A Deterministic Polynomial Time Algorithm for Heilbronn's Problem in Dimension Three (Extended Abstract)

Hanno Lefmann* and Niels Schmitt†

Abstract

Heilbronn conjectured that among any n points in the 2-dimensional unit square $[0,1]$, there must be three points which form a triangle of area at most $O(1/n^2)$. This conjecture was disproved by Komlós, Pintz and Szemerédi [13] who showed that for every n there exists a configuration of n points in the unit square $[0,1]^2$ where all triangles have area at least $\Omega(\log n/n^2)$. Here we will consider a 3-dimensional analogue of this problem and we will give a deterministic polynomial time algorithm which finds n points in the unit cube $[0,1]^3$ such that the volume of every tetrahedron among these n points is at least $\Omega(\ln n/n^3)$.

1 Introduction

An old conjecture of Heilbronn stated that among any n points in the 2-dimensional unit square $[0,1]^2$ (or unit disc) there are three distinct points which form a triangle of area at most c/n^2 for some constant $c > 0$. Erdös observed that this conjecture, if true, would be best possible, as the points $(i \bmod n, i^2 \bmod n)_{i=0,\dots,n-1}$ on the moment-curve in the $n \times n$ grid would show after rescaling, cf. [2]. However, Komlós, Pintz and Szemerédi [13] disproved Heilbronn's conjecture by showing for every n the existence of a configuration of n points in the unit square $[0,1]$ with every three points forming a triangle of area at least $\Omega(\log n/n^2)$. Using derandomization techniques, this existence argument was made constructive in [5], where a polynomial time algorithm was given, which finds n points in $[0,1]^2$ achieving the lower bound $\Omega(\log n/n^2)$ on the minimum triangle area. Upper bounds on Heilbronn's triangle problem were given in a series of papers by Roth [15, 16, 18, 17, 19] and Schmidt [21], cf. Rothschild and Straus [20] for related results, and the currently best upper bound is due to Komlós, Pintz and Szemerédi [12] and is of the order $n^{-8/7+\varepsilon}$ for every fixed $\varepsilon > 0$.

If n points are dropped uniformly at random and independently of each other in the unit square $[0,1]^2$, then the expected value of the smallest area of a triangle among these n points is $\Theta(1/n^3)$, as recently was proved by Jiang, Li and Vitány [10].

A k-dimensional version of Heilbronn's problem was considered, also recently, by Barequet [3]. For given dimension $k \geq 3$ he showed, that for every $n \in \mathbf{N}$ there exist n points in the k-dimensional unit cube $[0,1]^k$ such that the minimum volume of every simplex spanned by any $(k+1)$ of these n points is at least $\Omega(1/n^k)$. Barequet gave two different approaches for proving this lower bound. One approach uses a random argument. The other approach is similar to Erdös' construction, namely taking the points $P_l = (l^j \bmod n/n)_{j=1,\dots,k}$ for $l = 0, 1, \dots, n-1$ on the moment-curve.

In [14] Barequet's lower bound was improved by a factor $\Theta(\ln n)$ for dimensions $k \geq 3$ by using a probabilistic existence argument, based on a variant of Theorem 2.1. For the proof the continuous structure of the unit cube $[0,1]^k$ was crucial.

THEOREM 1.1. **[14]** *For every fixed integer $k \geq 3$ and for every $n \in \mathbf{N}$ there exists a configuration of n points in the k-dimensional unit cube $[0,1]^k$ such that the volume of any simplex spanned by any $(k+1)$ points is at least $\Omega(\ln n/n^k)$.*

Here we will give for dimension $k = 3$ a deterministic polynomial time algorithm for the result in Theorem 1.1:

*Lehrstuhl Informatik II, Universität Dortmund, D-44221 Dortmund, Germany. lefmann@Ls2.informatik.uni-dortmund.de

†Lehrstuhl Informatik II, Universität Dortmund, D-44221 Dortmund, Germany. niels.schmitt@uni-bochum.de

THEOREM 1.2. *For every positive integer n one can find in polynomial time a configuration of n points in the unit cube $[0,1]^3$ such that the volume of any tetrahedron spanned by any four of these points is at least $\Omega(\ln n/n^3)$.*

The proof of Theorem 1.2 is based on techniques from combinatorics and algebraic number theory. Some of our arguments are given for the case of arbitrary dimension $k \geq 3$, where appropriate. However, so far we are only able to provide a polynomial time algorithm for the case $k = 3$.

2 Hypergraphs

In our arguments we use hypergraphs. The *independence number of a hypergraph* and 2-cycles is important in our considerations:

DEFINITION 2.1. *Let $\mathcal{G} = (V, \mathcal{E})$ be a* hypergraph *where each edge $E \in \mathcal{E}$ satisfies $E \subseteq V$. The hypergraph $\mathcal{G}$ is* k-uniform *if every edge $E \in \mathcal{E}$ contains exactly k vertices.*
A subset $I \subseteq V$ is called independent *if I contains no edge $E \in \mathcal{E}$. The largest size of an independent set in $\mathcal{G}$ is called the* independence number $\alpha(\mathcal{G})$.
In a k-uniform hypergraph $\mathcal{G} = (V, \mathcal{E})$, $k \geq 3$, a 2-*cycle is a pair $\{E_1, E_2\}$ of distinct edges $E_1, E_2 \in \mathcal{E}$ with $|E_1 \cap E_2| \geq 2$. A* 2-cycle *$\{E_1, E_2\}$ in $\mathcal{G}$ is called* $(2, j)$-cycle *if $|E_1 \cap E_2| = j$, where $j = 2, \ldots, k-1$.*

Let $B_k(T) = \{x \in \mathbf{R}^k | \; \|x\| \leq T\} \subset \mathbf{R}^k$ be the k-dimensional ball around the origin with radius T. We reformulate our combinatorial, geometrical problem as a problem of finding a large independent set in a suitably defined hypergraph. To do so, we discretize the 3-dimensional search space, namely we consider only grid points from the set $B_3(T) \cap \mathbf{Z}^3$, where T is of suitable size, i.e. polynomial in n. With this discretization, we must take care of degenerated tetrahedra with volume 0.
For some parameter $B > 0$ and for the set $B_3(T) \cap \mathbf{Z}^3$ of points we form a 4-uniform hypergraph $\mathcal{G} = \mathcal{G}(B) = (V, \mathcal{E})$ with the vertex set V being the set $B_3(T) \cap \mathbf{Z}^3$ of $\Theta(T^3)$ grid points. The edges $E \in \mathcal{E}$ are determined by all subsets of four points from the set $B_3(T) \cap \mathbf{Z}^3$, which form a tetrahedron of volume at most B. An independent set in this hypergraph $\mathcal{G}(B)$ corresponds to a subset of grid points in the set $B_3(T) \cap \mathbf{Z}^3$, where all tetrahedra have volume bigger than B. To find a large independent set, we use the following result due to Ajtai, Komlós, Pintz, Spencer and Szemerédi [1], compare [8], stated here in an algorithmic variant proven in [4], see [9]:

THEOREM 2.1. **[1], [8], [4]** *Let $k \geq 3$ be a fixed integer. Let $\mathcal{G} = (V, \mathcal{E})$ be a k-uniform hypergraph on $|V| = N$ vertices and with average degree $t^{k-1} := k \cdot |\mathcal{E}|/|V|$. If for some constant $\gamma > 0$ the hypergraph $\mathcal{G}$ contains at most $N \cdot t^{2k-j-1-\gamma}$ many (2,j)-cycles for $j = 2, \ldots, k-1$, then one can find in $\mathcal{G}$ in polynomial time an independent set of size at least $\Omega(N/t \cdot (\ln t)^{1/(k-1)})$.*

In recent years, several applications of Theorem 2.1 have been found, compare [4]. Here we give another application of this deep result.
A main part of the proof of Theorem 1.2 consists of counting the degenerated resp. non-degenerated tetrahedra in $\mathbf{Z}^3 \cap B_3(T)$.

3 Grids in $\mathbf{Z}^k$

DEFINITION 3.1. *A* grid *L of $\mathbf{Z}^k$ is a subset of $\mathbf{Z}^k$, which is generated by all linear combinations of some linearly independent vectors $q_1, \ldots, q_m \in \mathbf{R}^k$, where all coefficients are integers, i.e. $L = \mathbf{Z}q_1 + \ldots + \mathbf{Z}q_m$. The parameter m is called the* rank *of the grid L and the set $Q = \{q_1, \ldots, q_m\}$ is called a* basis *of L.*

DEFINITION 3.2. *Let $Q = \{q_1, \ldots, q_m\} \subset \mathbf{Z}^k$ be a set of linear independent vectors.*
i) The $k \times m$ generator matrix *of Q (up to the ordering of the vectors) is defined by $G(Q) := (q_1, \ldots, q_m)_{k \times m}$.*
ii) The fundamental parallelopiped *F_Q of Q is the following set*

$$F_Q := \left\{ \sum_{i=1}^{m} \alpha_i \cdot q_i \;|\; 0 \leq \alpha_1, \ldots, \alpha_m \leq 1 \right\} \subseteq \mathbf{R}^k.$$

iii) The volume *of the fundamental parallelopiped $F_Q \subseteq \mathbf{R}^k$ is given by*

$$vol(F_Q) := (\det(G(Q)^\top \cdot G(Q)))^{1/2}.$$

If Q and Q' are two basis of a grid $L \subseteq \mathbf{Z}^k$, then the volumes of the corresponding fundamental parallelopipeds are equal, i.e. $\mathrm{vol}(F_Q) = \mathrm{vol}(F_{Q'})$. The parameter $d(L) := \mathrm{vol}(F_Q)$ is called the *determinant* of the grid L, see [6].
For integers $a_1, \ldots, a_k \in \mathbf{Z}$, which are not all equal to 0, we denote by $\gcd(a_1, \ldots, a_k)$ the *greatest common divisor* of $a_1, \ldots, a_k$.
The next result, see [7], gives a recursive procedure to find a basis of a grid in $\mathbf{Z}^k$.

THEOREM 3.1. *Let* $a = (a_1, \ldots, a_k)^\top \in (\mathbf{Z} \setminus \{0\})^k$ *with* $\gcd(a_1, \ldots, a_k) = 1$. *Then the set L of all solutions in $\mathbf{Z}^k$ of the equation* $a_1 \cdot X_1 + \ldots + a_k \cdot X_k = 0$ *is a grid in* $\mathbf{Z}^k$ *with* $rank(L) = k - 1$.
The following procedure yields a basis Q of the grid L. For $k = 1$ let $Q = \{0\}$. For $k = 2$ let $Q = \{(a_2, -a_1)^\top\}$. *For $k \geq 3$ let* $d = \gcd(a_{k-1}, a_k) = a_{k-1} \cdot y_0 + a_k \cdot y_1$ *with* $y_0, y_1 \in \mathbf{Z}$. *If* $q_1^*, \ldots, q_{k-2}^*$ *is a basis of the set of all solutions of the equation* $a_1 \cdot X_1 + \ldots + a_{k-2} \cdot X_{k-2} + d \cdot Y = 0$ *in $k-1$ variables, then the vectors $q_1, \ldots, q_{k-1}$ form a basis of L:* $q_i = (q_{i,1}^*, \ldots, q_{i,k-2}^*, y_0 \cdot q_{i,k-1}^*, y_1 \cdot q_{i,k-1}^*)^\top$ *for* $i = 1, \ldots, k-2$, *and* $q_{k-1} = (0, \ldots, 0, a_k/d, -a_{k-1}/d)^\top)$.

We use the standard scalar product $< a, b > := \sum_{i=1}^k a_i \cdot b_i$ for vectors $a = (a_1, \ldots, a_k)^\top \in \mathbf{R}^k$ and $b = (b_1, \ldots, b_k)^\top \in \mathbf{R}^k$. The Euclidean distance $\mathrm{dist}(a, b)$ of the corresponding points is defined by $\mathrm{dist}(a, b) = (\sum_{i=1}^k (a_i - b_i)^2)^{1/2}$. The length of a vector $a \in \mathbf{R}^k$ is defined by $\|a\| := \sqrt{< a, a >}$. For a point $p \in \mathbf{R}^k$ and a real subspace $V \subseteq R^k$ let $\mathrm{dist}(p, V) := \min\{\mathrm{dist}(p, v) \mid v \in V\}$. For a set $\{q_1, \ldots, q_m\} \subseteq \mathbf{R}^k$ of vectors let $\mathrm{span}(\{q_1, \ldots, q_m\})$ be the linear space over $\mathbf{R}$, generated by $q_1, \ldots, q_m$. The following two results can be found in [11]:

LEMMA 3.1. *Let* $V \subseteq \mathbf{R}^k$ *be a* $(k-1)$*-dimensional linear subspace and let* $a \in \mathbf{R}^k \setminus \{0^k\}$ *be a nonzero vector which is orthogonal to V. For every point* $p \in \mathbf{R}^k$ *it is:* $dist(p, V) = | < p^\top, a > |/\|a\|$.

LEMMA 3.2. *Let* $q_1, \ldots, q_m \in \mathbf{R}^k$ *be linearly independent vectors over* $\mathbf{R}$. *Let* $U := span(\{q_1, \ldots, q_{m-1}\})$. *Then the fundamental parallelopiped* $F_{\{q_1, \ldots, q_m\}}$ *satisfies* $vol(F_{\{q_1, \ldots, q_m\}}) = dist(q_m, U) \cdot vol(F_{\{q_1, \ldots, q_{m-1}\}})$.

THEOREM 3.2. *Let* $k \in \mathbf{N}$ *be fixed. Let L be a grid in* $\mathbf{Z}^k$ *with* $rank(L) = m$ *and let* $a_1, \ldots, a_m \in L$ *be linearly independent. Then there exists a basis* $b_1, \ldots, b_m$ *of L with* $\|b_i\| = O(\max_j \|a_j\|)$ *for* $i = 1, \ldots, m$.

Proof. The proof is similar to that in [6], Lemma 8, p. 135-136, there based on Theorem I, p. 11-13 in [6], which can be adapted to our situation. □

3.1 Maximal Grids in $\mathbf{Z}^k$

DEFINITION 3.3. *A grid L in $\mathbf{Z}^k$ is called* m-maximal, *if* $rank(L) = m$ *and for every grid L' in $\mathbf{Z}^k$ with* $rank(L') = m$ *and* $L \subseteq L'$ *it is* $L = L'$.

A vector $a = (a_1, \ldots, a_k)^\top \in \mathbf{Z}^k \setminus \{0^k\}$ is called *primitive*, if $\gcd(a_1, \ldots, a_k) = 1$ and $a_j > 0$ for $j = \min\{i \mid a_i \neq 0\}$.

LEMMA 3.3. *There is a bijection between the set of all* $(k-1)$*-maximal grids L in $\mathbf{Z}^k$ and the set of all primitive vectors* $a_L \in \mathbf{Z}^k$, *i.e.* $a_L \in \mathbf{Z}^k$ *is the unique primitive normal vector of the grid L.*

Proof. Omitted. □

THEOREM 3.3. *Let L be a* $(k-1)$*-maximal grid in $\mathbf{Z}^k$ with primitive normal vector* $a_L = (a_1, \ldots, a_k) \in \mathbf{Z}^k$. *Then there exists a vector* $v \in \mathbf{Z}^k \setminus L$, *such that $\mathbf{Z}^k$ can be partitioned into the residue classes* $s \cdot v + L$, $s \in \mathbf{Z}$, *of L, i.e.,* $\mathbf{Z}^k = \biguplus_{s \in \mathbf{Z}} (s \cdot v + L)$, *and for each $x \in L$ it is* $dist(s \cdot v + x, span(L)) = |s|/\|a_L\|$.

Here, residue classes L' of a $(k-1)$-maximal grid L in $\mathbf{Z}^k$ are all of the form $L' = v + L$ for $v \in \mathbf{Z}^k$.

Proof. For the proof we use Lemma 3.1. Details can be found in the full version. □

THEOREM 3.4. *Let L be a* $(k-1)$*-maximal grid in $\mathbf{Z}^k$ with primitive normal vector* $a_L \in \mathbf{Z}^k$ *and with basis Q. Then the determinant $d(L)$ of L satisfies* $d(L) = vol(F_Q) = \|a_L\|$.

Proof. This can be proved with Lemma 3.2 and essentially, evaluating the determinant of the matrix $M = (q_1^*, \ldots, q_{k-1}^*, e_k^*)$, where $q_1^*, \ldots, q_{k-1}^* \in \mathbf{Z}^k$ is the basis of L from Theorem 3.1 and e_k^* is the k'th unit vector. Details are in the full version. □

3.2 Simplices and Maximal Grids in $\mathbf{Z^k}$

Definition 3.4. *For a subset $S = \{p_0, \dots, p_k\} \subset \mathbf{R}^k$ of $(k+1)$ points, the set $S^* = \{p_0 + \sum_{i=1}^k \lambda_i \cdot (p_i - p_0) \mid \sum_{i=1}^k \lambda_i \leq 1; \lambda_1, \dots, \lambda_k \in [0,1]\}$ is called a* simplex. *For short, we identify S and S^* and call both sets* simplex *and specify by calling the points $p_0, \dots, p_k$* extreme points *of the simplex.*
i) The rank *of the simplex S is defined by $rank(S) = \dim(span(\{p_1 - p_0, \dots, p_k - p_0\}))$.*
ii) The simplex S is non-degenerated, *if $rank(S) = k$. If $rank(S) < k$, we call S a* degenerated simplex.

Lemma 3.4. *Let $k \in \mathbf{N}$ be fixed. Let $S \subseteq \mathbf{Z}^k \cap B_k(T)$ be a set of points with $rank(S) \leq k-1$. Then there exists a $(k-1)$-maximal grid L of $\mathbf{Z}^k$ such that S is contained in some residue class $v + L$ of L for some $v \in \mathbf{Z}^k$, and L has a basis $q_1, \dots, q_{k-1} \subset \mathbf{Z}^k$ with $\max_i \|q_i\| = O(T)$.*

Proof. For $S = \{p_0, \dots, p_m\} \subseteq \mathbf{Z}^k \cap B_k(T)$ the vectors $p_1 - p_0, \dots, p_m - p_0$ span a grid L' in $\mathbf{Z}^k$ with $\text{rank}(L') = r \leq k-1$, and have length $\|p_i - p_0\| \leq 2 \cdot T$ for $i = 1, \dots, m$. Take $(k-1-r)$ unit vectors from $\mathbf{Z}^k \setminus L'$, add them to L', and obtain a grid L'' of $\mathbf{Z}^k$ with $\text{rank}(L'') = k-1$ and $L' \subseteq L''$. Then the grid L'' uniquely determines a $(k-1)$-maximal grid L of $\mathbf{Z}^k$ with $L' \subseteq L'' \subseteq L$ and with $S \subseteq p_0 + L$. By Theorem 3.2 there exists a basis $q_1, \dots, q_{k-1}$ of L with $\max_i \|q_i\| = O(T)$. □

The next result will be used frequently:

Theorem 3.5. *Let $k \in \mathbf{N}$ be fixed. Let L be a $(k-1)$-maximal grid of $\mathbf{Z}^k$ with primitive normal vector $a_L \in \mathbf{Z}^k$ and let $Q = \{q_1, \dots, q_{k-1}\}$ be a basis of L with $\max_i \|q_i\| = O(T)$. Then the following holds:*

i) The primitive normal vector a_L satisfies $\|a_L\| = O(T^{k-1})$.

ii) There are at most $O(T \cdot \|a_L\|)$ distinct residue classes $s \cdot v + L$ with $s \in \mathbf{Z}$ and $v \in \mathbf{Z}^k \setminus L$ with $(s \cdot v + L) \cap B_k(T) \neq \emptyset$.

iii) For every residue class $v + L$ with $v \in \mathbf{Z}^k$, we have $|(v + L) \cap B_k(T)| = O(T^{k-1}/\|a_L\|)$.

Proof. (i): By Theorem 3.4 we have $\|a_L\| = d(L) = \text{vol}(F_Q)$ and $F_Q \subseteq B_k(c \cdot T)$ for some constant $c > 0$. Since $\dim(F_Q) = k-1$, the volume of the parallelopiped F_Q at at most $O(T^{k-1})$, hence $\|a_L\| = \text{vol}(F_Q) = O(T^{k-1})$.
(ii): By Theorem 3.3 distances between distinct residue classes of L are multiples of $1/\|a_L\|$. The distance between two points in $B_k(T)$ is at most $2 \cdot T$, hence at most $O(T \cdot \|a_L\|)$ residue classes have a nonempty intersection with $B_k(T)$.
(iii): The volume of a $(k-1)$-dimensional space S intersected with $B_k(T)$ is at most $O(T^{k-1})$. Since $F_Q \subseteq B_k(c \cdot T)$ and $\text{vol}(F_Q) = \|a_L\|$, we can cover the set $S \cap B_k(c \cdot T)$ by at most $O(T^{k-1}/\|a_L\|)$ distinct translates of the parallelopiped F_Q. As L is maximal, the interior of F_Q contains no points from L. □

3.3 Representations by Sums of Squares

We mention without proofs some estimates concerning the number of representations of an integer as a sum of squares. Let $r_k(d)$ be the number of vectors $(x_1, \dots, x_k)^\top \in \mathbf{Z}^k$ with $x_1^2 + \ldots + x_k^2 = d$.

Lemma 3.5. *For fixed $k, r \in \mathbf{N}$ and for every $n \in \mathbf{N}$ it is $\sum_{d=1}^n r_k(d) = O(n^{k/2})$, and*

$$\sum_{d=1}^{n} \frac{r_k(d)}{d^r} = \begin{cases} O(n^{k/2-r}) & \text{if } k/2 - r > 0 \\ O(\ln n) & \text{if } k/2 - r = 0 \\ O(1) & \text{if } k/2 - r < 0. \end{cases}$$

For a $(k-1)$-maximal grid L in $\mathbf{Z}^k$ with primitive normal vector $a_L \in \mathbf{Z}^k$ and $d \in \mathbf{N}$, we denote by $r_k(d; a_L)$ the number of points $P \in L$ with $(\text{dist}(O, P))^2 = d$, where O is the origin.

Lemma 3.6. *Let $k, r \in \mathbf{N}$ be fixed. Let L be a $(k-1)$-maximal grid in $\mathbf{Z}^k$ with primitive normal vector $a_L \in \mathbf{Z}^k$. For every $n \in \mathbf{N}$ it is $\sum_{d=1}^n r_k(d; a_L) = O(n^{(k-1)/2}/\|a_L\|)$, and*

$$\sum_{d=1}^{n} \frac{r_k(d; a_L)}{d^r} = \begin{cases} O\left(\frac{n^{\frac{k-1}{2}-r}}{\|a_L\|}\right) & \text{if } \frac{k-1}{2} - r > 0 \\ O\left(\frac{\ln n}{\|a_L\|}\right) & \text{if } \frac{k-1}{2} - r = 0 \\ O\left(\frac{1}{\|a_L\|}\right) & \text{if } \frac{k-1}{2} - r < 0. \end{cases}$$

4 Degenerated Simplices in $\mathbf{Z^k} \cap B_k(T)$

Theorem 4.1. *For fixed $k \in \mathbf{N}$ the number $D_k(T)$ of degenerated simplices in $\mathbf{Z}^k \cap B_k(T)$ satisfies $D_k(T) = O(T^{k^2} \cdot \ln T)$.*

All these degenerated simplices in $\mathbf{Z}^k \cap B_k(T)$ can be constructed in time polynomial in T.

Proof. By Lemma 3.4 each degenerated $(k+1)$-element subset of points in $\mathbf{Z}^k \cap B_k(T)$ is contained in a residue class L' for some $(k-1)$-maximal grid L in $\mathbf{Z}^k$, where L has a basis $q_1, \ldots, q_{k-1} \in \mathbf{Z}^k$ with $\|q_i\| = O(T)$ for $i = 1, \ldots, k-1$. By Theorem 3.5 (i) it suffices to consider all primitive normal vectors $a_L \in \mathbf{Z}^k$ of length $\|a_L\| = O(T^{k-1})$. By Theorem 3.5 (ii) there are at most $O(T \cdot \|a_L\|)$ residue classes L' of L with $L' \cap B_k(T) \neq \emptyset$.
Fix a residue class L' of a $(k-1)$-maximal grid L in $\mathbf{Z}^k$, determined by its primitive normal vector $a_L \in \mathbf{Z}^k$ with $\|a_L\| = O(T^{k-1})$. By Theorem 3.5 (iii) the set $L' \cap B_k(T)$ contains at most $O(T^{k-1}/\|a_L\|)$ points, hence from $L' \cap B_k(T)$ we can select $(k+1)$ points in at most $O(\binom{T^{k-1}/\|a_L\|}{k+1})$ ways, to obtain a degenerated simplex. This implies

$$D_k(T) = O\left(\sum_{\substack{a \in \mathbf{Z}^k \\ \|a\| = O(T^{k-1})}} T \cdot \|a\| \cdot \binom{\frac{T^{k-1}}{\|a\|}}{k+1} \right)$$

$$= O\left(T^{k^2} \cdot \sum_{d=1}^{O(T^{2k-2})} \frac{r_k(d)}{d^{k/2}} \right) = O\left(T^{k^2} \cdot \ln T \right),$$

as $\sum_{d=1}^{n} r_k(d)/d^{k/2} = O(\ln n)$ by Lemma 3.5. □
For fixed $k \in \mathbf{N}$, by inspecting every $(k+1)$-element subset $S \subset \mathbf{Z}^k \cap B_k(T)$ we can determine easily all the volumes in polynomial time $O(T^{k+1})$.

5 Non-degenerated Tetrahedra in $\mathbf{Z}^3 \cap B_3(T)$

From now on we consider only the case of dimension $k = 3$. We estimate for reals $B > 0$ the number $N_3(T; B)$ of non-degenerated tetrahedra $S \subset \mathbf{Z}^3 \cap B_3(T)$ with volume $\mathrm{vol}(S) \leq B$. The *volume* of the tetrahedron $S = \{p_0, \ldots, p_3\}$ is defined by $\mathrm{vol}(S) = 1/3 \cdot h \cdot \mathrm{area}(p_0, p_1, p_2)$, where h is the distance of p_3 to the affine real space generated by p_0, p_1, p_2 and $\mathrm{area}(p_0, p_1, p_2)$ is the area of the triangle p_0, p_1, p_2.

Theorem 5.1. *The number $N_3(T; B)$ of non-degenerated tetrahedra $S \subseteq B_3(T) \cap \mathbf{Z}^3$ with $vol(S) \leq B$ satisfies $N_3(T; B) = O(B \cdot T^9)$.*
All these tetrahedra in $B_3(T) \cap \mathbf{Z}^3$ can be constructed in time polynomial in T.

To prove this, first we count the non-degenerated triangles S in $L \cap B_3(T)$ for a 2-maximal grid L in $\mathbf{Z}^3$ with (a) $\mathrm{area}(S) \geq v$ or (b) $\mathrm{area}(S) \leq v$.

Lemma 5.1. *Let L be a 2-maximal grid L in $\mathbf{Z}^3$ with primitive normal vector $a_L \in \mathbf{Z}^3$. Let $v, B > 0$ be real numbers. For every residue class L' of L the number of non-degenerated tetrahedra $S = \{p_0, p_1, p_2, p_3\}$ in $B_3(T) \cap \mathbf{Z}^3$ with $S \setminus \{p_3\} \subseteq L'$ and area$(S \setminus \{p_3\}) \geq v$ and vol$(S) \leq B$ is at most $O(B \cdot T^8/(v \cdot \|a_L\|^3))$.*

Proof. By Lemma 3.4 we can assume that the 2-maximal grid L in $\mathbf{Z}^3$ with primitive normal vector $a_L \in \mathbf{Z}^3$ has a basis $q_1, q_2 \in \mathbf{Z}^3$, with $\|q_1\|, \|q_2\| = O(T)$. For every residue class L' of L the set $L' \cap B_3(T)$ contains by Theorem 3.5 (iii) at most $O(T^2/\|a_L\|)$ points. We can choose from the set $L' \cap B_3(T)$ three points (with triangle-area at least v) in at most $O(\binom{T^2/\|a_L\|}{3})$ ways. Since the tetrahedra have volume at most B, the distance between the corresponding fourth point and the span of L' is at most $O(B/v)$. By Theorem 3.3 the distance between distinct residue classes of L is a multiple of $1/\|a_L\|$, and since $|L'' \cap B_3(T)| = O(T^2/\|a_L\|)$ for every residue class L'' of L, a fourth point can be chosen in at most

$$O(B/v \cdot \|a_L\| \cdot T^2/\|a_L\|) = O(B \cdot T^2/v)$$

ways, and we obtain for the number of tetrahedra

$$O\left(\binom{T^2/\|a_L\|}{3} \cdot \frac{B}{v} \cdot T^2 \right) = O\left(\frac{B \cdot T^8}{v \cdot \|a_L\|^3} \right). \quad \square$$

Lemma 5.2. *Let L be a 2-maximal grid in $\mathbf{Z}^3$ with primitive normal vector $a_L \in \mathbf{Z}^3$. Let L' be any residue class of L, and let P and Q be distinct points in L'. The number of non-degenerated triangles S in the set $L' \cap B_3(T)$ with extreme points P and Q and with area$(S) \leq v$ is at most $O(v \cdot T/(dist(P,Q) \cdot \|a_L\|))$.*

Proof. By an affine mapping $f: L \longrightarrow \mathbf{Z}^2$ with $P' := f(P)$ for $P \in L$, we transform the 2-maximal grid L in $\mathbf{Z}^3$ with primitive normal vector $a_L \in \mathbf{Z}^3$ (or any residue class L' of L) into the standard 2-dimensional rectangular grid $\mathbf{Z}^2$ with basis $(0,1)^\top$

and $(1,0)^\top$. Points $P,Q,R \in L' \cap B_3(T)$ become the grid points $P',Q',R' \in E \cap \mathbf{Z}^2$ in an ellipsoid E. If $\text{area}(P,Q,R) = v$, then $\text{area}(P',Q',R') = v/\|a_L\|$, as can be easily seen.
Let $a = (a_1,a_2,a_3)^\top \in \mathbf{Z}^3$, $b = (b_1,b_2,b_3)^\top \in \mathbf{Z}^3$ be a basis of the grid L. We can assume that $L' = L$ and that $P = (0,0,0)$ and $Q = \lambda \cdot a + \mu \cdot b$ for some $\lambda, \mu \in \mathbf{Z}$ are the two given points. With the mapping $f\colon L \longrightarrow \mathbf{Z}^2$ we obtain the points $f(P) = P' = (0,0)$ and $f(Q) = Q' = (\lambda,\mu)$ which are contained in the ellipsoid E. Let $g = \gcd(\lambda,\mu)$ and set $\lambda' := \lambda/g$ and $\mu' := \mu/g$, where $\mu' > 0$. The line L_1 through the points P' and Q' in $\mathbf{Z}^2$ has the primitive normal vector $a_N := (\mu', -\lambda')^T$.
To estimate the number of points $R \in L' \cap B_3(T)$ such that $\text{area}(P,Q,R) \le v$, we compute the number of points $R' \in E \cap \mathbf{Z}^2$ such that $\text{area}(P',Q',R') \le v/\|a_L\|$. The distance of R' to the line L_1 is at most $2 \cdot v/(\|a_L\| \cdot \text{dist}(P',Q'))$. By Theorem 3.3 lines L_1' (residue classes of L_1) parallel to the line L_1 have distance a multiple of $1/\|a_N\|$, and the distance between points in $L_1' \cap \mathbf{Z}^2$ is a multiple of $\|a_N\|$, thus we consider at most $O(v \cdot \|a_N\|/(\|a_L\| \cdot \text{dist}(P',Q'))$ lines L_1'. As two points in $B_3(T)$ have distance at most $2 \cdot T$, the line L_1 intersects the ellipsoid E in two points with distance D, where

$$D = O(T \cdot \text{dist}(P',Q')/\text{dist}(P,Q)).$$

Every line L_1' parallel to L_1 intersects the ellipsoid E in two points whose distance is at most D, hence $|L_1' \cap E \cap \mathbf{Z}^2| = O(D/\|a_N\|)$. Thus, we have the following upper bound on the number of triangles in $L' \cap B_3(T)$ with fixed extreme points P and Q:

$$\begin{aligned} & O\left(\frac{v \cdot \|a_N\|}{\|a_L\| \cdot \text{dist}(P',Q')} \cdot \frac{D}{\|a_N\|}\right) \\ = \; & O\left(\frac{v \cdot T}{\text{dist}(P,Q) \cdot \|a_L\|}\right) . \end{aligned}$$ □

COROLLARY 5.1. *Let L be a 2-maximal grid in $\mathbf{Z}^3$ with primitive normal vector $a_L \in \mathbf{Z}^3$. For every residue class L' of L the number of non-degenerated triangles S in the set $L' \cap B_3(T)$ with area$(S) \le v$ is at most $O(v \cdot T^4/\|a_L\|^3)$.*

Proof. Let w.l.o.g. $L' = L$. There are at most $O(T^2/\|a_L\|)$ choices to fix a point $P \in L \cap B_3(T)$. For every $d \in \mathbf{N}$ there are $r_3(d; a_L)$ points $Q \in L$ with $(\text{dist}(P,Q))^2 = d$. By Lemma 5.2 and Lemma 3.6 the number of non-degenerated triangles $S \subset L \cap B_3(T)$ with $\text{area}(S) \le v$ is at most

$$\begin{aligned} & O\left(\frac{T^2}{\|a_L\|} \cdot \sum_{d=1}^{O(T^2)} \frac{v \cdot T}{d^{1/2} \cdot \|a_L\|} \cdot r_3(d;a_L)\right) \\ = \; & O\left(\frac{v \cdot T^3}{\|a_L\|^2} \cdot \sum_{d=1}^{O(T^2)} \frac{r_3(d;a_L)}{d^{1/2}}\right) = O\left(\frac{v \cdot T^4}{\|a_L\|^3}\right) . \end{aligned}$$ □

LEMMA 5.3. *Let $L \subseteq \mathbf{Z}^3$ be a 2-maximal grid with primitive normal vector $a_L \in \mathbf{Z}^3$. For every non-degenerated triangle $S \subset L$ it is area$(S) \ge \|a_L\|/2$.*

Proof. The minimum area of a non-degenerated triangle in L is half of the volume of a fundamental parallelopiped F_Q, where $\text{vol}(F_Q) = \|a_L\|$. □

LEMMA 5.4. *Let L be a 2-maximal grid in $\mathbf{Z}^3$ with primitive normal vector $a_L \in \mathbf{Z}^3$. For every residue class L' of L, and for any real numbers $B, v > 0$, the number of non-degenerated tetrahedra $S = \{p_0, \ldots, p_3\}$ in $B_3(T)$ with $S \setminus \{p_3\} \subseteq L'$ and with area$(S \setminus \{p_3\}) \le v$ and with vol$(S) \le B$ is at most $O(B \cdot v \cdot T^6/\|a_L\|^4)$.*

Proof. Let $S' = \{p_0,p_1,p_2\}$ be one of the by Corollary 5.1 at most $O(v \cdot T^4/\|a_L\|^3)$ many non-degenerated triangles in L' with $\text{area}(S') \le v$. By Lemma 5.3 we have $\text{area}(S') \ge \|a_L\|/2$. Since $\text{vol}(S' \cup \{p_3\}) \le B$, the distance between the fourth point p_3 of a tetrahedron with volume at most B and the real space generated by S' is at most $O(B/\|a_L\|)$. By Theorem 3.3 the distance between the residue classes L' of L is a multiple of $1/\|a_L\|$, and also $|L' \cap B_3(T)| = O(T^2/\|a_L\|)$. Thus the number of choices for the point p_3 is at most

$$O(B/\|a_L\| \cdot T^2/\|a_L\| \cdot \|a_L\|) = O(B \cdot T^2/\|a_L\|) .$$

With Corollary 5.1, the number of tetrahedra $S = \{p_0, \ldots, p_3\}$ in $B_3(T) \cap \mathbf{Z}^3$ with $\text{vol}(S) \le B$ and $S \setminus \{p_3\} \subseteq L'$ and $\text{area}(S \setminus \{p_3\}) \le v$ is at most

$$O\left(\frac{v \cdot T^4}{\|a_L\|^3} \cdot \frac{B \cdot T^2}{\|a_L\|}\right) = O\left(\frac{B \cdot v \cdot T^6}{\|a_L\|^4}\right) .$$ □

We infer from Lemmas 5.1 and 5.4 that for $v := (T^2 \cdot \|a_L\|)^{1/2}$ the number of tetrahedra $S = \{p_0, \dots, p_3\}$ in $B_3(T) \cap \mathbf{Z}^3$ with $S \setminus \{p_3\} \subseteq L$ and with $\mathrm{vol}(S) \leq B$ is at most

$$O\left(\frac{B \cdot v \cdot T^6}{\|a_L\|^4} + \frac{B \cdot T^8}{v \cdot \|a_L\|^3}\right) \quad = \quad O\left(\frac{B \cdot T^7}{\|a_L\|^{7/2}}\right) .$$

We summarize as follows:

Lemma 5.5. *Let L be a 2-maximal grid in $\mathbf{Z}^3$ with primitive normal vector $a_L \in \mathbf{Z}^3$. For every residue class L' of L the number of non-degenerated tetrahedra $S = \{p_0, p_1, p_2, p_3\} \subseteq B_3(T) \cap \mathbf{Z}^3$ with $S \setminus \{p_3\} \subseteq L'$ and vol$(S) \leq B$ is at most $O(B \cdot T^7/\|a_L\|^{7/2})$.*

Proof. To prove Theorem 1.2, for a fixed primitive normal vector $a_L \in \mathbf{Z}^3$ there are at most $O(T \cdot \|a_L\|)$ residue classes L' of the grid L with $L' \cap B_3(T) \neq \emptyset$. By Lemma 5.5 and Lemma 3.5 we infer:

$$N_3(T; B) = O\Bigg(\sum_{\substack{a \in \mathbf{Z}^3 \\ \|a\| = O(T^2)}} T \cdot \|a\| \cdot \frac{B \cdot T^7}{\|a\|^{7/2}}\Bigg)$$

$$= \quad O\Bigg(B \cdot T^8 \cdot \sum_{d=1}^{O(T^4)} \frac{r_3(d)}{d^{5/4}}\Bigg) = O\left(B \cdot T^9\right) . \quad \square$$

6 2-Cycles

As mentioned before, for integers $n \in \mathbf{N}$ and for some value $B > 0$ and for the given set of points in the set $B_3(T) \cap \mathbf{Z}^3$, we construct a 4-uniform hypergraph $\mathcal{G} = \mathcal{G}(B) = (V, \mathcal{E})$ with the vertex set being the set $B_3(T) \cap \mathbf{Z}^3$ of $\Theta(T^3)$ grid-points. The edges are given by all subsets of four points, which form a tetrahedron of volume at most B including the degenerated tetrahedra. This hypergraph $\mathcal{G}(B)$ can be constructed in time $O(T^{12})$. An independent set in $\mathcal{G}(B)$ corresponds to a set of points in $B_3(T) \cap \mathbf{Z}^3$, where all tetrahedra have volume bigger than B.
To apply Theorem 2.1 we have to show that its assumptions are satisfied. Set

$$(6.1) \quad B := T^3 \cdot \ln n / n^3 \quad \text{and} \quad T := n^{1+\varepsilon}$$

for some $\varepsilon > 0$, thus $B = \Omega\left(T^{3 \cdot \varepsilon} \cdot \ln T\right)$. By Theorems 4.1 and 5.1 we bound the number of edges of the hypergraph $\mathcal{G} = \mathcal{G}(B) = (V, \mathcal{E})$ by $|\mathcal{E}| = O(T^9 \cdot \ln T + B \cdot T^9) = O(B \cdot T^9)$, and with $|V| = \Theta(T^3)$, the average degree t^3 of $\mathcal{G}(B)$ satisfies

$$(6.2) \quad t^3 = O(B \cdot T^9 / T^3) = (B \cdot T^6) .$$

Next we consider the 2-cycles in our hypergraph, where we distinguish $(2,2)$-cycles and $(2,3)$-cycles.

6.1 $(2,2)$-cycles

Here we estimate the number $s_{2,2}(\mathcal{G})$ of $(2,2)$-cycles in our hypergraph $\mathcal{G} = \mathcal{G}(B)$, that is the number of pairs of tetrahedra in $B_3(T) \cap \mathbf{Z}^3$, which have exactly two extreme points in common, and both tetrahedra have volume at most B. We distinguish three cases: (a) both tetrahedra are degenerated, or (b) one tetrahedron is degenerated and the other one is non-degenerated or (c) both tetrahedra are non-degenerated. The corresponding numbers of $(2,2)$-cycles are denoted by $s_{2,2}(\mathcal{G}; dd)$, $s_{2,2}(\mathcal{G}; dn)$, $s_{2,2}(\mathcal{G}; nn)$, respectively.
Case (a): Both tetrahedra are degenerated. By Theorem 4.1 there are at most $O\left(T^9 \cdot \ln T\right)$ degenerated tetrahedra in the set $B_3(T) \cap \mathbf{Z}^3$. Fix one of these tetrahedra. The second degenerated tetrahedron is contained in a 2-maximal grid and has two extreme points, say $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$, in common with the first one. Fix a primitive normal vector $b_M := (b_1, b_2, b_3) \in \mathbf{Z}^3$ with $\|b_M\| = O(T^2)$ by Theorem 3.5 (i), which belongs to a 2-maximal grid M. There is at most one residue class M' of M such that $P, Q \in M'$, and with $y_i := p_i - q_i$ for $i = 1, 2, 3$ it must hold that

$$(6.3) \qquad b_1 \cdot y_1 + b_2 \cdot y_2 + b_3 \cdot y_3 = 0 .$$

The set $M' \cap B_3(T)$ contains by Theorem 3.5 (iii) at most $O\left(T^2/\|b_M\|\right)$ points. Two further points can be chosen from $M' \cap B_3(T)$ in at most $O(\binom{T^2/\|b_M\|}{2})$ ways. Hence we infer for the number $s_{2,2}(\mathcal{G}; dd)$ of pairs of degenerated tetrahedra in $B_3(T) \cap \mathbf{Z}^3$ which have two extreme points in common, where $y = (y_1, y_2, y_3)^\top \neq 0^3$ refers to the chosen points P and Q of the first tetrahedron:

$$s_{2,2}(\mathcal{G}; dd) \quad = \quad O\left(T^9 \cdot \ln T \cdot \sum_{\substack{b \in \mathbf{Z}^3; \|b\| = O(T^2) \\ <b, y> = 0}} \binom{\frac{T^2}{\|b\|}}{2}\right)$$

$$= O\left(T^{13} \cdot \ln T \cdot \sum_{\substack{b\in\mathbf{Z}^3;\|b\|=O(T^2)\\ <b,y>=0}} \frac{1}{\|b\|^2}\right)$$

$$(6.4) \qquad = O(T^{13} \cdot (\ln T)^2)$$

since with Lemma 3.5 we have:

$$\sum_{\substack{b\in\mathbf{Z}^3;\|b\|=O(T^2)\\ <b,y>=0}} \frac{1}{\|b\|^2}$$

$$= O\left(\sum_{\substack{b_1,b_2\in\mathbf{Z}\\ |b_1|,|b_2|=O(T^2)}} \frac{1}{b_1^2+b_2^2+\left(\frac{b_1\cdot y_1}{y_3}+\frac{b_2\cdot y_2}{y_3}\right)^2}\right)$$

$$= O\left(\sum_{\substack{b_1,b_2\in\mathbf{Z}\\ |b_1|,|b_2|=O(T^2)}} \frac{1}{b_1^2+b_2^2}\right) = O\left(\sum_{d=1}^{O(T^4)} \frac{r_2(d)}{d}\right)$$

$$= O(\ln T)\ .$$

Case (b): One tetrahedron is degenerated and the other one is non-degenerated with volume at most B. By Theorem 5.1 the number of non-degenerated tetrahedra in the set $\mathbf{Z}^3 \cap B_3(T)$ with volume at most B is at most $O(B\cdot T^9)$. Fix such a tetrahedron and fix two of its extreme points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ where $y_i := p_i - q_i$ for $i = 1, 2, 3$. The further calculations are exactly as in case (a), see (6.4), and we obtain:

$$s_{2,2}(\mathcal{G}; dn) = O\left(B \cdot T^9 \cdot \sum_{\substack{b\in\mathbf{Z}^3;\|b\|=O(T^2)\\ <b,y>=0}} \binom{\frac{T^2}{\|b\|}}{2}\right)$$

$$(6.5) \qquad = O(B \cdot T^{13} \cdot \ln T)\ .$$

Case (c): Both simplices have volume at most B, are non-degenerated and have two extreme points in common. To count this number $s_{2,2}(\mathcal{G}; nn)$ of $(2, 2)$-cycles, we fix a 2-maximal grid L in $\mathbf{Z}^3$ with primitive normal vector $a_L \in \mathbf{Z}^3$ with $\|a_L\| = O(T^2)$. Then fix a point $P \in \mathbf{Z}^3 \cap B_3(T)$. There is exactly one residue class L' of L with $P \in L'$. For $d \in \mathbf{N}$ consider the at most $r_3(d; a_L)$ many points $Q \in L' \cap B_3(T)$ with $(\text{dist}(P, Q))^2 = d$. The points P and Q are the two common extreme points of both tetrahedra.

By Lemma 5.2 there are at most $O(v \cdot T/(\|a_L\| \cdot d^{1/2}))$ points $R \in L' \cap B_3(T)$ with $\text{area}(P, Q, R) \leq v$. The fourth point in $B_3(T) \cap \mathbf{Z}^3$ of such a tetrahedron with volume at most B can be chosen in at most $O(B \cdot T^2/\|a_L\|)$ ways. Moreover, there are at most $O(T^2/\|a_L\|)$ third points $R \in L' \cap B_3(T)$ such that $\text{area}(P, Q, R) > v$, and a fourth point in $B_3(T) \cap \mathbf{Z}^3$ of a tetrahedron with volume at most B can be chosen in at most $O(B \cdot T^2/v)$ ways. With $v := T^{1/2} \cdot \|a_L\|^{1/2} \cdot d^{1/4}$ we have at most

$$O\left(\frac{B\cdot v\cdot T^3}{\|a_L\|^2\cdot d^{\frac{1}{2}}} + \frac{B\cdot T^4}{v\cdot\|a_L\|}\right)$$

$$(6.6) \qquad = O(B \cdot T^{\frac{7}{2}}/(\|a_L\|^{\frac{3}{2}} \cdot d^{\frac{1}{4}}))$$

possibilities for the third and fourth extreme point of the first tetrahedron.

Concerning the second tetrahedron with the extreme points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$, let $y_i := p_i - q_i$ for $i = 1, 2, 3$. For a primitive normal vector $b_M = (b_1, b_2, b_3) \in \mathbf{Z}^3$ with $\|b_M\| = O(T^2)$ and for the case that $P, Q \in M'$ for some residue class M' of a 2-maximal grid M in $\mathbf{Z}^3$ equation (6.3) must hold, where w.l.o.g. $y_3 \neq 0$. This would leave at most $O(T^4)$ possibilities for $b_M \in \mathbf{Z}^3$. However, since the third point $R \in M$ of the tetrahedron satisfies $R \in B_3(T) \cap \mathbf{Z}^3$, there are at most $O(T^3)$ choices for $b_M \in \mathbf{Z}^3$. Let $C \subseteq \mathbf{Z}^3$ be the set of all these primitive normal vectors b_M. The number of possibilities to extend these two points P and Q to a second tetrahedron in $\mathbf{Z}^3 \cap B_3(T)$ is by (6.6) at most

$$(6.7) \quad O\left(\frac{B\cdot T^{\frac{7}{2}}}{d^{\frac{1}{4}}} \cdot \sum_{b\in C} \frac{1}{\|b\|^{\frac{3}{2}}}\right)$$

$$= O\left(\frac{B\cdot T^{\frac{7}{2}}}{d^{\frac{1}{4}}} \cdot \sum_{(b_1,b_2,b_3)\in C} \frac{1}{(b_1^2+b_2^2)^{\frac{3}{4}}}\right)$$

$$= O\left(\frac{B\cdot T^{\frac{7}{2}}}{d^{\frac{1}{4}}} \cdot \sum_{e=1}^{O(T^3)} \frac{r_2(e)}{e^{\frac{3}{4}}}\right) = O(B \cdot T^{\frac{17}{4}}/d^{\frac{1}{4}})\ .$$

With fixed P and Q, by (6.6) and (6.7) we have $O(B^2 \cdot T^{31/4}/(\|a_L\|^{3/2} \cdot d^{1/2}))$ possibilities for the two tetrahedra, and summing over the choices of P and Q we obtain:

$$(6.8) \ s_{2,2}(\mathcal{G}; nn)$$

$$= O\left(T^3 \cdot \sum_{\substack{a\in\mathbf{Z}^3\\ \|a\|=O(T^2)}} \sum_{d=1}^{O(T^2)} r_3(d;a)\cdot \frac{B^2\cdot T^{\frac{31}{4}}}{\|a\|^{\frac{3}{2}}\cdot d^{\frac{1}{2}}}\right)$$

$$= O\left(B^2\cdot T^{\frac{43}{4}} \cdot \sum_{\substack{a\in\mathbf{Z}^3\\ \|a\|=O(T^2)}} \frac{1}{\|a\|^{\frac{3}{2}}}\cdot \sum_{d=1}^{O(T^2)} \frac{r_3(d;a)}{d^{\frac{1}{2}}}\right)$$

$$= O\left(B^2\cdot T^{\frac{47}{4}} \cdot \sum_{a\in\mathbf{Z}^3;\|a\|=O(T^2)} \frac{1}{\|a\|^{\frac{5}{2}}}\right)$$

$$= O\left(B^2\cdot T^{\frac{47}{4}} \cdot \sum_{d=1}^{O(T^4)} \frac{r_3(d)}{d^{\frac{5}{4}}}\right) = O(B^2\cdot T^{\frac{51}{4}})\ .$$

To apply Theorem 2.1 we must have for some suitable constant $\gamma > 0$ that

$$\begin{aligned} s_{2,2}(\mathcal{G}) &= s_{2,2}(\mathcal{G};dd)+s_{2,2}(\mathcal{G};dn)+s_{2,2}(\mathcal{G};nn)\\ &= O(T^3\cdot t^{5-\gamma}) = O(T^{13-2\gamma}\cdot B^{5/3-\gamma/3}), \end{aligned} \tag{6.9}$$

since $t = O(B^{1/3}\cdot T^2)$ and $B := T^3\cdot \ln n/n^3$ by (6.1) and (6.2) with $T := n^{1+\varepsilon}$ for fixed $\varepsilon > 0$. Using the estimates (6.4), (6.5) and (6.8) and

$$T^{13}\cdot(\ln T)^2 + B^2\cdot T^{51/4} = O(B\cdot T^{13}\cdot \ln T)$$

for $0 < \varepsilon < 1/11$, it will suffice to consider only case (b). Using (6.5), we have in this case

$$\frac{B\cdot T^{13}\cdot \ln T}{T^{13-2\gamma}\cdot B^{5/3-\gamma/3}} = O\left(\frac{(\ln n)^{1/3+\gamma/3}}{n^{2\varepsilon-2\varepsilon\gamma-3\gamma}}\right) = o(1)\ ,$$

for $0 < \gamma < 2\varepsilon/(3+2\varepsilon)$ and with $0 < \varepsilon < 1/11$ (6.9) holds.

6.2 $(2,3)$-cycles

Next we estimate the number $s_{2,3}(\mathcal{G})$ of $(2,3)$-cycles in our hypergraph $\mathcal{G} = \mathcal{G}(B)$, that is, the number of pairs of tetrahedra in $B_3(T)\cap \mathbf{Z}^3$, both with volume at most B, having exactly three extreme points in common. As in the case of $(2,2)$-cycles we distinguish three cases (a), (b) and (c). The corresponding numbers of $(2,3)$-cycles are denoted by $s_{2,3}(\mathcal{G};dd)$, $s_{2,3}(\mathcal{G};dn)$, $s_{2,3}(\mathcal{G};nn)$, respectively.
Case (a): both tetrahedra are degenerated and have three extreme points in common. The two tetrahedra are contained in a residue class L' of a 2-maximal grid L in $\mathbf{Z}^3$ with $L'\cap B_3(T)\neq\emptyset$ and with primitive normal vector $a_L\in\mathbf{Z}^3$ with $\|a_L\| = O(T^2)$. Since $|L'\cap B_3(T)| = O(T^2/\|a_L\|)$, we can choose five extreme points of the two tetrahedra in at most $O(\binom{T^2/\|a_L\|}{5})$ ways. By Theorem 3.5 (ii) there are at most $O(T\cdot\|a_L\|)$ residue classes L' of L with $L'\cap B_3(T)\neq\emptyset$, hence with Lemma 3.5:

$$\begin{aligned} s_{2,3}(\mathcal{G};dd) &= O\left(\sum_{a\in\mathbf{Z}^3;\|a\|=O(T^2)} T\cdot\|a\|\cdot\binom{\frac{T^2}{\|a\|}}{5}\right)\\ &= O\left(T^{11}\cdot\sum_{d=1}^{O(T^4)}\frac{r_3(d)}{d^2}\right) = O(T^{11}). \end{aligned} \tag{6.10}$$

Case (b): One tetrahedron is non-degenerated with volume at most B and the other one is degenerated, and they have three extreme points in common. The three common extreme points of the two tetrahedra lie in a residue class L' of a 2-maximal grid L in $\mathbf{Z}^3$ with primitive normal vector $a_L\in\mathbf{Z}^3$ with $\|a_L\| = O(T^2)$. By Lemma 5.5 the number of tetrahedra $S = \{p_0,\ldots,p_3\}$ in $B_3(T)\cap\mathbf{Z}^3$ with $S\setminus\{p_3\}\subseteq L'$ and $\mathrm{vol}(S)\le B$ is at most $O(B\cdot T^7/\|a_L\|^{7/2})$. There are at most $O(T^3)$ possibilities for the fourth point in $\mathbf{Z}^3\cap B_3(T)$ of a degenerated tetrahedron. Summing over all possible 2-maximal grids L in $\mathbf{Z}^3$, we obtain with Lemma 3.5:

$$s_{2,3}(\mathcal{G},dn) = O\left(\sum_{\substack{a\in\mathbf{Z}^3\\ \|a\|=O(T^2)}} T^3\cdot\frac{B\cdot T^7}{\|a\|^{7/2}}\right) \tag{6.11}$$

$$= O\left(B\cdot T^{10}\cdot\sum_{d=1}^{O(T^4)}\frac{r_3(d)}{d^{7/4}}\right) = O(B\cdot T^{10})\ .$$

Case (c): Both tetrahedra are non-degenerated, each with volume at most B. They have three extreme points in common, which are contained in a residue class L' of a 2-maximal grid L in $\mathbf{Z}^3$ with primitive normal vector $a_L\in\mathbf{Z}^3$.
By Corollary 5.1 the number of triangles in $L'\cap B_3(T)$ with area at most v is at most $O(v\cdot T^4/\|a_L\|^3)$. The two fourth points can be chosen in at most $O((B\cdot T^2/\|a_L\|)^2)$ ways.
Moreover, the number of triangles in $L'\cap B_3(T)$ with area at least v is at most $O(\binom{T^2/\|a_L\|}{3})$. The two fourth points of the tetrahedra can be chosen

in at most $O((B \cdot T^2/v)^2)$ ways. With $v := T^{2/3} \cdot \|a_L\|^{2/3}$, the number of such pairs of tetrahedra is at most

$$\begin{aligned} & O\left(\frac{v \cdot T^4}{\|a_L\|^3} \cdot \left(\frac{B \cdot T^2}{\|a_L\|}\right)^2 + \frac{T^6}{\|a_L\|^3} \cdot \left(\frac{B \cdot T^2}{v}\right)^2\right) \\ = \; & O(B^2 \cdot T^{26/3}/\|a_L\|^{13/3}) \; . \end{aligned}$$

Summing over all at most $O(T \cdot \|a_L\|)$ residue classes L' of 2-maximal grids L in $\mathbf{Z}^3$ with $L' \cap B_3(T) \neq \emptyset$, we obtain with Lemma 3.5:

$$\begin{aligned} s_{2,3}(\mathcal{G}, nn) \; &= \; O\left(\sum_{\substack{a \in \mathbf{Z}^3 \\ \|a\| = O(T^2)}} T \cdot \|a\| \cdot \frac{B^2 \cdot T^{26/3}}{\|a\|^{13/3}}\right) \\ &= \; O(B^2 \cdot T^{29/3}) \; . \qquad (6.12) \end{aligned}$$

As in the case of $(2,2)$-cycles considering the estimates (6.10), (6.11) and (6.12), with $t = O(B \cdot T^6)^{1/3}$ and $B := T^3 \cdot \ln n/n^3$ and $T := n^{1+\varepsilon}$ for fixed $0 < \varepsilon < 1/9$ and $0 < \gamma < 4\varepsilon/(2+3\varepsilon)$ we have

$$s_{2,3}(\mathcal{G}) \; = \; O(T^3 \cdot t^{4-\gamma}) = O(T^{11-2\gamma} \cdot B^{4/3-\gamma/3}) \; .$$

With (6.9) the assumptions of Theorem 2.1 are fulfilled. We apply it to our 4-uniform hypergraph $\mathcal{G} = \mathcal{G}(B) = (V, \mathcal{E})$ with average degree $t^3 = O\left(B \cdot T^6\right)$, where $B := T^3 \cdot \ln n/n^3$, and we find in $\mathcal{G}(B)$ in time polynomial in $T = n^{1+\varepsilon}$, hence in n, an independent set of size at least

$$\begin{aligned} & \Omega\left(\frac{T^3}{B^{1/3} \cdot T^2} \cdot (\ln t)^{1/3}\right) \\ = \; & \Omega\left(\frac{T}{(\ln n)^{1/3} \cdot n^{\varepsilon}} \cdot (\ln n)^{1/3}\right) = \Omega\left(n\right) \; . \end{aligned}$$

Thus, we have found at least $\Omega(n)$ points in time polynomial in n such that the volume of every tetrahedron is at least $\Omega(\ln n/n^3)$.

References

[1] M. Ajtai, J. Komlós, J. Pintz, J. Spencer and E. Szemerédi, *Extremal Uncrowded Hypergraphs*, J. Comb. Th. A, 32, 1982, 321-335.

[2] N. Alon and J. Spencer, *The Probabilistic Method*, Wiley & Sons, 1992.

[3] G. Barequet, *A Lower Bound for Heilbronn's Triangle Problem in d Dimensions.* Proc. '10th ACM-SIAM Symp. on Discrete Algorithms SODA'99', 76–81, 1999.

[4] C. Bertram–Kretzberg and H. Lefmann, *The Algorithmic Aspects of Uncrowded Hypergraphs,* Proc. '8th ACM-SIAM Symp. on Discrete Algorithms SODA'97', 1997, 296–304.

[5] C. Bertram-Kretzberg, T. Hofmeister and H. Lefmann, *An Algorithm for Heilbronn's Problem,* Proc. '3rd Ann. Int. Computing and Combinatorics Conference COCOON'97', LNCS 1276, eds. T. Jiang and D. T. Lee, Springer, 1997, 23-31.

[6] J. W. S. Cassels, *An Introduction to the Geometry of Numbers,* volume 99, Springer, 1971.

[7] H. Cohen, *A Course in Computational Algebraic Number Theory,* Springer, 1993.

[8] R. A. Duke, H. Lefmann and V. Rödl, *On Uncrowded Hypergraphs,* Rand. Struct. & Alg. 6, 1995, 209-212.

[9] A. Fundia, *Derandomizing Chebychev's Inequality to find Independent Sets in Uncrowded Hypergraphs*, Rand. Struct. & Alg., 8, 1996, 131–147.

[10] T. Jiang, M. Li and P. Vitány, *Kolmogorov Complexity and a Triangle Problem of the Heilbronn Type,* preprint, 2000.

[11] M. Koecher, *Lineare Algebra und analytische Geometrie,* 4th edition, Springer, 1997.

[12] J. Komlós, J. Pintz and E. Szemerédi, *On Heilbronn's Triangle Problem,* J. of the London Math. Soc., 24, 1981, 385–396.

[13] J. Komlós, J. Pintz and E. Szemerédi, *A Lower Bound for Heilbronn's Problem,* J. of the London Math. Soc., 25, 1982, 13–24.

[14] H. Lefmann, *On Heilbronn's Problem in Higher Dimension,* Proc. '11th ACM-SIAM Symp. on Discrete Algorithms SODA'00', 2000, 60-64.

[15] K. F. Roth, *On a Problem of Heilbronn,* J. of the London Math. Soc. 26, 1951, 198–204.

[16] K. F. Roth, *On a Problem of Heilbronn, II,* Proc. of the London Math. Soc. (3), 25, 1972, 193–212.

[17] K. F. Roth, *On a Problem of Heilbronn, III,* Proc. of the London Math. Soc. (3), 25, 1972, 543–549.

[18] K. F. Roth, *Estimation of the Area of the Smallest Triangle Obtained by Selecting Three out of n Points in a Disc of Unit Area,* Proc. of Symp. in Pure Math., 24, 1973, AMS, Providence, 251–262.

[19] K. F. Roth, *Developments in Heilbronn's Triangle Problem,* Adv. in Math., 22, 1976, 364–385.

[20] B. L. Rothschild and E. G. Straus, *On Triangulations of the Convex Hull of n Points,* Combinatorica 5, 1985, 167-179.

[21] W. M. Schmidt, *On a Problem of Heilbronn,* J. of the London Math. Soc. (2), 4, 1972, 545–550.