beta}$ for constants $C^*_{k+1}, C^{**}_{k+1}, c', c_{k+1}, c^* > 0$:

$$\begin{aligned}
&\alpha(\mathcal{G}) \geq \alpha(\mathcal{G}^*) \geq C^*_{k+1} \cdot \frac{|V^*|}{t} \cdot (\log t)^{\frac{1}{k}} \geq C^*_{k+1} \cdot \frac{|V^*|}{t_1} \cdot (\log t_1)^{\frac{1}{k}} \geq \\
&\geq C^*_{k+1} \cdot \frac{N/2}{(4 \cdot k^2 \cdot c_{k+1})^{\frac{1}{k}} \cdot V_0^{\frac{d-k+1}{k}} \cdot N} \cdot \left(\log \left(4 \cdot k^2 \cdot c_{k+1} \cdot V_0^{d-k+1} \cdot N^k\right)^{\frac{1}{k}} \right)^{\frac{1}{k}} \\
&\geq C^{**}_{k+1} \cdot \frac{n}{c^* \cdot (\log n)^{\frac{1}{k}}} \cdot \left(\log \left((4 \cdot k^2 \cdot c_{k+1})^{\frac{1}{k}} \cdot c^* \cdot n^\beta \cdot (\log n)^{\frac{1}{k}} \right) \right)^{\frac{1}{k}} \\
&\geq n \,, \qquad (23)
\end{aligned}$$

where the last inequality follows by choosing in (19) a sufficiently small constant $c^* > 0$, i.e., $c^* < (C^*_{k+1} \cdot \beta^{1/k})/(4 \cdot k^2 \cdot c_{k+1})^{1/k}$. This choice is possible as the constants $\beta, c_{k+1}, C^*_{k+1}$ do not depend on c^*. Thus the hypergraph $\mathcal{G}$ contains an independent set $I \subseteq V$ with $|I| = n$. These n vertices yield n points among the N points in $[0,1]^d$, such that the volume of each $(k+1)$-point simplex among these n points is bigger than V_0, i.e., $\Delta_{k,d}(n) = \Omega((\log n)^{1/(d-k+1)}/n^{k/(d-k+1)})$, which proves the lower bound (1) in Theorem 1. □

4 Upper Bounds on $\Delta_{k,d}(n)$

Here we show the upper bounds (1) and (2) in Theorem 1, namely, that for fixed $1 \leq k \leq d$ and constants $c'_{k,d}, c''_{k,d} > 0$ the inequalities $\Delta_{k,d}(n) \leq c'_{k,d}/n^{k/d}$, and, moreover, $\Delta_{k,d}(n) \leq c''_{k,d}/n^{k/d+(k-1)/(2d(d-1))}$ for odd $k \geq 1$ hold. The first upper bound can be obtained by the pigeonhole-principle, as has been done in [4]. Our arguments for proving the second upper bound also give a proof for the first upper bound.

Proof. First we prove for fixed $1 \leq k \leq d$ the general upper bound $\Delta_{k,d}(n) \leq c'_{k,d}/n^{k/d}$ for constants $c'_{k,d} > 0$. Given any n points $P_1, P_2, \dots, P_n \in [0,1]^d$, for some number D with $0 < D \leq 1$ we construct a graph $G = G(D) = (V, E)$ with vertex-set $V = \{1, \dots, n\}$, where vertex i corresponds to the point $P_i \in [0,1]^d$, and with edge-set E, where $\{i, j\} \in E$ if and only if dist $(P_i, P_j) \leq D$.

An independent set $I \subseteq V$ in this graph $G = G(D)$ yields a subset $I' \subseteq \{P_1, P_2, \ldots, P_n\}$ of points in $[0,1]^d$, such that the Euclidean distance between any two distinct points from I' is bigger than D. Each ball $B_r(P)$ with center $P \in [0,1]^d$ and radius $r \leq 1$ satisfies vol $(B_r(P) \cap [0,1]^d) \geq$ vol $(B_r(P))/2^d$. The balls with radius $D/2$ and centers from an independent set I' have pairwise empty intersection. Each ball $B_{D/2}(P)$ has volume $C_d \cdot (D/2)^d$, hence $(|I'| \cdot C_d \cdot (D/2)^d)/2^d \leq$ vol $([0,1]^d) = 1$, and we infer for the independence number $\alpha(G)$:

$$\alpha(G) \leq \frac{4^d}{C_d \cdot D^d}. \tag{24}$$

Set $D := c/n^{1/d}$ with $c \geq ((k \cdot 2 \cdot 4^d)/C_d)^{1/d}$ a constant. Let $t := (2 \cdot |E|)/n$ denote the average degree of the graph G. If $t < 1$, then we have $|E| < n/2$, and by deleting one vertex from each edge in E we obtain $\alpha(G) > n/2$. But then (24) yields $n/2 < 4^d/(C_d \cdot D^d)$, hence $k < 1$ by the choice of the constant c, which is a contradiction. Thus, we have $t \geq 1$ and Turán's theorem for graphs yields for the independence number the lower bound $\alpha(G) \geq n/(2 \cdot t)$. With (24) this implies

$$\frac{n}{2 \cdot t} \leq \alpha(G) \leq \frac{4^d}{C_d \cdot D^d} \quad \Longrightarrow \quad t \geq \frac{C_d}{2 \cdot 4^d} \cdot n \cdot D^d = k, \tag{25}$$

hence the average degree of the graph G is at least k. Then, there exists a vertex $i_1 \in V$ and k edges $\{i_1, i_2\}, \ldots, \{i_1, i_{k+1}\} \in E$, which are incident to i_1. By construction, each point $P_{i_j} \in [0,1]^d$, $j = 2, \ldots, k+1$, satisfies dist $(P_{i_1}, P_{i_j}) \leq D$, in particular, dist $(P_{i_j}; \langle P_{i_1}, P_{i_2}, \ldots, P_{i_{j-1}} \rangle) \leq c/n^{1/d}$, which implies vol $(P_{i_1}, \ldots, P_{i_{k+1}}) \leq ((1/k!) \cdot c^k)/n^{k/d}$, i.e., $\Delta_{k,d}(n) = O(1/n^{k/d})$.
For odd integers $k \geq 3$ we are able to improve this upper bound by taking into account also the angles between the directions, which are determined by the edges in the graph G. From (25) we obtain $|E| = n \cdot t/2 \geq (C_d \cdot n^2 \cdot D^d)/4^{d+1}$. We use a modification of an argument of Brass [8]. Each edge $\{i,j\} \in E$ determines an affine line $\langle P_i P_j \rangle$ and this line determines a direction $(P_i P_j)$, which is viewed as a vector of length 1. The volume of the surface of the d dimensional unit ball is equal to $d \cdot C_d$. Let ϕ be such that $|E| \cdot (\sin(\phi/2))^{d-1} \cdot C_{d-1} \geq \binom{k+1}{2} \cdot d \cdot C_d$, say, $\sin \phi = c_{k,d}/n^{1/(d-1)}$ for a constant $c_{k,d} > 0$. Then there exist $\binom{k+1}{2}$ directions $(P_i P_j)$, $\{i,j\} \in E$, with pairwise angle between them at most ϕ. The corresponding set $E^* \subseteq E$ of $\binom{k+1}{2}$ edges covers a subset $S \subseteq V$ of at least $(k+1)$ vertices. Consider a minimum subset $E^{**} \subseteq E^*$ of edges, which covers a subset $S^* \subseteq S$ of exactly $(k+1)$ vertices in G. Since $(k+1)$ is even, this set E^{**} exists and contains only isolated edges and stars, thus $|E^{**}| \geq (k+1)/2$. We pick one vertex from each isolated edge $E \in E^{**}$ and the center of each star. Let $S^{**} \subseteq S^*$ be the set of chosen vertices with $|S^{**}| = s \leq (k+1)/2$.
For each vertex $v \in S^* \setminus S^{**}$ there exists an edge $\{v, w\} \in E^{**}$ for some vertex $w \in S^{**}$, hence dist $(P_v, P_w) \leq D$. Having fixed such vertices $v \in S^* \setminus S^{**}$ and $w \in S^{**}$, for each vertex $u \in S^* \setminus (S^{**} \cup \{v\})$ there exists some vertex $t \in S^{**}$

such that $\{u,t\} \in E^{**}$ and the angle between the directions (P_uP_t) and (P_vP_w) is at most ϕ. Thus, the Euclidean distance of the point P_u from the affine space generated by the points P_r, $r \in S^{**} \cup \{v\}$, is at most $D \cdot \sin\phi$. The vertices in S^{**} pairwise have Euclidean distance at most $\sqrt{d}$, hence the $(s-1)$-dimensional volume of the corresponding simplex satisfies $(s-1)! \cdot \text{vol}\,(p_q \;:\; q \in S^{**}) \leq (\sqrt{d})^{s-1}$.
With $D = c/n^{1/d}$ and $\sin\phi = c_{k,d}/n^{1/(d-1)}$ we obtain for the volume of the simplex determined by the $(k+1)$ points P_s, $s \in S^*$, for a constant $c''_{k,d} > 0$ the following upper bound

$$\text{vol}\,(P_{s^*}; s^* \in S^*) \leq \frac{1}{k!} \cdot (\sqrt{d})^{s-1} \cdot D \cdot (D \cdot \sin\phi)^{k-s} \leq$$

$$\leq \frac{1}{k!} \cdot d^{\frac{k-1}{4}} \cdot D \cdot \left(\frac{c_{k,d} \cdot D}{n^{\frac{1}{d-1}}}\right)^{\frac{k-1}{2}} = \frac{c''_{k,d}}{n^{\frac{k}{d}+\frac{k-1}{2d(d-1)}}},$$

which finishes the proof of Theorem 1. □

5 Concluding Remarks

The arguments, which were presented here, together with an algorithmic version of Theorem 2, see [6], yield a randomized polynomial in n time algorithm for obtaining a distribution of n points in $[0,1]^d$, which shows the lower bound $\Delta_{k,d}(n) = \Omega((\log n)^{1/k}/n^{k/(d-k+1)})$ for fixed $2 \leq k \leq d$. In view of the recent results of Barequet and Shaikhet [5] it might be of some interest to achieve similar lower bounds, say $\Omega(1/n^{k/(d-k+1)})$ in the on-line situation. Moreover, it might be of interest to get a deterministic polynomial in n time algorithm, which achieves the lower bound $\Delta_{k,d}(n) = \Omega((\log n)^{1/k}/n^{k/(d-k+1)})$, as well as investigating the case $k > d+1$, but so far concerning this case only partial results are known, compare [15] for dimension $d = 2$, where the area of the convex hull of $(k+1)$ points among n points in $[0,1]^2$ has been considered.

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