Distributions of Points and Large Convex Hulls of k Points

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Abstract. We consider a variant of Heilbronn's triangle problem by asking for fixed integers $d, k \geq 2$ and any integer $n \geq k$ for a distribution of n points in the d-dimensional unit cube $[0,1]^d$ such that the minimum volume of the convex hull of k points among these n points is as large as possible. We show that there exists a configuration of n points in $[0,1]^d$, such that, simultaneously for $j = 2, \ldots, k$, the volume of the convex hull of any j points among these n points is $\Omega(1/n^{(j-1)/(1+|d-j+1|)})$. Moreover, for fixed $k \geq d+1$ we provide a deterministic polynomial time algorithm, which finds for any integer $n \geq k$ a configuration of n points in $[0,1]^d$, which achieves, simultaneously for $j = d+1, \ldots, k$, the lower bound $\Omega(1/n^{(j-1)/(1+|d-j+1|)})$ on the minimum volume of the convex hull of any j among the n points.

1 Introduction

For integers $n \geq 3$, Heilbronn's problem asks for the supremum $\Delta_2(n)$ of the minimum area of a triangle formed by three of n points over all distributions of n points in the unit square $[0,1]^2$. It has been observed by Erdős, see [16], that $\Delta_2(n) = \Omega(1/n^2)$, which can be seen by considering for primes n the points $P_k = 1/n \cdot (k \bmod n, k^2 \bmod n)$, $k = 0, 1, \ldots, n-1$. Komlós, Pintz and Szemerédi [10] improved this lower bound to the currently known best lower bound $\Delta_2(n) = \Omega(\log n/n^2)$, see [4] for a deterministic polynomial time algorithm achieving this lower bound. Upper bounds were given in a series of papers by Roth [16–19] and Schmidt [20], and the currently known best upper bound is due to Komlós, Pintz and Szemerédi [9], who proved that $\Delta_2(n) = O(2^{c\sqrt{\log n}}/n^{8/7})$ for some constant $c > 0$. We remark that for n points, which are chosen uniformly at random in $[0,1]^2$, the expected value of the minimum area of a triangle is $\Theta(1/n^3)$, as was shown recently by Jiang, Li and Vitany [8].

A variant of Heilbronn's problem in dimension $d \geq 2$, which has been considered by Barequet, asks for the supremum $\Delta_{d+1,d}(n)$ – over all distributions of n points in the d-dimensional unit cube $[0,1]^d$ – of the minimum volume of a $(d+1)$-point simplex among n points. Barequet showed in [2] the lower bound $\Delta_{d+1,d}(n) = \Omega(1/n^d)$ for fixed $d \geq 2$, see [3] for an on-line version for dimensions $d = 3, 4$. His lower bound was improved in [11] to $\Delta_{d+1,d}(n) = \Omega(\log n/n^d)$, and in [15] for dimension $d = 3$ a deterministic polynomial time algorithm was given, which achieves $\Delta_{4,3}(n) = \Omega(\log n/n^3)$. Recently, Brass [5] improved the upper bound $\Delta_{d+1,d}(n) = O(1/n)$ to $\Delta_{d+1,d}(n) = O(1/n^{(2d+1)/(2d)})$ for odd $d \geq 3$.

Here we consider the following generalization of Heilbronn's problem: for fixed integers $d, k \geq 2$ and any integer $n \geq k$ find n points in the d-dimensional unit cube $[0,1]^d$, such that the minimum volume of the convex hull of any k points among these n points is as large as possible. Let the corresponding supremum values – over all distributions of n points in $[0,1]^d$ – on the minimum volumes of the convex hull of k points among n points be denoted by $\Delta_{k,d}(n)$.
This problem has been investigated also by Chazelle, who considered it in connection with lower bounds on the query complexity of range searching problems. He proved in [7] that for any fixed dimension $d \geq 2$ there exists a constant $c > 0$ such that a random set of n points in the unit cube $[0,1]^d$ satisfies with probability greater than $1 - 1/n$, that the volume of the convex hull of any $k \geq \log n$ points is $\Omega(k/n)$, indeed it holds $\Delta_{k,d}(n) = \Theta(k/n)$ for $\log n \leq k \leq n$ for fixed $d \geq 2$. An extension of the range of k might also improve his lower bounds on the query complexity, see [7].
Here we consider the case of fixed values k and d. Areas of triangles arising from n points in $[0,1]^d$ have been investigated in [12], where for fixed dimension $d \geq 2$ it has been shown that $\Delta_{3,d}(n) = \Omega((\log n)^{1/(d-1)}/n^{2/(d-1)})$ and $\Delta_{3,d}(n) = O(1/n^{2/d})$. Moreover, for fixed $k \leq d+1$ it has been proved recently in [14] that $\Delta_{k,d}(n) = \Omega((\log n)^{1/(d-k+2)}/n^{(k-1)/(d-k+2)})$. For the special case of dimension $d = 2$ and arbitrary $k \geq 3$ it was shown in [13] that $\Delta_{k,2}(n) = \Omega((\log n)^{1/(k-1)}/n^{(k-1)/(k-2)})$.
Here we prove the following lower bounds, in particular for $k > d$.

Theorem 1. *Let $d, k \geq 2$ be fixed integers.*

(i) Then, for any integer $n \geq k$ there exists a configuration of n points in the unit cube $[0,1]^d$, such that, simultaneously for $j = 2, \ldots, k$, the volume of the convex hull of any j points among these n points is

$$\Omega(1/n^{(j-1)/(1+|d-j+1|)}). \tag{1}$$

(ii) Moreover, for fixed $k \geq d+1$ there is a deterministic polynomial time algorithm, which finds for any integer $n \geq k$ a configuration of n points in $[0,1]^d$, which, simultaneously for $j = d+1, \ldots, k$, achieves the lower bound $\Omega(1/n^{(j-1)/(1+|d-j+1|)})$ on the volume of the convex hull of any j among the n points in $[0,1]^d$.

Our arguments remain valid if d and k are functions of n, but then the lower bound (1) will depend on d and j. Notice that for fixed integers $d, j \geq 2$, Theorem 1 yields $\Delta_{j,d} = \Omega(1/n^{(j-1)/(1+|d-j+1|)})$. Concerning upper bounds, for fixed integers $d, j \geq 2$ a partition of $[0,1]^d$ into d-dimensional subcubes each of volume $\Theta(n^{-1/j})$, yields $\Delta_{j,d}(n) = O(1/n^{(j-1)/d})$ for $j \leq d+1$ and $\Delta_{j,d}(n) = O(1/n)$ for $j \geq d+1$. Moreover, for even integers j, $2 \leq j \leq d+1$, the upper bound can be improved to $\Delta_{j,d}(n) = O(1/n^{(j-1)/d+(j-2)/(2d(d-1))})$, see [14].
Somewhat surprisingly, achieving by a deterministic polynomial time algorithm for the same n points in $[0,1]^d$ the lower bound $\Delta_{j,d}(n) = \Omega(1/n^{(j-1)/(1+|d-j+1|)})$, simultaneously for $j = 2, \ldots, k$, where $d, k \geq 2$ are fixed integers, causes so far some difficulties w.r.t. the lower dimensional simplices, i.e., for $4 \leq j \leq d$.

2 Lower Bounds

Let dist (P_i, P_j) be the *Euclidean distance* between the points $P_i, P_j \in [0,1]^d$. A *simplex* given by the points $P_1, \ldots, P_j \in [0,1]^d$, $2 \le j \le d+1$, is the set of all points $P_1 + \sum_{i=2}^{j} \lambda_i \cdot (P_i - P_1)$ with $\sum_{i=2}^{j} \lambda_i \le 1$ and $\lambda_2, \ldots, \lambda_j \ge 0$. The (($j-1$)-dimensional) *volume of a simplex* given by j points $P_1, \ldots, P_j \in [0,1]^d$, $2 \le j \le d+1$, is defined by vol $(P_1, \ldots, P_j) := 1/(j-1)! \cdot \prod_{i=2}^{j}$ dist $(P_i; \langle P_1, \ldots, P_{i-1}\rangle)$, where dist $(P_i; \langle P_1, \ldots, P_{i-1}\rangle)$ is the Euclidean distance of the point P_i from the affine real space $\langle P_1, \ldots, P_{i-1}\rangle$ generated by the vectors $P_2^\top - P_1^\top, \ldots, P_{i-1}^\top - P_1^\top$ attached at P_1. For j points $P_1, \ldots, P_j \in [0,1]^d$, $j \ge d+1$, let vol $(P_1, \ldots, P_j)$ be the (d-dimensional) volume of the convex hull of the points $P_1, \ldots, P_j$.
First we prove part (i) of Theorem 1.

Proof. Let $d, k \ge 2$ be fixed integers. For arbitrary integers $n \ge k$, we select uniformly at random and independently of each other $N := k \cdot n$ points $P_1, P_2, \ldots, P_N$ from the unit cube $[0,1]^d$.
Set $v_j := \beta_j / n^{\gamma_j}$ for constants $\beta_j, \gamma_j > 0$, $j = 2, \ldots, k$, which will be fixed later. Let $V := \{P_1, P_2, \ldots, P_N\}$ be the random set of chosen points in $[0,1]^d$. For $j = 2, \ldots, k$, let $\mathcal{E}_j$ be the set of all j-element subsets $\{P_{i_1}, \ldots, P_{i_j}\} \in [V]^j$ of points in V such that vol $(P_{i_1}, \ldots, P_{i_j}) \le v_j$. We estimate the expected numbers $E(|\mathcal{E}_j|)$ of j-element sets in $\mathcal{E}_j$, $j = 2, \ldots, k$, and we show that for a suitable choice of the parameters $v_2, \ldots, v_k$ all numbers $E(|\mathcal{E}_j|)$ are not too big, i.e., $E(|\mathcal{E}_2|) + \cdots + E(|\mathcal{E}_k|) \le (k-1) \cdot n$. Thus, there exists a choice of N points $P_1, P_2, \ldots, P_N \in [0,1]^d$ such that $|\mathcal{E}_2| + \cdots + |\mathcal{E}_k| \le (k-1) \cdot n$. Then, for $j = 2, \ldots, k$, we delete one point from each j-element set of points in $\mathcal{E}_j$. The remaining points yield at least n points such that the volume of the convex hull of any j points of these at least n points is at least v_j.

Lemma 1. *Let $d, k \ge 2$ be fixed integers. For $j = 2, \ldots, k$, there exist constants $c_{j,d} > 0$ such that for every real $v_j > 0$ it is*

$$E(|\mathcal{E}_j|) \le c_{j,d} \cdot N^j \cdot v_j^{1+|d-j+1|}. \tag{2}$$

Proof. For reals $v_j > 0$ and random points $P_1, \ldots, P_j \in [0,1]^d$ we give an upper bound on the probability Prob (vol $(P_1, \ldots, P_j) \le v_j$). We assume that the points $P_1, \ldots, P_j$ are numbered such that for $2 \le g \le h \le j$ and $g \le d+1$ it is

$$\text{dist}\ (P_g; \langle P_1, \ldots, P_{g-1}\rangle) \ge \text{dist}\ (P_h; \langle P_1, \ldots, P_{g-1}\rangle)\,. \tag{3}$$

The point P_1 can be anywhere in $[0,1]^d$. Given the point P_1, the probability, that the point $P_2 \in [0,1]^d$ has from P_1 a Euclidean distance within the infinitesimal range $[r_1, r_1 + dr_1]$, is at most the difference of the volumes of the d-dimensional balls with center P_1 and with radii $(r_1 + dr_1)$ and r_1, respectively, hence

$$\text{Prob}\ (r_1 \le \ \text{dist}\ (P_1, P_2) \le r_1 + dr_1) \le d \cdot C_d \cdot r_1^{d-1} dr_1\,,$$

where C_d denotes the volume of the d-dimensional unit ball in $\mathbb{R}^d$.

Given the points P_1 and P_2 with dist $(P_1, P_2) = r_1$, the probability that the Euclidean distance of the point $P_3 \in [0,1]^d$ from the affine line $\langle P_1, P_2 \rangle$ is within the infinitesimal range $[r_2, r_2 + dr_2]$ is at most the difference of the volumes of two cylinders centered at the line $\langle P_1, P_2 \rangle$ with radii $r_2 + dr_2$ and r_2, respectively, and, by assumption (3), with height $2 \cdot r_1 = 2 \cdot \text{dist } (P_1, P_2)$, thus

$$\text{Prob } (r_2 \leq \text{ dist } (P_3; \langle P_1, P_2 \rangle) \leq r_2 + dr_2) \leq 2 \cdot r_1 \cdot (d-1) \cdot C_{d-1} \cdot r_2^{d-2} dr_2 \, .$$

In general, let the points $P_1, \ldots, P_g$, $g < j$ and $g < d+1$, be given with dist $(P_x; \langle P_1, \ldots, P_{x-1} \rangle) = r_{x-1}$ for $x = 2, \ldots, g$. For $g \leq j-2$ and $g \leq d-1$, by (3) the projection of the point P_{g+1} onto the affine space $\langle P_1, \ldots, P_g \rangle$ is contained in a $(g-1)$-dimensional box with volume $2^{g-1} \cdot r_1 \cdots r_{g-1}$, hence

$$\begin{aligned} &\text{Prob } (r_g \leq \text{dist } (P_{g+1}; \langle P_1, \ldots, P_g \rangle) \leq r_g + dr_g) \\ &\leq 2^{g-1} \cdot r_1 \cdots r_{g-1} \cdot (d-g+1) \cdot C_{d-g+1} \cdot r_g^{d-g} dr_g \, . \end{aligned} \tag{4}$$

For $g = j-1 < d$, to satisfy vol $(P_1, \ldots, P_j) \leq v_j$, we must have $1/(j-1)! \cdot \prod_{i=2}^{j} \text{dist } (P_i; \langle P_1, \ldots, P_{i-1} \rangle) \leq v_j$. By (3) the projection of the point P_j onto the affine space $\langle P_1, \ldots, P_{j-1} \rangle$ is contained in a $(j-2)$-dimensional box with volume $2^{j-2} \cdot r_1 \cdots r_{j-2}$, and the point P_j has Euclidean distance at most $((j-1)! \cdot v_j)/(r_1 \cdots r_{j-2})$ from the affine space $\langle P_1, \ldots, P_{j-1} \rangle$, which happens with probability at most

$$2^{j-2} \cdot r_1 \cdots r_{j-2} \cdot C_{d-j+2} \cdot \left(\frac{(j-1)! \cdot v_j}{r_1 \cdots r_{j-2}} \right)^{d-j+2} . \tag{5}$$

For $d \leq g \leq j-1$, the projection of the point P_{g+1} onto the affine space $\langle P_1, \ldots, P_d \rangle$ is contained in a $(d-1)$-dimensional box with volume at most $2^{d-1} \cdot r_1 \cdots r_{d-1}$. Since vol $(P_1, \ldots, P_d, P_{g+1}) \leq v_j$ by monotonicity, the point P_{g+1} has Euclidean distance at most $(d! \cdot v_j)/(r_1 \cdots r_{d-1})$ from the affine space $\langle P_1, \ldots, P_d \rangle$, which happens with probability at most

$$2^{d-1} \cdot r_1 \cdots r_{d-1} \cdot \frac{2 \cdot d! \cdot v_j}{r_1 \cdots r_{d-1}} = d! \cdot 2^d \cdot v_j \, . \tag{6}$$

Thus, for $j \leq d$ with (4) and (5) and some constants $c_{j,d}^*, c_{j,d}^{**} > 0$, we obtain

$$\begin{aligned} &\text{Prob } (\text{vol } (P_1, \ldots, P_j) \leq v_j) \\ &\leq \int_{r_{j-2}=0}^{\sqrt{d}} \cdots \int_{r_1=0}^{\sqrt{d}} 2^{j-2} \cdot \frac{C_{d-j+2} \cdot ((j-1)!)^{d-j+2} \cdot v_j^{d-j+2}}{(r_1 \cdots r_{j-2})^{d-j+1}} \cdot \\ &\quad \cdot \prod_{g=1}^{j-2} \left(2^{g-1} \cdot r_1 \cdots r_{g-1} \cdot (d-g+1) \cdot C_{d-g+1} \cdot r_g^{d-g} \right) dr_{j-2} \ldots dr_1 \\ &\leq c_{j,d}^{**} \cdot v_j^{d-j+2} \cdot \int_{r_{j-2}=0}^{\sqrt{d}} \cdots \int_{r_1=0}^{\sqrt{d}} \prod_{g=1}^{j-2} \left(r_g^{2j-2g-3} \right) dr_{j-2} \ldots dr_1 \\ &\leq c_{j,d}^* \cdot v_j^{d-j+2} \qquad\qquad \text{as } 2 \cdot j - 2 \cdot g - 3 \geq 1 \\ &= c_{j,d}^* \cdot v_j^{1+|d-j+1|} \qquad\qquad \text{as } j \leq d. \end{aligned} \tag{7}$$

Moreover, for $j = d+1, \ldots, k$, by (4) and (6) for constants $c^*_{j,d}, c^{**}_{j,d} > 0$ we infer

$$\begin{aligned}
&\text{Prob}\,(\text{vol}\,(P_1, \ldots, P_j) \leq v_j) \\
\leq &\int_{r_{d-1}=0}^{\sqrt{d}} \cdots \int_{r_1=0}^{\sqrt{d}} (d! \cdot 2^d \cdot v_j)^{j-d} \cdot \\
&\cdot \prod_{g=1}^{d-1} \left(2^{g-1} \cdot r_1 \cdots r_{g-1} \cdot (d-g+1) \cdot C_{d-g+1} \cdot r_g^{d-g}\right) dr_{d-1} \ldots dr_1 \\
\leq &\, c^{**}_{j,d} \cdot v_j^{j-d} \cdot \int_{r_{d-1}=0}^{\sqrt{d}} \cdots \int_{r_1=0}^{\sqrt{d}} \prod_{g=1}^{d-1} \left(r_g^{2d-2g-1}\right) dr_{d-1} \ldots dr_1 \\
\leq &\, c^*_{j,d} \cdot v_j^{j-d} \qquad\qquad \text{as } 2 \cdot d - 2 \cdot g - 1 \geq 1 \\
= &\, c^*_{j,d} \cdot v_j^{1+|d-j+1|} \qquad \text{as } j \geq d+1.
\end{aligned} \tag{8}$$

By (7) and (8) we have $\text{Prob}\,(\text{vol}\,(P_1, \ldots, P_j) \leq v_j) \leq c^*_{j,d} \cdot v_j^{1+|d-j+1|}$ for constants $c^*_{j,d} > 0$, $j = 2, \ldots, k$. Since there are $\binom{N}{j}$ choices for j out of the N random points $P_1, \ldots, P_N \in [0,1]^d$, inequality (2) follows. □

By (2) and Markov's inequality there exist $N = k \cdot n$ points $P_1, \ldots, P_N$ in the unit cube $[0,1]^d$ such that for $j = 2, \ldots, k$:

$$|\mathcal{E}_j| \leq k \cdot c_{j,d} \cdot N^j \cdot v_j^{1+|d-j+1|}. \tag{9}$$

Lemma 2. *Let $d, k \geq 2$ be fixed integers. Then, for every β_j, γ_j with $0 < \beta_j \leq 1/(c_{j,d} \cdot k^{j+1})^{1/(1+|d-j+1|)}$ and $\gamma_j \geq (j-1)/(1+|d-j+1|)$, $j = 2, \ldots, k$, it is*

$$|\mathcal{E}_j| \leq N/k\,. \tag{10}$$

Proof. For $j = 2, \ldots, k$, by (9) and using $v_j = \beta_j / n^{\gamma_j}$ we infer

$$\begin{aligned}
&|\mathcal{E}_j| \leq N/k \\
\Longleftarrow\ & k \cdot c_{j,d} \cdot N^j \cdot v_j^{1+|d-j+1|} \leq N/k \\
\Longleftrightarrow\ & k^{j+1} \cdot c_{j,d} \cdot \beta_j^{1+|d-j+1|} \cdot n^{j-1-\gamma_j(1+|d-j+1|)} \leq 1\,,
\end{aligned}$$

which holds for $j - 1 \leq \gamma_j \cdot (1 + |d-j+1|)$ and $k^{j+1} \cdot c_{j,d} \cdot \beta_j^{1+|d-j+1|} \leq 1$. □

Fix $\gamma_j := (j-1)/(1+|d-j+1|)$ and $\beta_j := 1/(c_{j,d} \cdot k^{j+1})^{1/(1+|d-j+1|)}$, $j = 2, \ldots, k$. By Lemma 2 we have $|\mathcal{E}_2| + \cdots + |\mathcal{E}_k| \leq ((k-1)/k)) \cdot N$. For $j = 2, \ldots, k$, we discard one point from each j-element set in $\mathcal{E}_j$. Then, the set $I \subseteq V$ of remaining points contains a subset of size $N/k = n$. These n points in $[0,1]^d$ satisfy, simultaneously for $j = 2, \ldots, k$, that the volume of the convex hull of each j of these n points is bigger than $v_j = \beta_j / n^{(j-1)/(1+|d-j+1|)}$, which finishes the proof of part (i) and (1) in Theorem 1. □

3 A Deterministic Algorithm

Here we derandomize the probabilistic arguments from Section 2 to show Theorem 1, part (ii). Throughout this section, let $k \geq d+1$. Let $B^d(T)$ denote the d-dimensional ball with radius T around the origin. Then $B^d(T) \cap \mathbb{Z}^d$ is the set of all points $P \in \mathbb{Z}^d$, which have Euclidean distance at most T from the origin. To provide a deterministic polynomial time algorithm which, for any integer $n > 0$, finds a configuration of n points in $[0,1]^d$, such that the volume of the convex hull of small sets of points is large, we discretize the unit cube $[0,1]^d$ by considering, for T large enough, but bounded from above by a polynomial in n, all points in $B^d(T) \cap \mathbb{Z}^d$. This set $B^d(T) \cap \mathbb{Z}^d$ will be rescaled later by the factor T^d. However, with this discretization we have to take care of degenerate sets of points, where a set $\{P_1, \ldots, P_j\} \subset [0,1]^d$ with $j \geq d+1$ is called *degenerate*, if all points $P_1, \ldots, P_j$ are contained in a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$, otherwise $\{P_1, \ldots, P_j\}$ is called *non-degenerate*.

Set $v_j := \beta_j \cdot T^d / n^{(j-1)/(j-d)}$ for suitable constants $\beta_j > 0$, $j = d+1, \ldots, k$, which will be fixed later. We construct for $j = d+1, \ldots, k$ two types of j-element edges. For points $P_{i_1}, \ldots, P_{i_j} \in B^d(T) \cap \mathbb{Z}^d$, let $\{P_{i_1}, \ldots, P_{i_j}\} \in \mathcal{E}_j$ if and only if vol $(P_{i_1}, \ldots, P_{i_j}) \leq v_j$ and $\{P_{i_1}, \ldots, P_{i_j}\}$ is not contained in a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$, i.e., the set $\{P_{i_1}, \ldots, P_{i_j}\}$ is non-degenerate. Moreover, let $\{P_{i_1}, \ldots, P_{i_j}\} \in \mathcal{E}_j^0$ if and only if $\{P_{i_1}, \ldots, P_{i_j}\}$ is contained in a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$.

To give upper bounds on these numbers $|\mathcal{E}_j|$ and $|\mathcal{E}_j^0|$ of j-element sets, $j = d+1, \ldots, k$, we use *lattices* in $\mathbb{Z}^d$.

A *lattice* L in $\mathbb{Z}^d$ is a subset of $\mathbb{Z}^d$, which is generated by all integral linear combinations of some linearly independent vectors $b_1, \ldots, b_m \in \mathbb{Z}^d$, hence $L = \mathbb{Z}b_1^\top + \cdots + \mathbb{Z}b_m^\top$. The parameter $m = \text{rank}(L)$ is called the *rank* of the lattice L, and the set $\mathcal{B} = \{b_1, \ldots, b_m\}$ is called a *basis* of L. The set $F_\mathcal{B} := \{\sum_{i=1}^m \alpha_i \cdot b_i \mid 0 \leq \alpha_i \leq 1,\ i = 1, \ldots, m\} \subseteq \mathbb{R}^d$ is called the *fundamental parallelepiped* $F_\mathcal{B}$ of $\mathcal{B}$, its *volume* is $\text{vol}(F_\mathcal{B}) := (\det(G(\mathcal{B})^\top \cdot G(\mathcal{B})))^{1/2}$, where $G(\mathcal{B}) := (b_1, \ldots, b_m)_{d \times m}$ is the $d \times m$ *generator matrix* of $\mathcal{B}$ (up to the ordering of the vectors). If $\mathcal{B}$ and $\mathcal{B}'$ are two bases of a lattice L in $\mathbb{Z}^d$, then the volumes of the fundamental parallelepipeds are equal, i.e., $\text{vol}(F_\mathcal{B}) = \text{vol}(F_{\mathcal{B}'})$, see [6].

For integers $a_1, \ldots, a_n \in \mathbb{Z}$, which are not all equal to 0, let $\gcd(a_1, \ldots, a_n)$ denote the *greatest common divisor* of $a_1, \ldots, a_n$. For vectors $a = (a_1, \ldots, a_d)^\top \in \mathbb{R}^d$ and $b = (b_1, \ldots, b_d)^\top \in \mathbb{R}^d$ let $\langle a, b \rangle := \sum_{i=1}^d a_i \cdot b_i$ be the standard scalar product. The length of a vector $a \in \mathbb{R}^d$ is defined by $\|a\| := \sqrt{\langle a, a \rangle}$. For a lattice L in $\mathbb{Z}^d$ let span(L) be the linear space over the reals, which is generated by the vectors in L. For a subset $S = \{P_1, \ldots, P_k\} \subset \mathbb{R}^d$ of points the *rank* of S is the dimension of the linear space over the reals, which is generated by the vectors $P_2^\top - P_1^\top, \ldots, P_k^\top - P_1^\top$.

A vector $a = (a_1, \ldots, a_d)^\top \in \mathbb{Z}^d \setminus \{0^d\}$ is called *primitive*, if $\gcd(a_1, \ldots, a_d) = 1$ and $a_j > 0$ with $j = \min\{i \mid a_i \neq 0\}$. A lattice L in $\mathbb{Z}^d$ is called *m-maximal*, if $\text{rank}(L) = m$ and no other lattice $L' \neq L$ in $\mathbb{Z}^d$ with $rank(L') = m$ contains L as a proper subset. There is a one-to-one correspondence between *m-maximal* lattices in $\mathbb{Z}^d$ and primitive vectors $a = (a_1, \ldots, a_d)^\top \in \mathbb{Z}^d \setminus \{0^d\}$:

(i) For each lattice L in $\mathbb{Z}^d$ with $\operatorname{rank}(L) = d-1 \geq 1$ there is exactly one primitive vector $a_L = (a_1, \ldots, a_d)^\top \in \mathbb{Z}^d \setminus \{0^d\}$ with $\langle a_L, x^\top \rangle = 0$ for every $x \in L$. This vector $a_L \in \mathbb{Z}^d \setminus \{0^d\}$ is called the *primitive normal vector* of the lattice L.

(ii) For each lattice L' in $\mathbb{Z}^d$ with $\operatorname{rank}(L') = d-1$ there is exactly one $(d-1)$-maximal lattice L in $\mathbb{Z}^d$ with $L' \subseteq L$.

(iii) There exists a bijection between the set of all $(d-1)$-maximal lattices L in $\mathbb{Z}^d$ and the set of all primitive vectors a_L in $\mathbb{Z}^d$.

For a $(d-1)$-maximal lattice L in $\mathbb{Z}^d$, a *residue class* of L is a set L' of the form $L' = x + L$ with $x \in \mathbb{Z}^d$.
The proofs of Lemmas 3 – 6 concerning lattices can be found in [15].

Lemma 3 ([15]). *Let L be a $(d-1)$-maximal lattice in $\mathbb{Z}^d$ with primitive normal vector $a_L \in \mathbb{Z}^d$ and with basis $\mathcal{B}$.*

(i) *There exists a point $v \in \mathbb{Z}^k \setminus L$ such that $\mathbb{Z}^d$ can be partitioned into the* residue classes *$s \cdot v + L$, $s \in \mathbb{Z}$, and, for each point $x \in L$, it is* $\operatorname{dist}(s \cdot v + x, \operatorname{span}(L)) = |s|/\|a_L\|$.

(ii) *The volume of the fundamental parallelepiped $F_{\mathcal{B}}$ fulfills* $\operatorname{vol}(F_{\mathcal{B}}) = \|a_L\|$.

Lemma 4 ([15]). *Let $d \in \mathbb{N}$ be fixed. Let $S \subseteq B^d(T) \cap \mathbb{Z}^d$ be a set of points with* $\operatorname{rank}(S) \leq d-1$*. Then there exists a $(d-1)$-maximal lattice L of $\mathbb{Z}^d$ such that S is contained in some residue class $L' = v + L$ of L for some $v \in \mathbb{Z}^d$, and L has a basis $b_1, \ldots, b_{d-1} \in \mathbb{Z}^d$ with* $\max_{i=1,\ldots,d-1} \|b_i\| = O(T)$.

The next lemma is crucial in our considerations to estimate the numbers $|\mathcal{E}_j|$ and $|\mathcal{E}_j^0|$ of j-element sets, $j = d+1, \ldots, k$.

Lemma 5 ([15]). *Let $d \in \mathbb{N}$ be fixed. Let L be a $(d-1)$-maximal lattice of $\mathbb{Z}^d$ with primitive normal vector $a_L \in \mathbb{Z}^d$, and let $\mathcal{B} = \{b_1, \ldots, b_{d-1}\}$ be a basis of L with* $\max_{i=1,\ldots,d-1} \|b_i\| = O(T)$*. Then the following hold:*

(i) *The primitive normal vector a_L satisfies $\|a_L\| = O(T^{d-1})$.*

(ii) *For every residue class L' of L it is $|L' \cap B^d(T)| = O\big(T^{d-1}/\|a_L\|\big)$.*

For integers $g, l \in \mathbb{N}$ let $r_g(l)$ be the number of representations $x_1^2 + \cdots + x_g^2 = l$ with $x_1, \ldots, x_g \in \mathbb{Z}$.

Lemma 6 ([15]). *Let $g, r \in \mathbb{N}$ be fixed integers. Then, for all integers $m \in \mathbb{N}$:*

$$\sum_{l=1}^{m} \frac{r_g(l)}{l^r} = \begin{cases} O\left(m^{g/2-r}\right) & \text{if } g/2 - r > 0 \\ O\left(\log m\right) & \text{if } g/2 - r = 0 \\ O(1) & \text{if } g/2 - r < 0. \end{cases}$$

Lemma 7. *Let $d, k \geq 2$ be fixed integers with $k \geq d+1$. For $j = d+1, \ldots, k$, there exist constants $c_{j,0} > 0$, such that the numbers $|\mathcal{E}_j^0|$ of j-element degenerate sets of points in $B^d(T) \cap \mathbb{Z}^d$ satisfy*

$$|\mathcal{E}_j^0| \leq c_{j,0} \cdot T^{(d-1)j+1} \cdot \log T. \tag{11}$$

Proof. By Lemma 4, each degenerate j-element subset of points in $B^d(T)\cap\mathbb{Z}^d$ is contained in a residue class L' of some $(d-1)$-maximal lattice L in $\mathbb{Z}^d$, and L has a basis $b_1,\ldots,b_{d-1}\in\mathbb{Z}^d$ with $\|b_i\|=O(T)$, $i=1,\ldots,d-1$. By Lemma 5(i), it suffices to consider all $(d-1)$-maximal lattices L with primitive normal vectors $a_L\in\mathbb{Z}^d$ of length $\|a_L\|=O(T^{d-1})$.
Having fixed a $(d-1)$-maximal lattice L in $\mathbb{Z}^d$, which is determined by its primitive normal vector $a_L\in\mathbb{Z}^d$, by Lemma 3(i), there are $O(T\cdot\|a_L\|)$ residue classes L' of the lattice L with $L'\cap B^d(T)\neq\emptyset$. By Lemma 5(ii), each set $L'\cap B^d(T)$ contains $O(T^{d-1}/\|a_L\|)$ points. From each set $L'\cap B^d(T)$ we can select j points in $\binom{O(T^{d-1}/\|a_L\|)}{j}$ ways to obtain a degenerate set of j points. This implies

$$\begin{aligned}|\mathcal{E}_j^0| &= O\left(\sum_{a\in\mathbb{Z}^d,\,\|a\|=O(T^{d-1})} T\cdot\|a\|\cdot\binom{T^{d-1}/\|a\|}{j}\right)\\ &= O\left(T^{(d-1)j+1}\cdot\sum_{a\in\mathbb{Z}^d,\,\|a\|=O(T^{d-1})}\frac{1}{\|a\|^{j-1}}\right)\\ &= O\left(T^{(d-1)j+1}\cdot\sum_{l=1}^{O(T^{2d-2})}\frac{r_d(l)}{l^{(j-1)/2}}\right) = O\left(T^{(d-1)j+1}\cdot\log T\right),\end{aligned}$$

since, by Lemma 6, we have $\sum_{l=1}^{m} r_d(l)/l^{(j-1)/2} = O(\log m)$ for $j=d+1$ and $\sum_{l=1}^{m} r_d(l)/l^{(j-1)/2}=O(1)$ for $j=d+2,\ldots,k$. □

Lemma 8. *Let $d,k\geq 2$ be fixed integers with $k\geq d+1$. For $j=d+1,\ldots,k$, there exist constants $c_j>0$, such that the numbers $|\mathcal{E}_j|$ of j-element non-degenerate sets of points in $B^d(T)\cap\mathbb{Z}^d$ with the volume of their convex hull at most v_j, fulfill*

$$|\mathcal{E}_j|\leq c_j\cdot T^{d^2}\cdot v_j^{j-d}. \tag{12}$$

Proof. For $j=d+1,\ldots,k$, consider j points $P_1,\ldots,P_j\in B^d(T)\cap\mathbb{Z}^d$ with vol $(P_1,\ldots,P_j)\leq v_j$, where $\{P_1,\ldots,P_j\}$ is non-degenerate. Let these points be numbered such that for $2\leq g\leq h\leq j$ and $g\leq d+1$ it is

$$\text{dist}\,(P_g;\langle P_1,\ldots,P_{g-1}\rangle)\geq \text{dist}\,(P_h;\langle P_1,\ldots,P_{g-1}\rangle)\,. \tag{13}$$

By Lemma 4, the points $P_1,\ldots,P_d\in B^d(T)\cap\mathbb{Z}^d$ are contained in a residue class L' of some $(d-1)$-maximal lattice L in $\mathbb{Z}^d$ with primitive normal vector $a_L\in\mathbb{Z}^d$, where L has a basis $b_1,\ldots,b_{d-1}\in\mathbb{Z}^d$ with $\|b_i\|=O(T)$ for $i=1,\ldots,d-1$. By Lemma 5(i), it suffices to consider all $(d-1)$-maximal lattices L with primitive vectors $a_L\in\mathbb{Z}^d$ of length $\|a_L\|=O(T^{d-1})$.
We fix a $(d-1)$-maximal lattice L in $\mathbb{Z}^d$, which is determined by its primitive normal vector $a_L\in\mathbb{Z}^d$. By Lemma 3(i), there are $O(T\cdot\|a_L\|)$ residue classes L' of L with $L'\cap B^d(T)\neq\emptyset$. By Lemma 5(ii), from each set $L'\cap B^d(T)$ we

can select d points $P_1, \ldots, P_d$ in $\binom{O(T^{d-1}/\|a_L\|)}{d}$ ways. By (13) we infer for the $(d-1)$-dimensional volume vol $(P_1, \ldots, P_d) > 0$, as otherwise $\{P_1, \ldots, P_j\}$ is degenerate. Also by (13) the projection of each point $P_i \in B^d(T) \cap \mathbb{Z}^d$, $i = d+1, \ldots, j$, onto the residue class L' is contained in a $(d-1)$-dimensional box of volume $2^{d-1} \cdot (d-1)! \cdot$ vol $(P_1, \ldots, P_d)$, which, by Lemma 3(ii), contains at most

$$2^{d-1} \cdot (d-1)! \cdot 2^{d-1} \cdot \text{vol } (P_1, \ldots, P_d)/\|a_L\| \tag{14}$$

points of L', since $P_1, \ldots, P_d \in L'$. With vol $(P_1, \ldots, P_d, P_i) \leq v_j$ it follows that dist $(P_i, \langle P_1, \ldots, P_d \rangle) \leq d \cdot v_j/\text{vol } (P_1, \ldots, P_d)$, and, by Lemma 3(i), each point $P_i \in B^d(T) \cap \mathbb{Z}^d$, $i = d+1, \ldots, j$, is contained in one of at most

$$\|a_L\| \cdot d \cdot v_j/\text{vol } (P_1, \ldots, P_d) \tag{15}$$

residue classes L'' of L. By (14) in each residue class L'' we can choose at most $(d-1)! \cdot 2^{2d-2} \cdot$ vol $(P_1, \ldots, P_d)/\|a_L\|$ points $P_i \in B^d(T) \cap \mathbb{Z}^d$, hence with (15) each point P_i, $i = d+1, \ldots, j$, can be chosen in at most $d! \cdot 2^{2d-2} \cdot v_j$ ways. Applying this to each point $P_{d+1}, \ldots, P_j \in B^d \cap \mathbb{Z}^d$, we infer the upper bound

$$\begin{aligned}
|\mathcal{E}_j| &= O\left(\sum_{a \in \mathbb{Z}^d,\, \|a\| = O(T^{d-1})} T \cdot \|a\| \cdot \binom{T^{d-1}/\|a\|}{d} \cdot v_j^{j-d} \right) \\
&= O\left(T^{d^2-d+1} \cdot v_j^{j-d} \cdot \sum_{a \in \mathbb{Z}^d,\, \|a\| = O(T^{d-1})} \frac{1}{\|a\|^{d-1}} \right) \\
&= O\left(T^{d^2-d+1} \cdot v_j^{j-d} \cdot \sum_{l=1}^{O(T^{2d-2})} \frac{r_d(l)}{l^{(d-1)/2}} \right) = O(T^{d^2} \cdot v_j^{j-d}) \,,
\end{aligned}$$

since, by Lemma 6, we have $\sum_{l=1}^{m} r_d(l)/l^{(d-1)/2} = O(m^{1/2})$. □

For fixed integers $d, j, k \geq 2$ the sets $\mathcal{E}_j$ and $\mathcal{E}_j^0$, can easily be constructed in time polynomial in T. Namely, by considering every j-element subset $S \subset B^d(T) \cap \mathbb{Z}^d$ of points, we determine all degenerate sets of j points in $B^d(T) \cap \mathbb{Z}^d$ and all non-degenerate sets of j points in $B^d(T) \cap \mathbb{Z}^d$ with volume of their convex hulls at most v_j in time $O(T^{dj})$, since there are $\binom{O(T^d)}{j}$ j-element subsets in $B^d(T) \cap \mathbb{Z}^d$. Let $|B^d(T) \cap \mathbb{Z}^d| = C'_d \cdot T^d$, where $C'_d > 0$ is a constant. We enumerate the points in $B^d(T) \cap \mathbb{Z}^d$ by $P_1, \ldots, P_{C'_d \cdot T^d}$. To each point P_i associate a parameter $p_i \in [0,1]$, $i = 1, \ldots, C'_d \cdot T^d$, and define a potential function $F(p_1, \ldots, p_{C'_d \cdot T^d})$:

$$\begin{aligned}
&F(p_1, \ldots, p_{C'_d \cdot T^d}) := 2^{pC'_d T^d/2} \cdot \prod_{i=1}^{C'_d T^d} \left(1 - \frac{p_i}{2}\right) + \\
&+ \sum_{j=d+1}^{k} \frac{\sum_{\{i_1, \ldots i_j\} \in \mathcal{E}_j} p_{i_1} \cdots p_{i_j}}{2 \cdot k \cdot p^j \cdot c_j \cdot T^{d^2} \cdot v_j^{j-d}} + \sum_{j=d+1}^{k} \frac{\sum_{\{i_1, \ldots, i_j\} \in \mathcal{E}_j^0} p_{i_1} \cdots p_{i_j}}{2 \cdot k \cdot p^j \cdot c_{j,0} \cdot T^{(d-1)j+1} \cdot \log T} \,.
\end{aligned}$$

With the initialisation $p_1 := \cdots := p_{C'_d \cdot T^d} := p = (2 \cdot k \cdot n)/(C'_d \cdot T^d) \le 1$, i.e., say $T^d = \omega(n)$, we infer by Lemmas 7 and 8 that $F(p, \ldots, p) < (2/e)^{pC'_d T^d/2} + (2k - 2d)/(2k)$, which is less than 1 for $p \cdot C'_d \cdot T^d \ge 7 \cdot \ln k$. Using the linearity of $F(p_1, \ldots, p_{C'_d \cdot T^d})$ in each p_i, we minimize $F(p_1, \ldots, p_{C'_d \cdot T^d})$ by choosing one after the other $p_i := 0$ or $p_i := 1$ for $i = 1, \ldots, C'_d \cdot T^d$, and finally we obtain $F(p_1, \ldots, p_{C'_d \cdot T^d}) < 1$. With $V^* = \{P_i \in B^d(T) \cap \mathbb{Z}^d \mid p_i = 1\}$ this yields a subset $V^* \subseteq B^d(T) \cap \mathbb{Z}^d$ of points and subsets $\mathcal{E}_j^{0*} := [V^*]^j \cap \mathcal{E}_j^0$ and $\mathcal{E}_j^* := [V^*]^j \cap \mathcal{E}_j$ of j-element sets, $j = d+1, \ldots, k$, such that

$$|V^*| \ge p \cdot C'_d \cdot T^d/2 \tag{16}$$

$$|\mathcal{E}_j^*| \le 2 \cdot k \cdot p^j \cdot c_j \cdot T^{d^2} \cdot v_j^{j-d} \tag{17}$$

$$|\mathcal{E}_j^{0*}| \le 2 \cdot k \cdot p^j \cdot c_{j,0} \cdot T^{(d-1)j+1} \cdot \log T\,. \tag{18}$$

By choice of the parameters v_j, $j = d+1, \ldots, k$, the running time of this derandomization is $O(T^d + \sum_{j=d+1}^{k} (|\mathcal{E}_j| + |\mathcal{E}_j^0|)) = O(T^{dk})$, which is polynomial in T for fixed integers $d, k \ge 2$.

Lemma 9. *For $j = d+1, \ldots, k$, and $0 < \beta_j \le (C'^j_d/(2^{j+2} \cdot k^{j+1} \cdot c_{j,d}))^{1/(j-d)}$, it is*

$$|\mathcal{E}_j^*| \le |V^*|/(2 \cdot k)\,.$$

Proof. By (16) and (17) with $v_j := \beta_j \cdot T^d/n^{\frac{j-1}{j-d}}$, and $p = (2 \cdot k \cdot n)/(C'_d \cdot T^d)$, and with $\beta_j > 0$ it is

$$\begin{aligned}
&|\mathcal{E}_j^*| \le |V^*|/(2 \cdot k)\\
\Longleftarrow\ & 2 \cdot k \cdot p^j \cdot c_j \cdot T^{d^2} \cdot v_j^{j-d} \le p \cdot C'_d \cdot T^d/(4 \cdot k)\\
\Longleftrightarrow\ & 8 \cdot k^2 \cdot \left(\frac{2 \cdot k \cdot n}{C'_d \cdot T^d}\right)^{j-1} \cdot c_j \cdot T^{d^2-d} \cdot \left(\frac{\beta_j \cdot T^d}{n^{\frac{j-1}{j-d}}}\right)^{j-d} \le C'_d\\
\Longleftrightarrow\ & 2^{j+2} \cdot k^{j+1} \cdot c_j \cdot \beta_j^{j-d} \le C'^j_d\,,
\end{aligned}$$

which holds for $\beta_j^{j-d} \le C'^j_d/(2^{j+2} \cdot k^{j+1} \cdot c_j)$, $j = d+1, \ldots, k$. □

Lemma 10. *For $j = d+1, \ldots, k$, and $T/(\log T)^{1/(j-1)} = \omega(n)$, it is*

$$|\mathcal{E}_j^{0*}| \le |V^*|/(2 \cdot k)\,.$$

Proof. By (16) and (18), with $p = (2 \cdot k \cdot n)/(C'_d \cdot T^d)$, $j = d+1, \ldots, k$, we infer

$$\begin{aligned}
&|\mathcal{E}_j^{0*}| \le |V^*|/(2 \cdot k)\\
\Longleftarrow\ & 2 \cdot k \cdot p^j \cdot c_{j,0} \cdot T^{(d-1)j+1} \cdot \log T \le p \cdot C'_d \cdot T^d/(4 \cdot k)\\
\Longleftrightarrow\ & 8 \cdot k^2 \cdot \left(\frac{2 \cdot k \cdot n}{C'_d \cdot T^d}\right)^{j-1} \cdot c_{j,0} \cdot T^{(d-1)j-d+1} \cdot \log T \le C'_d\\
\Longleftrightarrow\ & 2^{j+2} \cdot k^{j+1} \cdot c_{j,0} \cdot \frac{n^{j-1}}{T^{j-1}} \cdot \log T \le C'^j_d\,,
\end{aligned}$$

which holds for $T/(\log T)^{1/(j-1)} = \omega(n)$. □

With $T := n \cdot \log n$ and $\beta_j := (C_d'^j/(2^{j+2} \cdot k^{j+1} \cdot c_j)^{1/(j-d)}$, $j = d+1, \ldots, k$, the assumptions of Lemmas 9 and 10 are fulfilled. By deleting in time $O(|V^*| + \sum_{j=d+1}^{k} (|\mathcal{E}_j^*| + |\mathcal{E}_j^{0*}|))O(T^{kd})$ one point from each j-element set in $\mathcal{E}_j^*$ and $\mathcal{E}_j^{0*}$, $j = d+1, \ldots, k$, the remaining points yield a subset $V^{**} \subseteq V^*$ of size at least $|V^*|/k \geq p \cdot C_d' \cdot T^d/(2 \cdot k) = n$. Then these at least n points in $B^d(T) \cap \mathbb{Z}^d$ satisfy that the volume of the convex hull of any j of these points, $j = d+1, \ldots, k$, is at least v_j, i.e., $\Omega(T^d/n^{(j-1)/(j-d)})$. After rescaling by the factor T^d, we have at least n points in the unit cube $[0,1]^d$ such that the volume of the convex hull of any j of these points is $\Omega(1/n^{(j-1)/(j-d)})$, $j = d+1, \ldots, k$. Altogether the running time of this deterministic algorithm is $O((n \cdot \log n)^{dk})$ for fixed $d, k \geq 2$, hence polynomial in n, which finishes the proof of Theorem 1, part (ii).

4 Concluding Remarks

Our arguments yield a deterministic polynomial time algorithm for obtaining a distribution of n points in $[0,1]^d$, which, for fixed integers $j \geq d+1$, shows $\Delta_{j,d}(n) = \Omega(1/n^{(j-1)/(1+|d-j+1|)})$. With the results from [14], i.e., using a result of Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] on uncrowded hypergraphs we can improve Theorem 1 slightly (Details are omitted here), namely for fixed integers $d, k \geq 3$ and a fixed integer j_0 with $3 \leq j_0 \leq d+1$ one can find in polynomial time a configuration of n points in $[0,1]^d$, such that, simultaneously for $j = 2, \ldots, k$ but $j \neq j_0$, the volume of the convex hull of j points among these n points is at least $\Omega(1/n^{(j-1)/(1+|d-j+1|)})$ and $\Delta_{j_0,d}(n) = \Omega((\log n)^{1/(d-j_0+2)}/n^{(j_0-1)/(d-j_0+2)})$. It would be interesting to get such an improvement by a logarithmic factor for the same n points in $[0,1]^d$, simultaneously for $3 \leq j \leq k$, for fixed d, k.
Moreover, improvements of the existing upper bounds, which were given in the introduction, are desirable. Also investigations of this problem for non-constant values of k might be of interest in view of the results of Chazelle [7].

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