EDGE-COLORINGS AVOIDING RAINBOW STARS

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Abstract. We consider an extremal problem motivated by a paper of Balogh [J. Balogh, A remark on the number of edge colorings of graphs, European Journal of Combinatorics 27, 2006, 565–573], who considered edge-colorings of graphs avoiding fixed subgraphs with a prescribed coloring. More precisely, given $r \geq t \geq 2$, we look for n-vertex graphs that admit the maximum number of r-edge-colorings such that at most $t-1$ colors appear in edges incident with each vertex, that is, r-edge-colorings avoiding rainbow-colored stars with t edges. For large n, we show that, with the exception of the case $t=2$, the complete graph K_n is always the unique extremal graph. We also consider generalizations of this problem.

1. Introduction

We consider edge-colorings of graphs that satisfy a certain property. Given a number r of colors and a graph F, an r-*pattern* P of F is a partition of its edge set into r (possibly empty) classes. An edge-coloring (not necessarily proper) of a host graph G is said to be (F,P)-*free* if G does not contain a copy of F in which the partition of the edge set induced by the coloring is isomorphic to P. If at most r colors are used, we call it an (F,P)-free r-coloring of G. For example, if the pattern of F consists of a single class, no monochromatic copy of F should arise in G. We ask for the n-vertex host graphs G (among all n-vertex graphs) which allow the largest number of (F,P)-free r-colorings.

Questions of this type have been first considered by Erdős and Rothschild [8], who asked whether considering edge-colorings avoiding a monochromatic copy of F would lead to extremal configurations that are substantially different from those of the Turán problem. Indeed, F-free graphs on n vertices are natural candidates for admitting a large number of colorings, since any r-coloring of their edge set obviously does not produce a monochromatic copy of F (or a copy of F with any given pattern, for that matter), so that (Turán) F-extremal graphs admit $r^{\mathrm{ex}(n,F)}$ such colorings, where, as usual, $\mathrm{ex}(n,F)$ is the maximum number of edges in an n-vertex F-free graph. Erdős and Rothschild [8] conjectured that, for every $\ell \geq 3$ and $n > n_0(\ell)$, any n-vertex graph with the largest number of K_ℓ-free 2-colorings is isomorphic to the $(\ell-1)$-partite Turán graph, which was proven for $\ell = 3$ by Yuster [17] and for $\ell \geq 4$ by Alon, Balogh, Keevash, and Sudakov [1], who also showed that the same conclusion holds in the case $r=3$. However, for $r \geq 4$ colors, the Turán graph for K_ℓ is no longer optimal, and the situation becomes much more complicated; in fact, extremal configurations are not known unless $r=4$ and $F \in \{K_3, K_4\}$, see Pikhurko and Yilma [15]. A similar phenomenon, in which (Turán) extremal graphs admit the largest number of r-colorings if $r \in \{2,3\}$, but do not for $r \geq 4$, has been observed for a few other classes of graphs and hypergraphs, such as the 3-uniform Fano plane [14]. (See [11] for a more detailed account of instances where this phenomenon is known to hold.)

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Balogh [4] was the first to consider colorings avoiding fixed patterns that are not monochromatic. More precisely, he showed that the $(\ell-1)$-partite Turán graph is still optimal for $r=2$ colors when forbidding any 2-pattern of K_ℓ. On the other hand, he observed that this does not hold in general for $r=3$ colors and arbitrary 3-patterns of K_ℓ. Indeed, consider $F=K_3$ and let P be a partition of K_3 into three classes containing one edge each, so that we are looking for 3-colorings with no *rainbow* triangle. If we color the complete graph K_n with any two of the three colors available, there is no rainbow copy of K_3, which gives $3\cdot 2^{\binom{n}{2}}-3$ distinct (K_3,P)-free colorings of K_n, and is more than $3^{\mathrm{ex}(n,K_3)}\leq 3^{n^2/4}$. This suggests that the study of colorings that avoid general patterns, and in particular rainbow patterns, deserves more attention. In connection with this, we should mention that it was recently proven that, for large n, the complete graph is indeed optimal for rainbow triangles [6] in the case $r=3$. On the other hand, the balanced, complete bipartite Turán graph was shown to be optimal [13] for all $r\geq 10$. There has also been an extensive description of extremal graphs when one considers matchings with various forbidden patterns [12], which includes all rainbow cases.

Stars have played an important rôle in these developments. Monochromatic stars $F=S_t$ with $t\geq 3$ edges were the first instances for which it was shown [11] that F-extremal graphs (in this case, $(t-1)$-regular graphs for n even) do not admit the largest number of r-colorings with no monochromatic copy of F for all values of $r\geq 2$. In particular, this implies that the transition between the cases $r\in\{2,3\}$ and $r\geq 4$ described above does not hold for all graphs F. On the other hand, extremal n-vertex graphs for forbidden monochromatic stars S_t are not yet known for any $r\geq 2$ and $t\geq 3$.

In this paper, our initial motivation was to study r-colorings that avoid rainbow-colored stars S_t, that is, we let $F=S_t$ and we consider the pattern where each edge is in a different class (in particular, $r\geq t$). Colorings of this type have been introduced in the context of Ramsey Theory by Gyárfás, Lehel, Schelp and Tuza [10], who called them *local* $(t-1)$-*colorings*. In other words, the initial question was to find the n-vertex graphs that admits the largest number of local $(t-1)$-colorings using at most r-colorings. For $t=2$ and any given number of colors $r\geq 2$, it is easy to see that a matching of size $\lfloor n/2\rfloor$ yields the largest number of r-colorings with no rainbow S_2, as this restriction implies that any coloring must have monochromatic components (for odd n, both an additional isolated vertex and a connected component with three vertices generate extremal configurations). The same extremal configuration had been observed for monochromatic S_2 when $r=2$, but not for larger values of r. Note that the set of r-colorings avoiding a monochromatic S_2 is precisely the set of proper r-edge-colorings of a graph, and hence the problem would consist of finding n-vertex graphs with the largest number of proper colorings.

However, in contrast to the monocromatic case, we were able to find the optimal configuration in the rainbow case for large n and every fixed $r,t\geq 3$, which is always the complete graph K_n. Since the techniques used to prove this result may be adapted to other patterns, we state our results in greater generality. In particular, to derive this generalization, we show that in any graph with 'many' edges, there is an 'almost spanning' subgraph with a 'large' number of subgraphs of *any* bounded degree sequence satisfying a density constraint, which seems to be of independent interest (see Lemma 3.1 for a precise formulation).

Given positive integers t and $\ell_1\geq\cdots\geq\ell_t$, a *rainbow-*$S_{\ell_1,\ldots,\ell_t}$ is an edge-colored star with $\sum_{i=1}^t \ell_i$ edges such that t pairwise distinct colors $c_1,\ldots,c_t$ have the property that c_i is assigned to exactly ℓ_i edges. For a host graph G, a *rainbow-*$S_{\ell_1,\ldots,\ell_t}$*-free* r*-coloring* of a graph G is an edge-coloring of G with colors in $[r]=\{1,\ldots,r\}$ for which there is no rainbow $S_{\ell_1,\ldots,\ell_t}$. An important special case occurs when $\ell_1=\cdots=\ell_t=\ell$ for some constant $\ell\geq 1$, when we refer to a *rainbow-*$S_{t,\ell}$ and to *rainbow-*$S_{t,\ell}$*-free* r*-colorings of* G. Clearly, if $\ell=1$, we forbid rainbow stars S_t, and we call such colorings *rainbow-*S_t*-free*. This condition resembles the concept of an m*-good coloring*, which was introduced by Alon, Jiang, Miller and Pritikin [2] and consists of an edge-coloring where each color appears at most m times at any vertex. Hence, we consider colorings such that each vertex is $(\ell-1)$-good for all but $(t-1)$ colors.

For any host graph G, let $\mathcal{C}_{r,\ell_1,\dots,\ell_t}(G)$ be the set of all rainbow-$S_{\ell_1,\dots,\ell_t}$-free r-colorings of G. We write

$$c_{r,\ell_1,\dots,\ell_t}(n) = \max\{\,|\mathcal{C}_{r,\ell_1,\dots,\ell_t}(G)| \colon |V(G)| = n\,\},$$

and we say that an n-vertex graph G is $\mathcal{C}_{r,\ell_1,\dots,\ell_t}$-*extremal* if $|\mathcal{C}_{r,\ell_1,\dots,\ell_t}(G)| = c_{r,\ell_1,\dots,\ell_t}(n)$. We prove the following result.

Theorem 1.1. *For all $r, t \geq 3$ and $\ell_1 \geq \cdots \geq \ell_t \geq 1$, there exists n_0 such that, for all $n \geq n_0$, we have $c_{r,\ell_1,\dots,\ell_t}(n) = |\mathcal{C}_{r,\ell_1,\dots,\ell_t}(K_n)|$. Moreover, the complete graph K_n is the single $\mathcal{C}_{r,\ell_1,\dots,\ell_t}$-extremal graph on n vertices.*

As it turns out, this result will be a direct consequence of the following special case, where we consider rainbow-$S_{t,\ell}$-free r-colorings of G. For simplicity, let $\mathcal{C}_{r,t,\ell}(G)$ denote the set of rainbow-$S_{t,\ell}$-free r-colorings of G, and let $c_{r,t,\ell}(n)$ be the corresponding extremal number.

Theorem 1.2. *For all $r, t \geq 3$ and $\ell \geq 1$, there exists n_0 such that, for all $n \geq n_0$, we have $c_{r,t,\ell}(n) = |\mathcal{C}_{r,t,\ell}(K_n)|$. Moreover, the complete graph K_n is the single $\mathcal{C}_{r,t,\ell}$-extremal graph on n vertices.*

Note that the case $t = 1$ and $\ell \geq 3$ deals with colorings avoiding monochromatic stars S_ℓ, which have been considered in [11] and for which the extremal configurations are not known, even when $r = 2$. Moreover, our proof of Theorem 1.2 cannot be extended to the case $t = 2$, but we conjecture that the complete graph is also extremal in this case.

The remainder of this work is organized as follows. In Section 2, we deal with some easy cases and we prove our result for rainbow stars, which gives an overview of the general case. The proof of Theorems 1.2 and 1.2 is the subject of Section 3. Section 4 deals with the proof of our main auxiliary result. We conclude the paper with open questions.

2. Colorings avoiding a rainbow star

The main objective of this section is to prove Theorem 1.2 in the case $\ell = 1$. This case was the main motivation for our work and, as it turns out, its proof gives an accurate overview of the general case. Before proceeding with the proof, we first deal with some straightforward cases.

Recall that, for $t = 2$ and $\ell = 1$, the r-colorings of a graph G avoiding a rainbow-S_2 are such that adjacent edges have the same color. In such a graph edges in the same component have to be colored the same, but edges in different components might be colored differently. Thus, if the graph G has j components containing at least one edge, we have $|\mathcal{C}_{r,2,1}(G)| = r^j$. In order to maximize this, j has to be as large as possible. Hence, the number of such colorings is at most $|\mathcal{C}_{r,2,1}(M)|$, where M is a maximum matching in G. So the only $\mathcal{C}_{r,2,1}$-extremal graphs on n vertices are for n even a matching of size $n/2$, and, for n odd, a matching of size $(n-1)/2$ and an isolated vertex, or a matching of size $(n-3)/2$ and a vertex-disjoint connected component on three vertices. Clearly in this case we have $c_{r,2,1}(n) = r^{\lfloor \frac{n}{2} \rfloor}$. For $r < t$, no r-coloring can produce a rainbow-$S_{t,\ell}$, so that $c_{r,t,\ell}(n) = r^{\binom{n}{2}}$, and K_n is the only $\mathcal{C}_{r,t,\ell}$-extremal graph. Using this argument more carefully, we may extend this conclusion for some additional values of r and t.

Lemma 2.1. *Let $2t-3 \geq r \geq t \geq 3$, $\ell \geq 1$ and n be positive integers. Then $c_{r,t,\ell}(n) = \mathcal{C}_{r,t,\ell}(K_n)$ and the complete graph K_n is unique with this property among all n-vertex graphs.*

Proof. Let G be an n-vertex graph where some edge $e = \{v, w\}$ is missing. Consider a fixed rainbow-$S_{t,\ell}$-free r-coloring Δ of G. We show that we can extend Δ to a coloring of

$G' = G + e$. Let S_v and S_w be the sets of colors occuring on at least ℓ edges incident with v and w, respectively, hence $|S_v|, |S_w| \leq t - 1$. If $S_v \cap S_w \neq \emptyset$, then we can extend Δ to G' by coloring e with any color in $S_v \cap S_w$. Now let $S_v \cap S_w = \emptyset$, in particular $|S_v| + |S_w| \leq r \leq 2t-3$. If $t - 1 = |S_v| > |S_w|$, we can color e with any color in S_v. If $|S_v|, |S_w| < t - 1$, then we can color e with any color. In conclusion, Δ can be extended to a coloring of G'.

To finish the proof, we show that at least one of the colorings of G may be extended to G' in more than one way. As $t \geq 3$, any monochromatic coloring of G can be extended to a coloring of $G + e$ by coloring e with any color, so that $|\mathcal{C}_{r,t,\ell}(G)| < |\mathcal{C}_{r,t,\ell}(G + e)|$. □

Remark: The proof of Lemma 2.1 also yields the following for any $r \geq t \geq 3$ and $\ell \geq 2$. If

$$n \leq r(\ell - 1)/2 + \ell(t - 1) + 1,$$

then it is possible to extend any coloring to a missing edge $\{v, w\}$, even if $S_v \cap S_w = \emptyset$ and $|S_v| = |S_w| = t - 1$. Indeed, if a coloring cannot be extended under such conditions, all colors in S_v must have appeared for at least ℓ edges incident with v, and the same holds for w. Moreover, the colors in S_v must have been assigned to exactly $\ell - 1$ edges incident with w, and vice-versa. Moreover, any color in $\overline{S_v \cup S_w}$ must appear at $\ell - 1$ edges incident with v or $\ell - 1$ edges incident with w, otherwise it could be used to extend the coloring. However, the degrees of v and w (which are at most $n - 2$) are too small for all of these conditions to hold because of our bound on n.

We remark that the case of $r > 2t - 3$ is considerably more involved. Before dealing with the general case, we first focus on the proof of Theorem 1.2 in the case $\ell = 1$. The general idea of the proof is as follows. Consider a fixed rainbow-S_t-free r-edge coloring of a graph G. By definition, for every vertex v of G, the number of colors appearing on edges incident with v is at most $(t - 1)$. For sets $S_1, \ldots, S_n \subseteq [r]$, let $\mathcal{C}_{r,t,(S_1,\ldots,S_n)}(G)$ denote the set of all edge-colorings of G where no edges incident with vertex v_i are assigned colors from the set $[r] \setminus S_i$, for all $i = 1 \ldots, n$. Then the set $\mathcal{C}_{r,t}(G)$ of all rainbow-$S_{t,\ell}$-free r-colorings of G satisfies

$$\mathcal{C}_{r,t}(G) = \bigcup_{\substack{(S_1,\ldots,S_n) \\ |S_i|=t-1, i=1,\ldots,n}} \mathcal{C}_{r,t,(S_1,\ldots,S_n)}(G). \tag{1}$$

Observe that the union is not disjoint, as fewer than $t-1$ colors could appear in edges incident with some vertex.

Before proceeding, note that this decomposition can be easily generalized to any $\ell \geq 1$. The difference, for a fixed rainbow-$S_{t,\ell}$-free r-edge coloring of a graph G, is that the sets S_i contain the colors that appear at least ℓ times in edges incident with v_i, which we call *ordinary colors* with respect to v_i, while the remaining colors are said to be *rare* for v_i. In analogy to the above case, $\mathcal{C}_{r,t,\ell,(S_1,\ldots,S_n)}(G)$ denotes the set of all edge-colorings of G where fewer than ℓ edges incident with vertex v_i are assigned each color from the set $[r] \setminus S_i$, for all $i = 1 \ldots, n$. As in (1), the set $\mathcal{C}_{r,t,\ell}(G)$ of all rainbow-$S_{t,\ell}$-free r-colorings of G satisfies

$$\mathcal{C}_{r,t,\ell}(G) = \bigcup_{\substack{(S_1,\ldots,S_n) \\ |S_i|=t-1, i=1,\ldots,n}} \mathcal{C}_{r,t,\ell,(S_1,\ldots,S_n)}(G). \tag{2}$$

Our proof consists of four steps. We first show that any extremal graph must have a lot of edges, as otherwise it cannot beat the number of colorings achieved by the complete graph. Next we prove that most colorings in (1) arise from the cases when almost all sets S_i are the same. Using these facts, we can prove that extremal graphs have large minimum degree, which, in the last step, allows us to prove that any extremal graph coincides with K_n.

The following lemma is the first step in the above description, which may be easily proved for general ℓ.

Lemma 2.2. *For $r \geq t \geq 3$ and $\ell \geq 1$, there are constants n_0 and $D > 0$ such that if $G = (V, E)$ is a $\mathcal{C}_{r,t,\ell}$-extremal graph on $n \geq n_0$ vertices, then*

$$|E(G)| \geq \binom{n}{2} - Dn \log_{t-1} n.$$

Proof. Fix $r \geq t \geq 3$, $\ell \geq 1$ and let $G = (V, E)$ be an n-vertex $\mathcal{C}_{r,t,\ell}$-extremal graph with $V = \{v_1, \ldots, v_n\}$. Note that G has at least

$$(t-1)^{\binom{n}{2}} \tag{3}$$

rainbow-$S_{t,\ell}$-free r-edge colorings, as the complete graph K_n has at least these many colorings: choose a fixed $(t-1)$-subset S of $[r]$ and assign colors in S to all edges of K_n.

We consider the decomposition in (2), and fix sets $S_1, \ldots, S_n$. Colorings in $\mathcal{C}_{r,t,\ell,(S_1,\ldots,S_n)}(G)$ may be produced as follows: for each vertex v_i we choose at most $(r-t+1)(\ell-1)$ incident edges to be assigned colors that are not in S_i and color them with these colors. The remaining edges $\{v_i, v_j\} \in E$ are assigned colors in $S_i \cap S_j$. For n sufficiently large, this implies that

$$\begin{aligned} |\mathcal{C}_{r,t,\ell(S_1,\ldots,S_n)}(G)| &\leq \left(\sum_{j=0}^{(r-t+1)(\ell-1)} \binom{n-1}{j} \cdot r^j \right)^n \cdot \prod_{\{v_i,v_j\} \in E} |S_i \cap S_j| \\ &\leq n^{(r-t+1)(\ell-1)n} \cdot r^{(r-t+1)(\ell-1)n} \cdot (t-1)^{|E|}. \end{aligned} \tag{4}$$

As $(S_1, \ldots, S_n)$ can be chosen in $\binom{r}{t-1}^n$ ways,

$$\begin{aligned} |\mathcal{C}_{r,t,\ell}(G)| &\leq \sum_{\substack{(S_1,\ldots,S_n) \\ |S_i|=t-1, i=1,\ldots,n}} |\mathcal{C}_{r,t,\ell(S_1,\ldots,S_n)}(G)| \\ &\leq \binom{r}{t-1}^n \cdot n^{2(r-t+1)(\ell-1)n} \cdot (t-1)^{|E|} \\ &\leq (t-1)^{Dn \log_{t-1} n} \cdot (t-1)^{|E|}, \end{aligned} \tag{5}$$

where $D = \log_{t-1} \binom{r}{t-1} + 2(r-t+1)(\ell-1)$ is a constant. Combining (3) and (5), we have

$$(t-1)^{Dn \log_{t-1} n} \cdot (t-1)^{|E|} \geq (t-1)^{\binom{n}{2}} \Longrightarrow |E| \geq \binom{n}{2} - Dn \log_{t-1} n,$$

as required. □

To perform the second step of the proof, for a constant $A > 0$, let $\mathcal{S}_A$ denote the set of all collections $(S_1, \ldots, S_n)$ of $(t-1)$-subsets of $[r]$ where no set S_i appears more than $n - A \log_{t-1} n$ times, $i \in \{1, \ldots, n\}$. We prove that we may find A for which the number of colorings in $\mathcal{S}_A$ is negligible. As in the previous result, we prove this for general ℓ, as there is little additional work.

Lemma 2.3. *Let $r \geq t \geq 3$ and $\ell \geq 1$ be integers. For all $D > 0$ there exists a positive constant A with the following property. Given $\varepsilon > 0$ there is a constant n_0 such that, for all $n \geq n_0$, any n-vertex graph $G = (V, E)$ with at least $\binom{n}{2} - Dn \log_{t-1} n$ edges satisfies*

$$\left| \bigcup_{(S_1,\ldots,S_n) \in \mathcal{S}_A} \mathcal{C}_{r,t,\ell,(S_1,\ldots,S_n)}(G) \right| \leq \varepsilon (t-1)^{|E(G)|}.$$

Proof. With foresight, fix

$$A > \max \left\{ \frac{3(r-t+1)(\ell-1)}{1 - \log_{t-1}(t-2)}, 2D \right\},$$

and let B be an integer satisfying $n/\binom{r}{t-1} \leq B \leq n - A\log_{t-1} n$, where n will be chosen sufficiently large later in the proof. Given an n-vertex graph $G=(V,E)$ with $|E| \geq \binom{n}{2} - Dn\log_{t-1} n$, we provide an upper bound on the number of rainbow-$S_{t,\ell}$-free r-edge colorings in a set $\mathcal{C}_{r,t,\ell,(S_1,\ldots,S_n)}(G)$ such that $\max_S |\{v \in V \colon S_v = S \text{ and } |S| = t-1\}| = B$.

To generate these colorings, we choose a set $U \subset V$ such that $|U| = B$ and a $(t-1)$-subset S of $[r]$ which is assigned to all vertices in U. We then assign other $(t-1)$-subsets to the remaining $n-B$ vertices of G. Let $E(U, V\setminus U)$ denote the set of edges with one vertex in U and the other in $V \setminus U$. As in the proof of Lemma 2.2 (see (4)), for each vertex v_i we choose at most $(r-t+1)(\ell-1)$ edges in at most $\sum_{i=0}^{(r-t+1)(\ell-1)} \binom{n-1}{i} \leq n^{(r-t+1)(\ell-1)}$ ways for n sufficiently large. These edges are assigned colors that are not in S_i in at most $r^{(r-t+1)(\ell-1)}$ ways. The remaining edges $\{v_i, v_j\} \in E$ are assigned colors in $S_i \cap S_j$. Any such edge in $E(U, V\setminus U)$ may be assigned at most $t-2$ colors, since the sets assigned to their endvertices are distinct. Hence the number of rainbow-$S_{t,\ell}$-free r-colorings of G is bounded above by

$$\begin{aligned} & \binom{n}{B} \cdot \binom{r}{t-1}^{n-B+1} \cdot (nr)^{(r-t+1)(\ell-1)n} \cdot (t-2)^{|E(U,V\setminus U)|} \cdot (t-1)^{|E(G)|-|E(U,V\setminus U)|} \\ = & \binom{n}{B} \cdot \binom{r}{t-1}^{n-B+1} \cdot (nr)^{(r-t+1)(\ell-1)n} \cdot (t-1)^{|E(G)|} \cdot \left(\frac{t-2}{t-1}\right)^{|E(U,V\setminus U)|}. \end{aligned} \tag{6}$$

Note that for large n

$$\begin{aligned} |E(U, V\setminus U)| &\geq \binom{n}{2} - Dn\log_{t-1} n - \binom{B}{2} - \binom{n-B}{2} \\ &= -Dn\log_{t-1} n + Bn - B^2 \\ &\geq -Dn\log_{t-1} n + \min\left\{ An\log_{t-1} n - A^2\log^2_{t-1} n, \frac{n^2}{\binom{r}{t-1}} - \frac{n^2}{\binom{r}{t-1}^2} \right\} \\ &= (A-D)n\log_{t-1} n - A^2 \log^2_{t-1} n. \end{aligned}$$

As a consequence, since $A > 2D$ and n is sufficiently large (in particular n depends on A, r, ℓ and t), we obtain $|E(U, V\setminus U)| \geq (An\log_{t-1} n)/2$.

If we sum (6) over all possible values of B, we obtain at most

$$\begin{aligned} & 2^n \cdot \binom{r}{t-1}^n \cdot (nr)^{(r-t+1)(\ell-1)n} \cdot (t-1)^{|E(G)|} \cdot \left(\frac{t-2}{t-1}\right)^{\frac{An\log_{t-1} n}{2}} \\ \leq\; & 2^{n+n\log_2 \binom{r}{t-1}} \cdot (t-1)^{|E(G)|+n(r-t+1)(\ell-1)\log_{t-1}(rn)} \cdot \left(\frac{t-2}{t-1}\right)^{\frac{An\log_{t-1} n}{2}} \\ \leq\; & \varepsilon(t-1)^{|E(G)|} \end{aligned}$$

rainbow-$S_{t,\ell}$-free r-colorings of G, where $\varepsilon > 0$ is arbitrary as long as we choose n sufficiently large, since we have $A > \frac{3(r-t+1)(\ell-1)}{1-\log_{t-1}(t-2)}$. □

The next step in our proof of Theorem 1.2 for $\ell = 1$ is proving that any extremal graph has large minimum degree. Unlike the previous steps, we shall now deal exclusively with the case $\ell = 1$, as treating rare colors will require considerably more work.

Lemma 2.4. *For all integers $r \geq t \geq 3$ there is an n_0 such that the minimum degree of G satisfies $\delta(G) \geq 3n/4 - 1$ for all $\mathcal{C}_{r,t,1}$-extremal graphs G with $n \geq n_0$ vertices.*

Proof. Assume that an n-vertex $\mathcal{C}_{r,t,1}$-extremal graph G has a vertex v with degree $d(v) < (3n/4 - 1)$. Let $w_1, \ldots, w_{\lceil n/4 \rceil}$ be vertices in G that are not adjacent to v. Define the graph G' by adding the edges $\{v, w_1\}, \ldots, \{v, w_{\lceil n/4 \rceil}\}$ to G.

The basic idea of the proof is to show that G' admits more rainbow-S_t-free r-colorings than G, and we do this by showing that, if we compare the number of colorings 'created' and 'lost' with the addition of the new edges, there are more of the former. To be more precise, given a collection $(S_1, \ldots, S_n)$ of $(t-1)$-subsets of $[r]$, it is clear that we may extend all colorings in $\mathcal{C}_{(S_1,\ldots,S_n)}(G)$ to $\mathcal{C}_{(S_1,\ldots,S_n)}(G')$ whenever $S_v \cap S_{w_i} \neq \emptyset$ for all $i \in \{1, \ldots, \lceil n/4 \rceil\}$, as we may assign any color in the corresponding intersection to $\{v, w_i\}$ without producing a rainbow star S_t. Moreover, this extension may be done in several ways, depending on the sizes of the intersections, which leads to 'new colorings' of G', as opposed to colorings that are in one-to-one correspondence with colorings of G. On the other hand, colorings of G for which $S_v \cap S_{w_i} = \emptyset$ for some i may not be extended in this way, and we say that these colorings are lost when the new edges are added.

To find a lower bound on the number of colorings created, consider only those edge colorings of G where every edge is assigned a color from a fixed $(t-1)$-set S in $[r]$. Each such coloring can be extended to at least $(t-1)^{n/4} \cdot (t-1)^{|E(G)|}$ rainbow-S_t-free colorings of G' by assigning an arbitrary color of S to each new edge. This creates at least $\left((t-1)^{n/4} - 1\right) \cdot (t-1)^{|E(G)|}$ new colorings.

On the other hand, the rainbow-S_t-free r-colorings of G that cannot be extended to colorings of G' are those where the sets of colors available at v and at w_i do not intersect, for some $i \in \{1, \ldots, \lceil n/4 \rceil\}$. By Lemma 2.3 with $\varepsilon = 1$, the number of colorings of G where every $(t-1)$-set of colors is assigned to at most $n - A \log_{t-1} n$ vertices of G is at most $(t-1)^{|E(G)|}$. Hence we concentrate on colorings where some $(t-1)$-set S appears at least $n - A \log_{t-1} n$ times. The number of such colorings is at most

$$\binom{r}{t-1} \cdot n \cdot \binom{r-(t-1)}{t-1} \cdot \binom{r}{t-1} \cdot \binom{n}{n - A\log_{t-1} n} \cdot \binom{r}{t-1}^{A \log_{t-1} n - 2} \cdot (t-1)^{|E(G)|}, \quad (7)$$

since there are $\binom{r}{t-1}$ ways to choose S_v, a non-neighbor w_i can be chosen in at most n ways and it is assigned a set S_{w_i} of colors with $S_{w_i} \cap S_v = \emptyset$, which can be done in $\binom{r-(t-1)}{t-1}$ ways. The set S can be chosen in $\binom{r}{t-1}$ ways, the vertices which are assigned the set S can be chosen in at most $\binom{n}{n-A\log_{t-1} n} = \binom{n}{A \log_{t-1} n}$ ways and every remaining vertex is associated with some arbitrary $(t-1)$-set of colors. (Note that this upper bound takes care of all the colorings where the set S is assigned to m vertices, where $n - A\log_{t-1} n \leq m \leq n$.) Clearly, we have $\binom{n}{A\log_{t-1} n} \leq n^{A \log_{t-1} n}$, and, for large n

$$\binom{r}{t-1} \cdot n \cdot \binom{r-(t-1)}{t-1} \cdot \binom{r}{t-1} \cdot \binom{r}{t-1}^{A\log_{t-1} n - 2} < n^{A \log_{t-1} n}.$$

We conclude from (7) that, for sufficiently large n, the number of rainbow-S_t-free r-colorings of G that cannot be extended to such colorings of G' is at most

$$n^{2A\log_{t-1} n} \cdot (t-1)^{|E(G)|} + (t-1)^{|E(G)|} \ll ((t-1)^{n/4} - 1) \cdot (t-1)^{|E(G)|}.$$

In other words, by adding the edges $\{v, w_1\}, \ldots, \{v, w_{\lceil n/4 \rceil}\}$ to G, we increase the total number of colorings, which contradicts the choice of G. □

We remark that the previous proof may be easily adapted to the case where the condition on the minimum degree is replaced by $\delta(G) \geq \alpha n$ for any fixed $0 < \alpha < 1$.

We are now ready to perform the last step in the proof of Theorem 1.2 for $\ell = 1$, which shows that, in an extremal graph G no edge may be missing.

Theorem 2.5. *For $r \geq t \geq 3$, there exists n_0 such that $c_{r,t,1}(n) = |\mathcal{C}_{r,t,1}(K_n)|$ holds for $n \geq n_0$. Moreover, K_n is the unique n-vertex $\mathcal{C}_{r,t,1}$-extremal graph.*

Proof. Assume that there is a $\mathcal{C}_{r,t,1}$-extremal graph $G=(V,E)$ on n vertices with at least two non-adjacent vertices x and y. As in the proof of Lemma 2.4, we prove that $G'=G+\{x,y\}$ has more rainbow-S_t-free r-colorings than G if n is sufficiently large. By Lemma 2.2 we know that n may be chosen so that $|E(G)|\geq \binom{n}{2}-Dn\log_{t-1}n$, where D is a constant.

Every coloring of G for which only $(t-1)$ colors are used can be extended, assigning any of these colors to $\{x,y\}$, to a coloring of G', which increases the total number of colorings by

$$(t-2)\cdot(t-1)^{|E(G)|}. \tag{8}$$

We show that the number of all rainbow-$S_{t,1}$-free r-colorings of G that cannot be extended to a coloring of G' is smaller than (8).

By Lemma 2.3 with $A=A(r,t,\ell,D)$ and $\varepsilon=1/2$, we know that we may choose n_0 such that the number of colorings associated with assigments in $\mathcal{S}_A$ is at most $\frac{1}{2}(t-1)^{|E(G)|}$. Therefore, in the following we only need to consider colorings from the set

$$\mathcal{A}=\bigcup_{(S_1,\ldots,S_n)\in\overline{\mathcal{S}_A}}\mathcal{C}_{(S_1,\ldots,S_n)}(G). \tag{9}$$

The only colorings of G that cannot be extended to colorings of G' are those where the color sets S_x and S_y assigned to x and y, respectively, are disjoint, so that we are unable to assign a color to $\{x,y\}$. Fix $(S_1,\ldots,S_n)$ such that S is assigned to at least $n-A\log_{t-1}n$ vertices of G. Recall that both vertices x,y have degree at least $3n/4-1$ by Lemma 2.4.

The condition on the degrees implies that the common neighbourhood $N(\{x,y\})$ of x and y has size at least $n/2$. For any vertex w in $N(\{x,y\})$ we have $S_w\cap(S_x\cup S_y)\leq t-1$. More precisely, we have $|S_w\cap S_x|=a_w$ and $|S_w\cap S_y|\leq t-1-a_w$, so that there are at most $a_w(t-1-a_w)\leq((t-1)/2)^2$ ways to assign colors to the edges $\{x,w\}$ and $\{y,w\}$. Hence all edges between $\{x,y\}$ and their common neighbourhood $N(\{x,y\})$ may be colored in at most $((t-1)/2)^{2|N(\{x,y\})|}$ ways.

This leads to the following upper bound on the number of elements in (9) that cannot be extended to a coloring of G'. The set S may be chosen in $\binom{r}{t-1}$ ways and, for n large, there are $\binom{n}{A\log_{t-1}n}<2^{n/4}$ ways of choosing $n-A\log_{t-1}n$ vertices which are assigned S. For n sufficiently large, the remaining vertices may be assigned color sets in at most $\binom{r}{t-1}^{A\log_{t-1}n}<2^{n/4}$ ways, and we infer that

$$\begin{aligned}
|\mathcal{A}| &\leq \binom{n}{A\log_{t-1}n}\cdot\binom{r}{t-1}\cdot\binom{r}{t-1}^{A\log_{t-1}n}\cdot\frac{(t-1)^{|E(G)|}}{4^{|N(\{x,y\})|}}\\
&\leq \binom{n}{A\log_{t-1}n}\cdot\binom{r}{t-1}^{A\log_{t-1}n+1}\cdot\frac{(t-1)^{|E(G)|}}{2^n}\\
&\leq 2^{n/2}\cdot\binom{r}{t-1}\cdot\frac{(t-1)^{|E(G)|}}{2^n}\\
&\leq \frac{1}{2^{\frac{n}{2}}}\cdot\binom{r}{t-1}\cdot(t-1)^{|E(G)|}.
\end{aligned}$$

Altogether, the number of all colorings of G that cannot be extended by adding edge $\{x,y\}$ to the graph G is no more than

$$\frac{1}{2}(t-1)^{|E(G)|}+\frac{1}{2^{\frac{n}{2}}}\cdot\binom{r}{t-1}\cdot(t-1)^{|E(G)|},$$

which is smaller than (8) for n sufficiently large. □

3. Colorings avoiding a rainbow $S_{t,\ell}$

In this section, we consider the proof Theorem 1.2 for general $\ell \geq 2$. Recall, that an edge-colored star $S_{t\ell}$ with $t\ell$ edges such that t distinct colors are each assigned to exactly ℓ edges is called a rainbow-$S_{t,\ell}$. As we remarked before, the strategy for achieving this result is exactly the same as for the case $\ell = 1$, but the presence of rare colors will make the arguments more technical. Recall that first and second main steps of the proof, namely showing that extremal graphs have a large number of edges, and that most colorings have the property that almost all vertices have the same set of ordinary colors, have already been proved for general ℓ (Lemmas 2.2 and 2.3).

To perform the remaining steps, we use the strategy employed in Lemma 2.4, and show that the number of colorings created exceeds the number of colorings lost when edges are added. To reach this conclusion, we need a lower bound on the number of colorings created, and an upper bound on the number of colorings lost, with the property that the lower bound is larger than the upper bound. However, evaluating these bounds will be harder in this case because of the rare colors.

To describe the main ingredient needed to treat rare colors, first consider colorings for which $S_1 = \cdots = S_n$, so that sets of ordinary and rare colors are the same for all vertices. In any $S_{t,\ell}$-free r-coloring in $\mathcal{C}_{(S_1,\ldots,S_n)}(G)$, the graph induced by each rare color has maximum degree less than ℓ, so we need to count the numbers of subgraphs of G of this type. A classical result of Bender and Canfield [5] implies that the number $N_{K_n}(\mathbf{d})$ of labeled subgraphs of K_n with degree sequence $\mathbf{d} = (d_1,\ldots,d_n)$, where all components d_i are bounded above by some absolute constant d, satisfies

$$N_{K_n}(\mathbf{d}) \sim \frac{(2m)! \cdot \exp(-\lambda - \lambda^2)}{m! \cdot 2^m \cdot \prod_{i=1}^n d_i!} \sim \frac{\sqrt{2}}{\prod_{i=1}^n d_i!} \cdot \left(\frac{2m}{e}\right)^m \cdot \exp(-\lambda - \lambda^2), \tag{10}$$

where $2m = \sum d_i$ and $\lambda = \frac{1}{2m}\sum \binom{d_i}{2}$, and where the asymptotics are in n (the second approximation uses Stirling's formula). Here $A(n) \sim B(n)$ means that $\lim_{n\to\infty} A(n)/B(n) = 1$. As it turns out, the Bender and Canfield formula may be applied directly when finding an upper bound on the number of colorings lost, as we may assume that any edge needed when fixing the subgraph whose edges have some rare color lies in the graph. However, this may not be done for the lower bound on the number of colorings created, where we need an approximate version of Bender and Canfield's result. In the following, given a graph H on n vertices and an integer sequence $\mathbf{d} = (d_1,\ldots,d_n)$, let $N_H(\mathbf{d})$ be the number of subgraphs with degree sequence $\mathbf{d}$ in H. More generally, given an array $\vec{\mathbf{d}} = (\mathbf{d}^1,\ldots,\mathbf{d}^k)$, let $N_H(\vec{\mathbf{d}})$ be the number of ways of selecting a k-tuple $(H_1,\ldots,H_k)$ of edge-disjoint subgraphs of H such that each subgraph H_i has degree sequence $\mathbf{d}^i$, for all $i \in \{1,\ldots,k\}$. We clearly have

$$N_H(\vec{\mathbf{d}}) \leq N_H(\mathbf{d}^1)\cdots N_H(\mathbf{d}^k).$$

In the following, we say that an integer sequence $\mathbf{d} = (d_1,\ldots,d_n)$ is γ-*dense* if $\sum_{i=1}^n d_i \geq \gamma n$.

Lemma 3.1. *Given positive integers d and k, constants $D > 0$ and $\gamma > 0$, there exist positive constants n_0, M and α satisfying the following property for all $n \geq n_0$. For every graph H with $|V(H)| = n$ and $|E(H)| \geq \binom{n}{2} - Dn\ln n$, there exists $W \subseteq V(H)$ with $|W| \geq n - M\ln n$ such that, for all γ-dense degree sequences $\boldsymbol{d}^1,\ldots,\boldsymbol{d}^{k'} \in \{0,\ldots,d\}^{|W|}$, where $k' \leq k$, we have*

$$N_{H[W]}(\boldsymbol{d}^1,\ldots,\boldsymbol{d}^{k'}) \geq n^{-\alpha} \cdot \prod_{i=1}^{k'} N_{K_{|W|}}(\boldsymbol{d}^i).$$

Intuitively, this lemma states that, in any graph with 'many' edges, there is an 'almost spanning' subgraph with a 'large' number of subgraphs of *any* bounded degree sequence that is sufficiently dense. Note that this would trivially fail if we required $W = V$, as an opponent would be able to isolate vertices when removing $Dn \log_{t-1} n$ edges of K_n to produce G, so that $N_G(\mathbf{d}) = 0$ for any positive sequence $\mathbf{d} = (d_1, \ldots, d_n)$. The proof of Lemma 3.1, which adapts ideas of Gao [9], lies in Section 4.

Lemma 3.2. *For all integers $r \geq t \geq 3$ and $\ell \geq 1$, there is n_0 such that any $\mathcal{C}_{r,t,\ell}$-extremal graph G on $n \geq n_0$ vertices satisfies $\delta(G) \geq 3n/4 - 1$.*

Proof. Assume that an $\mathcal{C}_{r,t,\ell}$-extremal n-vertex graph $G = (V, E)$ has a vertex v with degree $d(v) < 3n/4 - 1$. We suppose that n is sufficiently large for all steps in the proof to hold. Let $w_1, \ldots, w_{\lceil n/4 \rceil}$ be $\lceil n/4 \rceil$ vertices in G that are not adjacent to v, and let G' be the graph obtained by adding all the edges $\{v, w_i\}$ to G, $i = 1, \ldots, \lceil n/4 \rceil$. Let $D > 0$ such that G has at least $\binom{n}{2} - Dn \log_{t-1} n$ edges (Lemma 2.2) and fix $A > 0$ with the property of Lemma 2.3.

We show that the number N' of new colorings of G' obtained by extending colorings of G is larger than the number N of colorings of G that cannot be extended to colorings of G'. By Lemma 2.3 with $\varepsilon = 1/3$, the number of colorings in $\bigcup_{(S_1,\ldots,S_n)\in\mathcal{S}_A} \mathcal{C}_{(S_1,\ldots,S_n)}(G)$ is at most $(t-1)^{|E(G)|}/3$ for sufficiently large n.

Let N'_A be the number of new colorings of G' associated with n-tuples $(S_1, \ldots, S_n)$ in $\overline{\mathcal{S}_A}$, and let N_A be the number of colorings associated with such collections that cannot be extended. In the remainder of the proof, we find a lower bound on N'_A and an upper bound on N_A to show that $N'_A \geq (t-1)^{n/8} N_A \geq 2N_A$. Moreover, it turns out that $N'_A \geq (t-2) \cdot (t-1)^{|E(G)|}$ (see (16)), so that

$$N' - N \geq N'_A - \left(N_A + \frac{(t-1)^{|E(G)|}}{3}\right) \geq \frac{N'_A}{2} - \frac{(t-1)^{|E(G)|}}{3} \geq \left(\frac{(t-2)}{2} - \frac{1}{3}\right) \cdot (t-1)^{|E(G)|} > 0,$$

as required.

Before proceeding, let $n_0^{(3.1)}$, M and α given by Lemma 3.1 applied for D, $d = \ell - 1$, $k = r - t + 1$ and $\gamma = 8/9$ (adjusting the constants so that the logarithms in the statement of the lemma have base $t-1$), and fix a set $W \subset V$ with $|W| \geq n - M \log_{t-1} n$ such that $G[W]$ satisfies the conclusion of the lemma. Note that we use $n_0^{(3.1)}$ to denote the value of n_0 obtained in this application of Lemma 3.1.

Upper bound: We give an upper bound on the number N_A of rainbow-$S_{t,\ell}$-free r-colorings of G that are associated with collections $(S_1, \ldots, S_n) \in \overline{\mathcal{S}_A}$ and cannot be extended to a coloring of G'. If $S_v \cap S_{w_i} \neq \emptyset$ for every i, the colorings of $\mathcal{C}_{r,t,\ell,(S_1,\ldots,S_n)}(G)$ can be easily extended to colorings of $\mathcal{C}_{r,t,\ell,(S_1,\ldots,S_n)}(G')$ using ordinary colors for each new edge, so that we may assume that the sets of colors available at v and available at w_i do not intersect for some $i \in \{1, \ldots, \lceil n/4 \rceil\}$. However, note that $S_v \cap S_{w_i} = \emptyset$ does not imply that there is no color available for the edge $\{v, w_i\}$, since it could possibly be colored with one of the rare colors. For the sake of simplicity, and since we are looking for an upper bound, we shall ignore this fact and assume that $S_v \cap S_{w_i} = \emptyset$ always makes it impossible to color the edge $\{v, w_i\}$.

To construct colorings of this type we proceed as follows. First, we fix a collection $(S_1, \ldots, S_n) \in \overline{\mathcal{S}_A}$ with the required properties: (i) choose a $(t-1)$-subset $S_v \subset [r]$; (ii) choose a vertex w that is not adjacent to v; (iii) choose a $(t-1)$-subset set $S_w \subset [r]$, which is disjoint from S_v, to be assigned to w; (iv) choose the $(t-1)$-subset $S \subset [r]$ that is assigned to at least $n - A \log_{t-1} n$ vertices of G; (v) choose $n - A \log_{t-1} n$ vertices that are assigned this set S; (vi) assign any $(t-1)$-subsets in $[r]$ to the $A \log_{t-1} n - 2$ remaining vertices. Note that steps (i), (ii) and (vi) allow us to choose S, so it might well be that more than $n - A \log_{t-1} n$ vertices are assigned S. The number of choices for the sets S_v, S_w, S above is at most $\binom{r}{t-1}^3$, while n is an upper bound on the number of choices of w. Steps (v) and (vi) may be performed in $\binom{n}{n-A\log_{t-1}n}\cdot\binom{r}{t-1}^{A\log_{t-1}n-2}$ ways, so that, for n large, an upper bound on the number of ways of fixing a collection $(S_1,\dots,S_n)$ with the required properties is

$$\binom{r}{t-1}^3\cdot n\cdot\binom{n}{n-A\log_{t-1}n}\cdot\binom{r}{t-1}^{A\log_{t-1}n-2}\le\binom{r}{t-1}^{A\log_{t-1}n+1}\cdot n^{A\log_{t-1}n}. \quad (11)$$

Now assume that such a collection $(S_1,\dots,S_n)$ is fixed. Let $Y=\{u\in V\colon S_u=S\}$ and let $H=G[W\cap Y]$. We now construct colorings in $\mathcal{C}_{(S_1,\dots,S_n)}(G)$. To this end, we proceed as follows:

(i) color edges incident with vertices $u\in V\setminus Y$ with rare colors with respect to u;
(ii) color edges incident with vertices $u\in(V\setminus W)\cap Y$ with rare colors with respect to u;
(iii) color edges incident with $V\setminus Y$ with ordinary colors (with respect to some endpoint in $V\setminus Y$);
(iv) color edges in H with rare colors (with respect to S);
(v) color edges with both ends in Y with ordinary colors (with respect to S).

Observe that the edges $e=\{u,v\}$ such that u is assigned S, but v is not, may be colored in (i) if e is assigned a rare color with respect to v or in (iii), if e is assigned an ordinary color with respect to v. For simplicity, we shall assume that edges colored in (i) may be recolored in (iii). Also note that edges $e=\{u,v\}$ such that u is in $W\cap Y$ and $v\in(V\setminus W)\cap Y$ are colored in (ii) or (v).

For step (i), there are at most $(n^{(\ell-1)(r-t+1)})^{A\log_{t-1}n}$ ways of choosing $(\ell-1)$ edges incident with each such vertex w for each of the $(r-t+1)$ rare colors. Note that we do not need to consider the possibility that fewer edges are assigned such colors because, with our estimates, the edges colored at this point could be recolored in later steps. Step (ii) may be performed in at most $n^{(\ell-1)(r-t+1)M\log_{t-1}n}$, while step (iii) may be performed in at most $(t-1)^\eta$ ways, where η is the number of edges incident with vertices in $V\setminus Y$. To assign rare colors (with respect to S) to the edges of H, we use the following procedure.

Procedure 3.3. Suppose we have a q-vertex input graph $H=(V,E)$ and a set $T\in\binom{[r]}{t-1}$. Assume that $H_{q,0}$ is a set of isolated vertices labeled by V. For $i\in\{1,\dots,r-t+1\}$, choose a graph $H_{q,i}$ in $E\setminus\bigcup_{j=1}^{i-1}E(H_{q,j})$ with the degree sequence $\mathbf{d}^i=(d_1^i,\dots,d_q^i)$, where $d_j^i\le\ell-1$ and assign the ith color in $[r]\setminus T$ to the edges of $H_{q,i}$.

As we are looking for an upper bound, we shall assume that $W\cap Y=W$, possibly being able to recolor some edges that have been colored in previous steps. For simplicity, assume that $|W|=p$. For $i\in\{1,\dots,r-t+1\}$, fix degree sequences $\mathbf{d}^i=(d_1^i,\dots,d_p^i)$, where the ith degree sequence is associated with the ith rare color: we find $r-t+1$ edge-disjoint subgraphs $H_1,\dots,H_{r-t+1}$ of H such that H_i has degree sequence $\mathbf{d}^i$, $i=1,\dots,r-t+1$. Note that $N_H(\mathbf{d}^1,\dots,\mathbf{d}^{r-t+1})$ is the number of ways in which this can be done. We denote $\vec{\mathbf{d}}=(\mathbf{d}^1,\dots,\mathbf{d}^{r-t+1})$, where for each $\mathbf{d}^i=(d_1^i,\dots,d_p^i)$ all components are bounded above by $\ell-1$. Moreover, we denote $u(\vec{\mathbf{d}})=\frac{1}{2}\sum_{i=1}^{r-t+1}\sum_{j=1}^{p}d_j^i$.

The number of ways of performing steps (iv) and (v) is bounded above by

$$\sum_{\vec{\mathbf{d}}}N_H(\vec{\mathbf{d}})\cdot(t-1)^{|E(G)|-u(\vec{\mathbf{d}})-\eta},$$

where the sum ranges over the arrays $\vec{\mathbf{d}}=(\mathbf{d}^1,\dots,\mathbf{d}^{r-t+1})$.

As a consequence, the number of colorings constructed above is at most

$$n^{(\ell-1)(r-t+1)A\log_{t-1}n}\cdot n^{(\ell-1)(r-t+1)M\log_{t-1}n}\cdot(t-1)^\eta\cdot\sum_{\vec{\mathbf{d}}}N_H(\vec{\mathbf{d}})\cdot(t-1)^{|E(G)|-u(\vec{\mathbf{d}})-\eta}. \quad (12)$$

It is easy to see that, choosing n sufficiently large, we may choose an arbitrarily small constant $\delta > 0$ such that the product of equations (11) and (12) is at most

$$(1+\delta)^n \cdot \sum_{\vec{\mathbf{d}}} N_H(\vec{\mathbf{d}}) \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}})}.$$

Recall that a sequence $\mathbf{d}^i$ is γ-dense if the components of $\mathbf{d}^i$ sum to γp or more, otherwise it is γ-sparse (it will be convenient for use $\gamma = 8/9$ in these calculations). We can write

$$(1+\delta)^n \cdot \sum_{\vec{\mathbf{d}}} N_H(\vec{\mathbf{d}}) \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}})} = (1+\delta)^n \cdot \sum_{\mathbf{s}} \sum_{\vec{\mathbf{d}}_-} \sum_{\vec{\mathbf{d}}_+} N_H(\vec{\mathbf{d}}) \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}})}, \quad (13)$$

where $\mathbf{s}\colon [r-t+1] \to \{0,1\}$ is a function that, for each i, indicates whether the degree sequence $\mathbf{d}^i$ is γ-*dense* (when $\mathbf{s}(i) = 1$) or *sparse* (when $\mathbf{s}(i) = 0$), while $\vec{\mathbf{d}}_+$ and $\vec{\mathbf{d}}_-$ are the arrays of dense and sparse degree sequences, respectively, that create an array $\vec{\mathbf{d}}$ with the distribution determined by $\mathbf{s}$. We split equation (13) according to whether all rare colors generate 'dense graphs', or whether this does not hold. For simplicity, we write $\sum_{|\vec{\mathbf{d}}_+|=j}$ to say that we sum over arrays of length j whose degree sequences are all dense. We obtain

$$(1+\delta)^n \left(\sum_{|\vec{\mathbf{d}}_+|=r-t+1} N_H(\vec{\mathbf{d}}_+) \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)} + \sum_{\mathbf{s}\neq\vec{1}} \sum_{\vec{\mathbf{d}}_-} \sum_{\vec{\mathbf{d}}_+} N_H(\vec{\mathbf{d}}) \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}})} \right). \quad (14)$$

Observe that the second summation may be estimated as

$$\begin{aligned} &\sum_{\mathbf{s}\neq\vec{1}} \sum_{\vec{\mathbf{d}}_-} \sum_{\vec{\mathbf{d}}_+} N_H(\vec{\mathbf{d}}) \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}})} \\ \leq\ & 2^{r-t+1} \cdot \ell^{p(r-t+1)} \cdot 2^{(r-t+1)/2} \cdot \sum_{j=0}^{r-t} \left(\frac{p}{e}\right)^{\gamma p(r-t+1-j)/2} \cdot \sum_{|\vec{\mathbf{d}}_+|=j} (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)} \cdot N_H(\vec{\mathbf{d}}_+) \\ \leq\ & \ell^{p(r-t+1)} \cdot \sum_{j=0}^{r-t} p^{\gamma p(r-t+1-j)/2} \cdot \sum_{|\vec{\mathbf{d}}_+|=j} (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)} \cdot \prod_{i=1}^{j} N_{K_p}(\mathbf{d}^i_+). \quad (15) \end{aligned}$$

To see why the first inequality is true, note that $\mathbf{s}$ ranges in a set with at most 2^{r-t+1} elements, the number of arrays $\vec{\mathbf{d}}_- = (\mathbf{d}^1, \ldots, \mathbf{d}^{r-t+1-j})$ is bounded above by $\ell^{p(r-t+1-j)}$ and

$$N_H(\vec{\mathbf{d}}_-) \leq \prod_{i=1}^{r-t+1-j} N_{K_p}(\mathbf{d}^i) \leq \left(\sqrt{2}\left(\frac{p}{e}\right)^{\gamma p/2}\right)^{r-t+1-j} \leq \frac{2^{(r-t+1)/2}}{e^{\gamma p(r-t+1-j)/2}} \cdot p^{\gamma p(r-t+1-j)/2},$$

where $N_{K_p}(\mathbf{d}^i)$ is bounded using (10) and the fact that $\sum_j d^i_j \leq \gamma p < p$.

Lower bound: Next we give a lower bound on the number N'_A of rainbow-$S_{t,\ell}$-free r-colorings we gain by adding the edges $\{v, w_1\}, \ldots, \{v, w_{\lceil n/4 \rceil}\}$ to our extremal graph G. The idea here is that, if we consider colorings in $\mathcal{C}_{r,t,\ell,(S,\ldots,S)}(G)$ for some fixed $(t-1)$-subset S of $[r]$, those colorings can be extended to colorings of G' by assigning any of these $(t-1)$ colors to each of the edges $\{v, w_i\}$, $i = 1, \ldots \lceil n/4 \rceil$. Hence the number of colorings of this type for G' is at least $(t-1)^{n/4} \cdot |\mathcal{C}_{r,t,\ell,(S,\ldots,S)}(G)|$. Removing colorings that are extensions of the corresponding colorings of G, the net gain of colorings is

$$\left((t-1)^{n/4} - 1\right) \cdot |\mathcal{C}_{r,t,\ell,(S,\ldots,S)}(G)|.$$

We now find a lower bound on $|\mathcal{C}_{r,t,\ell,(S,\ldots,S)}(G)|$. To assign rare colors to edges of G, we apply Procedure 3.3 to $H = G[W]$ and $T = S$. Since we now need a lower bound, we may not suppose that $G[W] = K_p$, but Lemma 3.1 guarantees that, for all k-tuples $(\mathbf{d}^1, \ldots, \mathbf{d}^k)$ such that $k \leq r-t+1$ and $\sum_{j=1}^p d_j^i \geq \gamma p$, for all i, we have $N_{G[W]}(\mathbf{d}^1, \ldots, \mathbf{d}^k) \geq n^{-\alpha} \cdot \prod_{i=1}^k N_{K_p}(\mathbf{d}^i)$. With this, a lower bound on the number of colorings gained by adding the edges $\{v, w_1\}, \ldots, \{v, w_{\lceil n/4 \rceil}\}$ to G is

$$((t-1)^{n/4} - 1) \cdot \sum_{\vec{\mathbf{d}}} (t-1)^{|E(G)|-u(\vec{\mathbf{d}})} \cdot N_{G[W]}(\vec{\mathbf{d}}), \tag{16}$$

where we again sum over arrays $\vec{\mathbf{d}} = (\mathbf{d}^1, \ldots, \mathbf{d}^{r-t+1})$, where each $\mathbf{d}^i = (d_1^i, \ldots, d_p^i)$ has components bounded by $\ell - 1$.

To conclude our argument, we compare the upper bound (14) and the lower bound (16). We have

$$\frac{(14)}{(16)} \leq \frac{(1+\delta)^n \cdot \sum_{|\vec{\mathbf{d}}_+|=r-t+1} N_{K_p}(\vec{\mathbf{d}}_+) \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)}}{((t-1)^{n/4}-1) \cdot \sum_{\vec{\mathbf{d}}} (t-1)^{|E(G)|-u(\vec{\mathbf{d}})} \cdot N_{G[W]}(\vec{\mathbf{d}})} \tag{17}$$

$$+ \quad \frac{(1+\delta)^n \ell^{p(r-t+1)} \cdot \sum_j p^{\gamma p(r-t+1-j)/2} \cdot \sum_{|\vec{\mathbf{d}}_+|=j} (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)} \prod_{i=1}^j N_{K_p}(\mathbf{d}_+^i)}{((t-1)^{n/4}-1) \cdot \sum_{\vec{\mathbf{d}}} (t-1)^{|E(G)|-u(\vec{\mathbf{d}})} \cdot N_{G[W]}(\vec{\mathbf{d}})}. \tag{18}$$

By our choice of W, the term (17) becomes

$$\begin{aligned}
& \frac{(1+\delta)^n \sum_{|\vec{\mathbf{d}}_+|=r-t+1} N_{K_p}(\vec{\mathbf{d}}_+) \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)}}{((t-1)^{n/4}-1) \cdot \sum_{|\vec{\mathbf{d}}_+|=r-t+1} (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)} N_{G[W]}(\vec{\mathbf{d}}_+)} \\
\leq \quad & \frac{(1+\delta)^n \cdot \sum_{|\vec{\mathbf{d}}_+|=r-t+1} \prod_{i=1}^{r-t+1} N_{K_p}(\mathbf{d}^i) \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)}}{((t-1)^{n/4}-1) \cdot n^{-\alpha} \cdot \sum_{|\vec{\mathbf{d}}_+|=r-t+1} (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)} \prod_{i=1}^{r-t+1} N_{K_p}(\mathbf{d}^i)} \\
= \quad & \frac{n^\alpha \cdot (1+\delta)^n}{((t-1)^{n/4}-1)} \leq \frac{1}{2(t-1)^{n/8}},
\end{aligned} \tag{19}$$

for sufficiently large n.

For the second term (18), we write

$$\begin{aligned}
& \frac{(1+\delta)^n \cdot \ell^{p(r-t+1)} \sum_j p^{\gamma p(r-t+1-j)/2} \sum_{|\vec{\mathbf{d}}_+|=j} (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)} \prod_{i=1}^j N_{K_p}(\mathbf{d}_+^i)}{((t-1)^{n/4}-1) \cdot \sum_{\vec{\mathbf{d}}} (t-1)^{|E(G)|-u(\vec{\mathbf{d}})} N_{G[W]}(\vec{\mathbf{d}})} \\
= \quad & \frac{(1+\delta)^n \cdot \ell^{p(r-t+1)}}{(t-1)^{n/4}-1} \sum_{j=0}^{r-t} \sum_{|\vec{\mathbf{d}}_+|=j} \frac{p^{\gamma p(r-t+1-j)/2} \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)} \prod_{i=1}^j N_{K_p}(\mathbf{d}_+^i)}{\sum_{\vec{\mathbf{d}}} (t-1)^{|E(G)|-u(\vec{\mathbf{d}})} N_{G[W]}(\vec{\mathbf{d}})} \\
\leq \quad & \frac{(1+\delta)^n \ell^{p(r-t+1)}}{(t-1)^{n/4}-1} \sum_{j=0}^{r-t} \sum_{|\vec{\mathbf{d}}_+|=j} \frac{p^{\gamma p(r-t+1-j)/2} (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)} \prod_{i=1}^j N_{K_p}(\mathbf{d}_+^i)}{(t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)-(r-t+1-j)p(\ell-1)/2} N_{G[W]}(\vec{\mathbf{d}}_+, \vec{\mathbf{d}}^*)},
\end{aligned} \tag{20}$$

where $\vec{\mathbf{d}}^*$ is an array of $r-t+1-j$ degree sequences equal to $\mathbf{d}^* = (\ell-1, \ldots, \ell-1)$ (one of the terms $\ell-1$ may be replaced by $\ell-2$ in these sequences to deal with parity constraints). Note that, to reach (20), we replaced the denominator within the sums by a single term, which depends on each particular term being added. Our choice of W implies that, when $|\vec{\mathbf{d}}_+| = j$,

$$\begin{aligned}
& N_{G[W]}(\vec{\mathbf{d}}_+, \vec{\mathbf{d}}^*) \\
\geq \; & n^{-\alpha} \prod_{i=1}^{j} N_{K_p}(\mathbf{d}_+^i) \prod_{i=1}^{r-t+1-j} N_{K_p}(\mathbf{d}^*), \\
\geq \; & n^{-\alpha} \left(\frac{\sqrt{2}}{(\ell-1)!^p} \left(\frac{p(\ell-1)}{e} \right)^{p(\ell-1)/2} \exp\left(\frac{1-(\ell-1)^2}{4} \right) \right)^{r-t+1-j} \prod_{i=1}^{j} N_{K_p}(\mathbf{d}_+^i) \\
\geq \; & \frac{n^{-\alpha} \cdot p^{p(\ell-1)(r-t+1-j)/2}}{(e(\ell-1)!)^{(\ell-1)p(r-t+1)}} \prod_{i=1}^{j} N_{K_p}(\mathbf{d}_+^i). \qquad (21)
\end{aligned}$$

To find an upper bound on (20), we combine (21) with the fact that the number of terms in the sums is at most $(r-t+1)\ell^{p(r-t+1)}$ and $\gamma = 8/9$ and that, for $\ell \geq 2$, the term $p^{(r-t+1-j)\cdot p(17/9-\ell)/2}$ is maximized for $j = r-t$, which leads to the following upper bound on (20):

$$\begin{aligned}
& \frac{n^{\alpha}(1+\delta)^n \cdot \ell^{2p(r-t+1)} \cdot (r-t+1) \cdot (e \cdot (t-1) \cdot (\ell-1)!)^{(r-t+1)(\ell-1)p}}{((t-1)^{n/4}-1)p^{(\ell-(17/9))p/2}} \\
= \; & (t-1)^{O(n)} \cdot p^{(17/9-\ell)p/2} < \frac{1}{2(t-1)^{n/8}}
\end{aligned}$$

for large n, as $p \geq n - M \log_{t-1} n$. Therefore, we obtain

$$\frac{(14)}{(16)} \leq \frac{1}{2(t-1)^{n/8}} + \frac{1}{2(t-1)^{n/8}} = \frac{1}{(t-1)^{n/8}}, \qquad (22)$$

for $t \geq 3$ and n sufficiently large, which implies that $N'_A \geq (t-1)^{n/8} N_A$, as required. □

Theorem 3.4. *For integers $r \geq t \geq 3$ and $\ell \geq 2$, there exists n_0 such that $c_{r,t,\ell}(n) = |\mathcal{C}_{r,t,\ell}(K_n)|$ holds for $n \geq n_0$. Moreover, K_n is the unique n-vertex $\mathcal{C}_{r,t,\ell}$-extremal graph.*

Proof. Fix $r \geq t \geq 3$ and $\ell \geq 2$. Assume that there is a $\mathcal{C}_{r,t,\ell}$-extremal graph G on n-vertices with at least two non-adjacent vertices x and y. We prove that $G' = G + \{x, y\}$ has more colorings than G if n is sufficiently large. Our proof uses the strategy employed for Lemma 3.2.

By Lemma 2.2 we know that n_0 may be chosen so that $|E(G)| \geq \binom{n}{2} - Dn \log_{t-1} n$, where $D = D(r, t)$ is a constant. Fix $A > 0$ with the property of Lemma 2.3. Let $n_0^{(3.1)}$, M and α given by Lemma 3.1 applied for D, $d = \ell - 1$, $k = r - t + 1$ and $\gamma = 8/9$ (adjusting the constants so that logarithms in the statement of the lemma have base $t-1$), and fix a set $W \subset V$ with $|W| \geq n - M \log_{t-1} n = p$ such that $G[W]$ satisfies the conclusion of the lemma.

We show that the number N' of new colorings of G' obtained by extending colorings of G is larger than the number N of colorings of G that cannot be extended to colorings of G'. By Lemma 2.3 with $\varepsilon = 1/3$, the number of colorings in $\bigcup_{(S_1,\ldots,S_n)\in\mathcal{S}_A} \mathcal{C}_{(S_1,\ldots,S_n)}(G)$ is at most $(t-1)^{|E(G)|}/3$ for sufficiently large n.

Let N'_A be the number of new colorings of G' associated with n-tuples $(S_1, \ldots, S_n)$ in $\overline{\mathcal{S}_A}$, and let N_A be the number of colorings associated with such collections that cannot be extended. In the remainder of the proof, we find a lower bound on N'_A and an upper bound on N_A to show that $N'_A \geq 2^{n/4} N_A \geq 2N_A$. Once again we have $N'_A \geq (t-2) \cdot (t-1)^{|E(G)|}$ (see (23)), which leads to the desired result:

$$N' - N \geq N'_A - \left(N_A + \frac{(t-1)^{|E(G)|}}{3} \right) \geq \left(\frac{(t-2)}{2} - \frac{1}{3} \right) \cdot (t-1)^{|E(G)|} > 0.$$

With the arguments used for (16), we may show that every coloring of G for which every vertex is assigned the same set of $(t-1)$ colors can be extended to colorings of G', which increases the total number of rainbow-$S_{t,\ell}$-free r-colorings by at least

$$(t-2)\cdot\sum_{\vec{\mathbf{d}}} N_{G[W]}(\vec{\mathbf{d}})\cdot(t-1)^{|E(G)|-u(\vec{\mathbf{d}})}, \tag{23}$$

where we again sum over arrays $\vec{\mathbf{d}}=(\mathbf{d}^1,\ldots,\mathbf{d}^{r-t+1})$, where each $\mathbf{d}^i=(d_1^i,\ldots,d_p^i)$ has components bounded by $\ell-1$. Recall that $u(\vec{\mathbf{d}})=\frac{1}{2}\sum_{i=1}^{r-t+1}\sum_{j=1}^{p}d_j^i$.

We show next that the number of all colorings of G that cannot be extended to a coloring of G' is smaller than (23). We give an upper bound on the number N_A of rainbow-$S_{t,\ell}$-free r-colorings of G that are associated with collections $(S_1,\ldots,S_n)\in\overline{\mathcal{S}_A}$ and cannot be extended to a coloring of G'. All such colorings have the property that the color sets S_x and S_y assigned to x and y, respectively, are disjoint. Fix $(S_1,\ldots,S_n)$ for which this holds, and where S is assigned to at least $n-A\log_{t-1}n$ vertices of G. The $(t-1)$-subset $S\subset[r]$ may chosen in $\binom{r}{t-1}$ ways and there are $\binom{n}{n-A\log_{t-1}n}$ ways of choosing $n-A\log_{t-1}n$ vertices which are assigned S. The remaining vertices may be assigned $(t-1)$-element color sets in at most $\binom{r}{t-1}^{A\log_{t-1}n}$ ways. As a consequence,

$$\binom{r}{t-1}\cdot\binom{n}{n-A\log_{t-1}n}\cdot\binom{r}{t-1}^{A\log_{t-1}n}\le\binom{r}{t-1}^{A\log_{t-1}n+1}\cdot n^{A\log_{t-1}n} \tag{24}$$

is an upper bound on the number of ways of fixing a collection $(S_1,\ldots,S_n)$ with the required properties.

We derive an upper bound on the number of colorings in $\mathcal{C}_{(S_1,\ldots,S_n)}(G)$ that cannot be extended to G'. Since x,y have degree at least $3n/4$ by Lemma 3.2, their common neighbourhood $N(\{x,y\})$ has size at least $n/2$. For any vertex w in $N(\{x,y\})$ we have $S_w\cap(S_x\cup S_y)\le t-1$. More precisely, we have $|S_w\cap S_x|=a_w$ and $|S_w\cap S_y|\le t-1-a_w$, so that there at most $a_w(t-1-a_w)\le((t-1)/2)^2$ ways to assign ordinary colors to the edges $\{x,w\}$ and $\{y,w\}$. Hence all edges between $\{x,y\}$ and their common neighbourhood $N(\{x,y\})$ may be colored in at most $((t-1)/2)^{2|N(\{x,y\})|}$ ways with ordinary colors (with respect to both ends).

As in the proof of Lemma 3.2, we proceed as follows: let $Y=\{u\in V\colon S_u=S\}$ and let $H=G[W\cap Y]$, (i) color edges incident with vertices $u\in V\setminus Y$ with rare colors with respect to u; (ii) color edges incident with vertices of $(V\setminus W)\cap Y$ with rare colors; (iii) color edges incident with $V\setminus Y$ with ordinary colors (with respect to some endpoint in $V\setminus Y$); (iv) color edges in H with rare colors (with respect to S); (v) color edges with both ends in Y with ordinary colors (with respect to S).

We obtain the following upper bound on the number of colorings of G that cannot be extended to G':

$$\binom{r}{t-1}^{A\log_{t-1}n+1}\cdot n^{A\log_{t-1}n}\cdot n^{(\ell-1)(r-t+1)A\log_{t-1}n}\cdot n^{(\ell-1)(r-t+1)M\log_{t-1}n}\cdot$$
$$\cdot\sum_{\vec{\mathbf{d}}}N_H(\vec{\mathbf{d}})\cdot(t-1)^{|E(G)|-u(\vec{\mathbf{d}})-2|N(x,y)|}\cdot\left(\frac{t-1}{2}\right)^{2|N(x,y)|}. \tag{25}$$

It is easy to see that, choosing n sufficiently large, equation (25) is at most

$$2^{n/2}\cdot\sum_{\vec{\mathbf{d}}}N_H(\vec{\mathbf{d}})\cdot(t-1)^{|E(G)|-u(\vec{\mathbf{d}})-2|N(x,y)|}\cdot\left(\frac{t-1}{2}\right)^{2|N(x,y)|}, \tag{26}$$

which may be rewritten as

$$2^{n/2} \cdot \sum_{\mathbf{s}} \sum_{\vec{\mathbf{d}}_-} \sum_{\vec{\mathbf{d}}_+} N_H(\vec{\mathbf{d}}) \cdot \frac{(t-1)^{|E(G)|-u(\vec{\mathbf{d}})}}{2^{2|N(x,y)|}}, \tag{27}$$

where, as in (13), $\mathbf{s}\colon [r-t+1] \to \{0,1\}$ is a function that, for each i, indicates whether the degree sequence $\mathbf{d}^i$ is γ-dense or not, while $\vec{\mathbf{d}}_+$ and $\vec{\mathbf{d}}_-$ are the arrays of γ-dense and sparse degree sequences, respectively, that create an array $\vec{\mathbf{d}}$ with the distribution determined by $\mathbf{s}$. We split equation (27) according to whether all rare colors generate dense graphs, or whether this does not hold, which leads to

$$\begin{aligned} &2^{n/2}\left(\sum_{|\vec{\mathbf{d}}_+|=r-t+1} N_H(\vec{\mathbf{d}}_+) \cdot \frac{(t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)}}{2^{2|N(x,y)|}} + \sum_{\mathbf{s}\neq\vec{1}} \sum_{\vec{\mathbf{d}}_-} \sum_{\vec{\mathbf{d}}_+} N_H(\vec{\mathbf{d}}) \cdot \frac{(t-1)^{|E(G)|-u(\vec{\mathbf{d}})}}{2^{2|N(x,y)|}}\right) \\ \leq\ & \sum_{|\vec{\mathbf{d}}_+|=r-t+1} N_H(\vec{\mathbf{d}}_+) \cdot \frac{(t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)}}{2^{n/2}} + \sum_{\mathbf{s}\neq\vec{1}} \sum_{\vec{\mathbf{d}}_-} \sum_{\vec{\mathbf{d}}_+} N_H(\vec{\mathbf{d}}) \cdot \frac{(t-1)^{|E(G)|-u(\vec{\mathbf{d}})}}{2^{n/2}}. \end{aligned} \tag{28}$$

Using the estimate (15) we have

$$\begin{aligned} &\sum_{\mathbf{s}\neq\vec{1}} \sum_{\vec{\mathbf{d}}_-} \sum_{\vec{\mathbf{d}}_+} N_H(\vec{\mathbf{d}}) \cdot \frac{(t-1)^{|E(G)|-u(\vec{\mathbf{d}})}}{2^{n/2}} \\ \leq\ & \ell^{p(r-t+1)} \cdot \sum_{j=1}^{r-t} p^{\gamma p(r-t+1-j)/2} \cdot \sum_{|\vec{\mathbf{d}}_+|=j} \frac{(t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)}}{2^{n/2}} \cdot \prod_{i=1}^{j} N_{K_p}(\mathbf{d}_+^i). \end{aligned} \tag{29}$$

To conclude our argument, we compare the upper bound (28) and the lower bound (23). We have

$$\frac{(28)}{(23)} \leq \frac{\sum_{|\vec{\mathbf{d}}_+|=r-t+1} N_H(\vec{\mathbf{d}}_+) \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)}}{2^{n/2}(t-2) \cdot \sum_{\vec{\mathbf{d}}} (t-1)^{|E(G)|-u(\vec{\mathbf{d}})} \cdot N_{G[W]}(\vec{\mathbf{d}})} \tag{30}$$

$$+ \frac{\ell^{p(r-t+1)} \cdot \sum_j p^{\gamma p(r-t+1-j)/2} \cdot \sum_{|\vec{\mathbf{d}}_+|=j} (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)} \cdot \prod_{i=1}^{j} N_{K_p}(\mathbf{d}_+^i)}{2^{n/2}(t-2) \cdot \sum_{\vec{\mathbf{d}}} (t-1)^{|E(G)|-u(\vec{\mathbf{d}})} \cdot N_{G[W]}(\vec{\mathbf{d}})}. \tag{31}$$

Using the estimates of (19), the term (30) may be bounded by

$$\begin{aligned} &\frac{\sum_{|\vec{\mathbf{d}}_+|=r-t+1} N_H(\vec{\mathbf{d}}_+) \cdot (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)}}{2^{n/2}(t-2) \cdot \sum_{|\vec{\mathbf{d}}_+|=r-t+1} (t-1)^{|E(G)|-u(\vec{\mathbf{d}}_+)} \cdot N_{G[W]}(\vec{\mathbf{d}}_+)} \\ \leq\ & \frac{n^{\alpha}}{2^{n/2}(t-2)} \leq \frac{1}{2^{n/4+1}} \end{aligned}$$

For the second term (31), we repeat the arguments used in the proof of Lemma 3.2. Therefore, we obtain

$$\frac{(28)}{(23)} \leq \frac{1}{2^{n/4+1}} + \frac{1}{2^{n/4+1}} = \frac{1}{2^{n/4}}, \tag{32}$$

for $t \geq 3$ and n sufficiently large, as required. □

To conclude this section, we show how the proof of Theorem 1.1 may be derived from our proof of Theorem 1.2.

Proof of Theorem 1.1. The basic idea is to split the set $\mathcal{C} = \mathcal{C}_{r,\ell_1,\ldots,\ell_t}(G)$ of all rainbow-$S_{\ell_1,\ldots,\ell_t}$-free r-colorings of a graph G, $\ell_1 \geq \ldots \geq \ell_t$, as a union similar to (2). However, given a coloring, we shall now consider that vertices may have two types: we will say that v has *type 1* if there is no rainbow S_{t,ℓ_t} centered at v, and that it has *type 2* otherwise. The crucial observation is that a vertex of type 2 cannot be the center of a rainbow S_{t-1,ℓ_1}, as otherwise there would be a rainbow $S_{\ell_1,\ldots,\ell_t}$ centered in it. This motivates us to modify the definition of an ordinary color: for a vertex v of type 1, a color is ordinary if it appears at least ℓ_t times among the with v adjacent edges; if v has type 2, an ordinary color must appear at least ℓ_1 times. As before, the remaining colors are said to be rare with respect to v. One important feature of this definition is that vertices of type 1 may have up to $t-1$ ordinary colors, while vertices of type 2 might have only $t-2$ ordinary colors. In particular, the analogue of (2) is given by

$$\mathcal{C}_{r,\ell_1,\ldots,\ell_t}(G) = \bigcup_{(S_1,\ldots,S_n)} \mathcal{C}_{r,\ell_1,\ldots,\ell_t,(S_1,\ldots,S_n)}(G),$$

where $(S_1,\ldots,S_n)$ is such that $|S_i| = t-1$ for vertices of type 1 and $|S_i| = t-2$ for vertices of type 2.

With this in mind, it is easy to see that Lemma 2.2 would hold with essentially the same arguments for this more general forbidden configuration. Moreover, we can easily prove a version of Lemma 2.3 where the class $\mathcal{S}_A$ shown to be negligible contains all colorings in $\mathcal{C}_{r,\ell_1,\ldots,\ell_t,(S_1,\ldots,S_n)}(G)$ with the property that no set S for which $|S| = t-1$ appears at least $n - A\log_{t-1} n$ times in the vector $(S_1,\ldots,S_n)$. In other words, colorings in $\overline{\mathcal{S}_A}$ have at least $n - A\log_{t-1} n$ vertices of type 1 which have the same set of ordinary colors.

As was done for $S_{t,\ell}$-free colorings, we show that n-vertex $\mathcal{C}_{r,\ell_1,\ldots,\ell_t}$-extremal graphs have minimum degree at least $3n/4$. To this end, we imitate the proof of Lemma 3.2, in which we let G be a graph containing some vertex v with degree at most $3n/4-1$, and we define G' by adding edges between v and all vertices not adjacent to it, which form a set W such that $|W| \geq n/4$.

Here, it is convenient to split the set $\mathcal{C} = \mathcal{C}_{r,\ell_1,\ldots,\ell_t}(G)$ as the union of $\mathcal{C}_1$, containing colorings for which all vertices have type 1, and $\mathcal{C}_2$, which contains the other colorings. We treat these colorings as in the proof of Lemma 3.2, where we assume that a coloring in G can be extended to a coloring in G' provided that v shares an ordinary color with any vertex $w \in W$. This is pessimistic, as some coloring of G might be extendible to a coloring of G' using rare colors.

The discussion in Lemma 3.2 applies directly to colorings in $\mathcal{C}_1(G)$, which allows us to conclude that

$$N'_A \geq (t-1)^{n/8} \cdot N_A^{(1)},$$

where N'_A is a lower bound on the number of 'new' colorings (in $\mathcal{C}_1(G')$) created by the extension and $N_A^{(1)}$ is an upper bound on the number of colorings in $\mathcal{C}_1(G)$ that cannot be extended to G'.

We still need to consider the number $N_A^{(2)}$ of colorings in $\mathcal{C}_2(G)$ that cannot be extended to G'. Note that a coloring in $\mathcal{C}_2(G)$ for which i vertices have type 2 can be turned into a coloring where each rare color appears fewer than ℓ_t times if we recolor at most $\ell_1 - \ell_t$ edges incident with each rare color. This may be done by assigning them a fixed ordinary color (say the ordinary color of least index among the ordinary colors associated with that vertex). This gives rise to colorings of G in classes $\mathcal{D}_j(G)$, which are colorings such that rare colors appear at most $\ell_t - 1$ times, each vertex is assigned at most $t-1$ ordinary colors, and exactly j vertices have fewer than $t-1$ ordinary colors (i.e. are incident with ℓ_t or more edges of at most $t-2$ different colors). Note that each coloring in $\mathcal{D}_j$ may be turned into at most

$$\binom{j}{i} n^{(\ell_1-\ell_t)(r-t+2)i} 2^{(\ell_1-\ell_t)(r-t+2)i} \le j^i (2n)^{(\ell_1-\ell_t)(r-t+2)i}$$

colorings in $\mathcal{C}_2(G)$ for which i vertices have type 2, since we first choose i vertices to have type 2, and then, for each such vertex w and each of the $r-t+2$ rare colors, we choose up to $\ell_1-\ell_t$ edges incident with w to be assigned the corresponding ordinary color (the factor $2^{(\ell_1-\ell_t)(r-t+2)i}$ takes care of the 'up to', as it allows us to decide, for each chosen edge, whether to recolor it or not). Summing over all $1 \le j \le n$ and all $1 \le i \le \min\{j, A \log_{t-1} n\}$, we deduce that the number of colorings in $\mathcal{C}_2(G)$ that cannot be extended to G' is at most $2^{O(\log_{t-1}^2 n)} D$, where D is the number of colorigs of $\mathcal{D} = \cup_{j=1}^{A \log_{t-1} n} \mathcal{D}_j(G)$ that cannot be extended to G'. But it is clear that D is bounded above by the number $N_A^{(1)}$ of colorings in $\mathcal{C}_1(G)$ that cannot be extended to G', as colorings in $\mathcal{D}$ can be viewed as colorings of $\mathcal{C}_1(G)$ (indeed, they are colorings of $\mathcal{C}_1(G)$ where some ordinary color fails to appear ℓ_t times for each vertex of type 2).

As a consequence, for n sufficiently large,

$$N_A^{(1)} + N_A^{(2)} \le 2^{O(\log_{t-1}^2 n)} \cdot N_A^{(1)} \le \frac{2^{O(\log_{t-1}^2 n)} \cdot N'_A}{(t-1)^{n/8}} < N'_A,$$

which proves the extension of Lemma 3.2 to the present context. Clearly, Theorem 3.4 may be extended analogously, which leads to the desired result. □

4. Auxiliary results

To conclude our paper, we prove Lemma 3.1. To do this, we need some preliminary definitions and results, which are based on ideas of Gao [9].

Let H_n be a graph on n vertices and let $e(H_n)$ denote the number of edges in H_n. Color all edges in H_n blue and all edges in its complement $\overline{H_n}$ red, so that K_n is the complete graph on n vertices with a 2-coloring $K_n = H_n \cup \overline{H_n}$. For any red edge $e = \{u, v\}$, define the red degree of e

$$d_r(e) = d_r(u) + d_r(v) - 2,$$

where $d_r(u)$ denotes the number of red edges incident with u. Let $\Delta_r(\overline{H_n}) = \max_{e \in E(\overline{H_n})} d_r(e)$, where we maximize over red edges.

Fix a degree sequence $\mathbf{d} = (d_1, \ldots, d_n)$, where $\max d_i \le d$ for some absolute constant $d \in \mathbb{N}$, and let F_n be a graph with degree sequence $\mathbf{d} = (d_1, \ldots, d_n)$ chosen uniformly at random from all subgraphs of K_n with this degree sequence. Let $X_n = X_n(F_n)$ be the random variable that accounts for the number of red edges in F_n.

For $k' = 1$, recall that Lemma 3.1 counts the number of subgraphs (with a given degree sequence) of a complete graph that remain after an opponent deletes a certain number of edges. In the current setting, edges that remain are represented by blue, while edges deleted are represented by red. As a consequence, treating subgraphs of the complete graph that are not affected by the deletion of edges is the same as treating colored subgraphs of a 2-coloring of K_n whose edges are all blue.

The following result is the main ingredient in the proof of Lemma 3.1, as it gives a lower bound on the probability of choosing a subgraph with no red edges for any 2-coloring such that the red degree is small.

Lemma 4.1. *For $\gamma > 0$ and a positive integer d, there exists a positive integer n_0 such that the following holds for $n \ge n_0$. Fix a γ-dense sequence $d = (d_1, \ldots, d_n)$ such that* $\max d_i \le d$.

Let H_n be a graph on n vertices and denote $t_n = \binom{n}{2} - e(H_n)$. Assume that:

$$a = \frac{\Delta_r(\overline{H_n})}{n} < \frac{\gamma}{d} \tag{33}$$

$$\sqrt{2}\left(\frac{e^2 t_n}{b\gamma n^2}\right)^{bn} \cdot (d!)^n \cdot \exp(d+d^2) < \frac{1}{2} \tag{34}$$

where $b = (\gamma - da)/8$. Then the random variable X_n defined above satisfies

$$\mathbf{P}(X_n = 0) \geq \frac{1}{2}\exp\left(\frac{-d^2 t_n}{bn}\right) = \frac{1}{2}\exp\left(\frac{-8d^2 t_n}{(\gamma - da)n}\right).$$

Before proving this result, we show that it implies Lemma 3.1. First, given a positive integer k, let $Y_n = Y_n(\mathbf{F}_n)$ denote the numbers of red edges contained in a k-tuple $\mathbf{F}_n = (F_n^1, \ldots, F_n^k)$ of edge-disjoint subgraphs of K_n such that F_n^i has degree sequence $\mathbf{d}^i$ bounded by d, $i = 1, \ldots, k$. We assume that these graphs are chosen sequentially, uniformly at random from the remaining graph.

Corollary 4.2. *For $\gamma > 0$ and positive integers d and k, there exists a positive integer n_0 such that the following holds for $n \geq n_0$. Fix γ-dense sequences $\mathbf{d}^1 = (d_1^1, \ldots, d_n^1), \ldots, \mathbf{d}^k = (d_1^k, \ldots, d_n^k)$ such that $\max d_j^i \leq d$. Let H_n be a graph on n vertices and denote $t_n = \binom{n}{2} - e(H_n)$. Assume that:*

$$\frac{\Delta_r(\overline{H_n}) + 2dk}{n} < \frac{\gamma}{d} \tag{35}$$

$$\sqrt{2}\left(\frac{e^2(t_n + kdn/2)}{b\gamma n^2}\right)^{bn} \cdot (d!)^n \cdot \exp(d+d^2) < \frac{1}{2} \tag{36}$$

where $a = \frac{\Delta_r(\overline{H_n})}{n}$ and $b = \frac{\gamma - da}{8}$. Then the random variable Y_n defined above satisfies

$$\mathbf{P}(Y_n = 0) \geq \exp\left(\frac{-4d^2 t_n}{bn}\right) \geq \min\left\{\exp\left(\frac{-24kd^2 t_n}{(\gamma - da)n}\right), \exp\left(\frac{-24k^2 d^3}{\gamma - da}\right)\right\}.$$

Proof. By assumption, given γ, d and k, there exists n_0 such that (35) and (36) hold for $n \geq n_0$. We iterate Lemma 4.1. Let $H_n^1 = H_n$ be a graph satisfying the hypotheses, and choose F_n^1 uniformly at random from all subgraphs of $K_n = H_n^1 \cup \overline{H_n^1}$ with degree sequence $\mathbf{d}^1$. Note that $a_1 = a < \frac{\Delta_r(\overline{H_n}) + 2dk}{n}$ and $t_n^{(1)} = t_n < t_n + kdn/2$, so (33) and (34) are satisfied for $n \geq n_0$. Then, by Lemma 4.1,

$$\mathbf{P}(X_n^1 = 0) \geq \frac{1}{2}\exp\left(\frac{-8d^2 t_n^{(1)}}{(\gamma - da_1)n}\right),$$

where $X_n^1 = X_n(F_n^1)$.

Assume that such a graph F_n^1 has been chosen so that $X_n^1 = 0$. Consider the graph $H_n^2 = H_n^1 \setminus E(F_n^1)$ and let $t_n^{(2)} = \binom{n}{2} - e(H_n^2)$. In other words, we consider a new coloring of K_n in which the edges in F_n^1 are recolored red. Note that $t_n^{(2)} \leq t_n + dn/2 < t_n + kdn/2$ and $\Delta_r(\overline{H_n^2}) \leq \Delta_r(\overline{H_n^1}) + 2d < \Delta_r(\overline{H_n}) + 2dk$. Therefore, for $n \geq n_0$, $a_2 = \Delta_r(\overline{H_n^2})/n$ and $t_n^{(2)}$ satisfy (33) and (34), respectively. As a consequence, if we choose F_n^2 uniformly at random from all subgraphs of $K_n = H_n^2 \cup \overline{H_n^2}$ with degree sequence $\mathbf{d}^2$, we may apply Lemma 4.1. We apply it to the new coloring induced by H_n^2, which we have shown to satisfy the hypotheses of Lemma 4.1 to obtain

$$\mathbf{P}(X_n^2 = 0 | X_n^1 = 0) \geq \frac{1}{2}\exp\left(\frac{-8d^2 t_n^{(2)}}{(\gamma - da_2)n}\right),$$

where $X_n^2 = X_n(F_n^2)$. Observe that the probability obtained by the lemma is a conditional probability, because we choose F_n^2 after F_n^1 has already been chosen.

Iteratively, for $3 \leq i \leq k$, we condition upon having chosen $F_n^1, \ldots, F_n^{i-1}$ with the corresponding degree sequences, all of which containing no red edges of the original graph K_n. We let $H_n^i = H_n^{i-1} \setminus E(F_n^{i-1})$, $t_n^{(i)} = \binom{n}{2} - e(H_n^i)$ and $a_i = \Delta_r(\overline{H_n^i})/n$. Observe that $t_n^{(i)} < t_n + kdn/2$ and $a_i < \frac{\Delta_r(\overline{H_n})+2dk}{n}$. Thus (34) and (33) have been satisfied for all $n \geq n_0$. And we choose F_n^i uniformly at random from all subgraphs of $K_n = H_n^i \cup \overline{H_n^i}$ with degree sequence $\mathbf{d}^i$. If we let $X_n^i = X_n(F_n^i)$, we obtain

$$\mathbf{P}\left(X_n^i = 0 | X_n^1 = \cdots = X_n^{i-1} = 0\right) \geq \frac{1}{2} \exp\left(\frac{-8d^2 t_n^{(i)}}{(\gamma - da_i)n}\right).$$

As a consequence,

$$\begin{aligned}
\mathbf{P}(Y_n = 0) &= \mathbf{P}\left(X_n^1 = \cdots = X_n^k = 0\right) \\
&= \mathbf{P}(X_n^1 = 0) \cdot \mathbf{P}(X_n^2 = 0 | X_n^1 = 0) \cdots \mathbf{P}(X_n^k = 0 | X_n^1 = \cdots = X_n^{k-1} = 0) \\
&\geq \frac{1}{2^k} \exp\left(\frac{-8d^2 t_n^{(1)}}{(\gamma - da_1)n}\right) \cdots \exp\left(\frac{-8d^2 t_n^{(k)}}{(\gamma - da_k)n}\right) \\
&\geq \frac{1}{2^k} \exp\left(\frac{-8kd^2(t_n + kdn/2)}{(\gamma - d(a + 2dk/n))n}\right). \qquad (37)
\end{aligned}$$

To conclude the proof, we treat the cases $t_n \geq kdn$ and $t_n < kdn$ separately. In the first case, we have $t_n + kdn/2 \leq 3t_n/2$, and the term (37) is bounded below by $\frac{1}{2^k} \exp\left(\frac{-12kd^2 t_n}{(\gamma - d(a+2dk/n))n}\right) \geq \exp\left(\frac{-24kd^2 t_n}{(\gamma - da)n}\right)$, for n sufficiently large. In the second case, we have $t_n + kdn/2 \leq 3kdn/2$, so that, with similar arguments, we see that (37) is bounded below by $\exp\left(\frac{-24k^2 d^3}{\gamma - da}\right)$. As a consequence, the term (37) is at most

$$\min\left\{\exp\left(\frac{-24kd^2 t_n}{(\gamma - da)n}\right), \exp\left(\frac{-24k^2 d^3}{\gamma - da}\right)\right\},$$

as required. □

Using the idea that the red edges are the edges missing from the graph G to turn it into a complete graph, we prove Lemma 3.1, which we now restate.

Lemma 3.1. *Given positive integers d and k, constants $D > 0$ and $\gamma > 0$, there exist positive constants n_0, M and α satisfying the following property for all $n \geq n_0$. For every graph H with $|V(H)| = n$ and $|E(H)| \geq \binom{n}{2} - Dn \ln n$, there exists $W \subseteq V(H)$ with $|W| \geq n - M \ln n$ such that, for all γ-dense degree sequences $\boldsymbol{d}^1, \ldots, \boldsymbol{d}^{k'} \in \{0, \ldots, d\}^{|W|}$, where $k' \leq k$, we have*

$$N_{H[W]}(\boldsymbol{d}^1, \ldots, \boldsymbol{d}^{k'}) \geq n^{-\alpha} \cdot \prod_{i=1}^{k'} N_{K_{|W|}}(\boldsymbol{d}^i).$$

Proof. Fix d, k, D and γ as in the statement of the lemma, and let H be a graph on n vertices with $|E(H)| \geq \binom{n}{2} - Dn \ln n$. As before, we shall assume that n is sufficiently large so that all inequalities in the proof are satisfied.

Consider the set

$$A = \left\{v \in V(H) \colon d(v) \leq \frac{3d - \gamma}{3d} n\right\},$$

so that

$$\begin{aligned} & |A| \cdot \frac{3d-\gamma}{3d} n + (n - |A|) \cdot (n-1) \geq n(n-1) - 2Dn \ln n \\ \implies & |A| \cdot \left(\frac{-\gamma n}{3d} + 1 \right) \geq -2Dn \ln n \\ \implies & |A| \cdot \left(\frac{\gamma n}{3d} - 1 \right) \leq 2Dn \ln n \\ \implies & |A| \leq \frac{6Dd \cdot n \ln n}{\gamma n - 3d} \leq M \ln n \end{aligned}$$

for $M = \frac{7dD}{\gamma}$ and n sufficiently large.

Let $W = V(H) \setminus A$ and fix bounded degree sequences $\mathbf{d}^1, \ldots, \mathbf{d}^{k'} \in \{0, \ldots, d\}^{|W|}$ where $k' \leq k$. We wish to apply Corollary 4.2 to the graph $H_{|W|} = H[W]$ on $|W|$ vertices, where the edges in $H_{|W|}$ are precisely the blue edges of $K_{|W|}$. To this end, let $a = \Delta_r(\overline{H_{|W|}})/|W|$ and $t_{|W|} = \binom{|W|}{2} - e(H_{|W|})$. We need to prove that $a + 2dk/|W| < \gamma/d$ and that $t_{|W|} + kd|W|/2$ satisfies (36). Observe that, for any vertex v in W, the number of red edges incident with v in $K_{|W|}$ is precisely the number of edges incident with v in the complement of $H[W]$ (with respect to $K_{|W|}$), which is at most the number of edges incident with v in the complement of H (with respect to K_n). By our choice of W, this is at most $\frac{\gamma n}{3d}$. Then $d_r(v) \leq \frac{\gamma n}{3d}$ for any vertex v in W, in particular, we have $\Delta_r(\overline{H_{|W|}}) \leq \frac{2\gamma n}{3d} - 2$. Thus for large n,

$$\frac{\Delta_r(\overline{H_{|W|}})}{|W|} < \frac{2\gamma n}{3d} \frac{1}{n - M \ln n} + \frac{2dk}{n - M \ln n} \leq \frac{\gamma}{d} \frac{4}{5} + \frac{\gamma}{10d} < \frac{\gamma}{d},$$

and (35) holds. Moreover, $e(H_{|W|}) \geq \binom{n}{2} - Dn \ln n - Mn \ln n = \binom{n}{2} - (D+M) n \ln n$, for large n. We have

$$t_{|W|} = \binom{|W|}{2} - e(H_{|W|}) \leq (D+M) n \ln n,$$

so that

$$\frac{t_{|W|} + kd|W|/2}{|W|^2} = \frac{(D+M) n \ln n}{(n - M \ln n)^2} + \frac{dk}{2(n - M \ln n)} < \frac{(D+M) \ln n}{n - 2M \ln n} + \frac{dk}{2(n - M \ln n)}.$$

Since this can be made arbitrarily close to 0 by choosing n sufficiently large, we have

$$\left(\frac{t_{|W|} + kd|W|/2}{2b\gamma |W|^2} \right)^b \ll \frac{1}{d!}$$

for large n (and hence large $|W|$), so that (36) is satisfied. As a consequence, Corollary 4.2 applies to $H_{|W|}$ and the degree sequences $\mathbf{d}^1, \ldots, \mathbf{d}^{k'}$. Since $k' \leq k$,

$$\mathbf{P}(Y_{|W|} = 0) \geq \min \left\{ \exp\left(\frac{-24kd^2 t_{|W|}}{(\gamma - da)|W|} \right), \exp\left(\frac{-24k^2 d^3}{\gamma - da} \right) \right\}.$$

If the minimum is attained by the second term, the result is immediate, as the probability is a constant. If the minimum is attained by the first term, because, $|W| \geq n - M \ln n$ and $t_{|W|} \leq (D+M) n \ln n$, after some straightforward calculations, we obtain

$$\mathbf{P}(Y_{|W|} = 0) \geq n^{-\beta},$$

for large n, where $\beta = \frac{48(D+M)d^2 k}{\gamma - da}$.

Hence, we infer that

$$N_{H[W]}(\mathbf{d}^1, \ldots, \mathbf{d}^{k'}) \geq n^{-\beta} \cdot N_{K_{|W|}}(\mathbf{d}^1, \ldots, \mathbf{d}^{k'}).$$

It is well-known that (for instance, using Brun's sieve, see [3])

$$N_{K_{|W|}}(\mathbf{d}^1,\dots,\mathbf{d}^{k'}) = \Omega\left(\prod_{i=1}^{k'} N_{K_{|W|}}(\mathbf{d}^i)\right)$$

if k is a constant. Consequently, for $\alpha = \frac{49(D+M)d^2k}{\gamma-da}$, we obtain

$$N_{H[W]}(\mathbf{d}^1,\dots,\mathbf{d}^{k'}) \geq n^{-\alpha}\cdot\prod_{i=1}^{k'} N_{K_{|W|}}(\mathbf{d}^i).$$

□

To conclude this section, we prove Lemma 4.1. The following *switching* operations will be particularly useful. We use the notation introduced at the beginning of this section, where we have a complete graph K_n whose edge set is 2-colored with respect of a given subgraph H_n.

(i) *r-switching*: Given a graph F_n containing at least one red edge, choose a red edge $x \in F_n$, label its end vertices u and v, and choose a blue edge $x' \in F_n$ that is not adjacent with x, label its end vertices u' and v'. Replace these two edges by $\{u,u'\}$ and $\{v,v'\}$. The r-switching is applicable if and only if $\{u,u'\}$ and $\{v,v'\}$ are blue edges and are not in F_n.

(ii) *inverse r-switching*: Given a graph F_n containing at least two blue edges, choose a blue edge in F_n and label its end vertices u and u', then choose another blue edge in F_n that is not adjacent with $\{u,u'\}$ and label its end vertices v and v'. Replace these two edges by $\{u,v\}$ and $\{u',v'\}$. The inverse r-switching is applicable if and only if $\{u,v\}$ and $\{u',v'\}$ are not in F_n and $\{u,v\}$ is red and $\{u',v'\}$ is blue.

Let $\mathbf{d} = (d_1,\dots,d_n)$ be a γ-dense sequence such that $\max d_i \leq d$. Let $\mathcal{R}(s)$ be the set of all subgraphs F_n of K_n with degree sequence $\mathbf{d}$ and s red edges. Note that, for every $s \geq 1$, an r-switching operation converts a graph $F_n \in \mathcal{R}(s)$ into a graph $F'_n \in \mathcal{R}(s-1)$. On the other hand, an inverse r-switching converts an $F'_n \in \mathcal{R}(s-1)$ into an $F_n \in \mathcal{R}(s)$.

Let $N(F_n)$ be the number of r-switchings applicable on F_n and $N'(F'_n)$ the number of inverse r-switching applicable on F'_n. Let $t_n = \binom{n}{2} - e(H_n)$.

Proof of Lemma 4.1. By hypotheses $2m = \sum_i d_i \geq \gamma n$ and $\frac{\Delta_r(\overline{H_n})}{n} = a < \frac{\gamma}{d}$, for all $n \geq n_0$.

Let $b = \frac{\gamma-da}{8}$. Then

$$\frac{\Delta_r(\overline{H_n})}{n} < a + \frac{\gamma-da}{2d} = \frac{\gamma-4b}{d}, \text{ and } bn \geq 2d^2.$$

Claim 4.3. *For all s such that $2m/d - 2s/d - 2d - \Delta_r(\overline{H_n}) > 0$, and given $F_n \in \mathcal{R}(s)$ and $F'_n \in \mathcal{R}(s-1)$, we have*

$$2s(2m-2s-2d^2-d\Delta_r(\overline{H_n})) \leq N(F_n) \leq 4sm,$$
$$0 \leq N'(F'_n) \leq 2d^2 t_n.$$

Proof. Given $F_n \in \mathcal{R}(s)$, the number of ways to choose the red edge x and label its end vertices is $2s$. The number of ways to choose the blue edge y and label its vertices is at most $2m$. So $N(F_n) \leq 4sm$. For the lower bound, once x has been chosen and its vertices have been labeled, the number of ways to choose y and label its end vertices such that y is a blue edge that is not incident with x, and where $\{u,u'\}$ and $\{v,v'\}$ are both blue edges that do not lie in F_n, is at least $2m - 2s - 2d^2 - d\Delta_r(\overline{H_n})$. Indeed, having chosen the red edge $\{u,v\}$, the edge $\{u',v'\}$ must be chosen from the $m-s$ blue edges in F_n, which gives $2(m-s)$ ways of fixing u' and v'. However, we must also make sure that they are not chosen from any of the at most $\Delta_r(\overline{H_n})$ endpoints of the red edges leaving u or v, nor from the endpoints of the at most $2d$ edges in F_n leaving u or v (note that these restriction apply only to the edges if they are labeled in a particular way). Thus, there are at least $2m - 2s - d(\Delta_r(\overline{H_n}) + 2d)$ possible choices to ensure the right connection between u, v and the $\{u', v'\}$.

Given $F'_n \in \mathcal{R}(s-1)$, the number of ways to choose the red edge x and label its end vertices is at most $2t_n$. Moreover, there are at most d ways to choose u' and d ways to choose v', so that $N'(F'_n) \leq 2d^2 t_n$. $\square$

In our case, the assumption that $2m \geq \gamma n$ implies with $s \leq bn$ and $b = \frac{\gamma - da}{8}$ that

$$2m - 2s - 2d^2 - d\Delta_r(\overline{H_n}) \geq \gamma n - 2bn - bn - dn\left(\frac{\gamma - 4b}{d}\right) = bn.$$

Given $s \leq bn$, consider the auxiliary bipartite graph with bipartition $\mathcal{R}(s) \cup \mathcal{R}(s-1)$ where an element in $\mathcal{R}(s)$ is adjacent to an element in $\mathcal{R}(s-1)$ if they can be obtained from each other by switching operations. By counting the number of edges in this auxiliary graph, using Claim 4.3, we obtain

$$2sbn|\mathcal{R}(s)| \leq 2d^2 t_n |\mathcal{R}(s-1)|,$$

so that

$$\frac{|\mathcal{R}(s)|}{|\mathcal{R}(s-1)|} \leq \frac{d^2 t_n}{bsn},$$

and hence

$$\frac{|\mathcal{R}(s)|}{|\mathcal{R}(0)|} \leq \left(\frac{d^2 t_n}{bn}\right)^s \cdot \frac{1}{s!}.$$

We shall also make use of the following fact about the probability of F_n having many red edges.

Claim 4.4. $\mathbf{P}(X_n \geq bn) < 1/2$.

This claim, combined with $\sum_{s=0}^{m} \mathbf{P}(X_n = s) = 1$, leads to $\sum_{s=0}^{bn} \mathbf{P}(X_n = s) \geq 1/2$. Thus,

$$\begin{aligned}
\frac{1}{\mathbf{P}(X_n = 0)} &\leq 2 \cdot \sum_{s=0}^{bn} \frac{\mathbf{P}(X_n = s)}{\mathbf{P}(X_n = 0)} \\
&\leq 2 \cdot \sum_{s=0}^{bn} \frac{|\mathcal{R}(s)|}{|\mathcal{R}(0)|} \\
&\leq 2 \cdot \sum_{s=0}^{bn} \left(\frac{d^2 t_n}{bn}\right)^s \cdot \frac{1}{s!} \\
&\leq 2 \cdot \exp\left(\frac{d^2 t_n}{bn}\right). \qquad (38)
\end{aligned}$$

Note that this implies the validity of Lemma 4.1, as

$$\mathbf{P}(X_n = 0) \geq \frac{1}{2}\exp\left(-\frac{d^2 t_n}{bn}\right) = \frac{1}{2}\exp\left(\frac{-8d^2 t_n}{(\gamma - da)n}\right)$$

for large n. $\square$

Proof of Claim 4.4. There are at most t_n red edges in K_n, so there are at most $\binom{t_n}{bn}$ ways to choose bn red edges to include in F_n. We need to choose the $m - bn$ remaining edges to form F_n. To find an upper bound on the number of ways in which this can be done, we consider the following way of producing subgraphs of K_n with degree sequence $(d_1, \ldots, d_n)$, where $\sum_{i=1}^{n} d_i = 2m$: consider n bins labeled $1, \ldots, n$ and put (labeled) balls into these bins in such a way that the ith bin contains d_i balls. A *pairing* of the set of balls is given by $P = \{a_1, \ldots, a_m\}$ such that each a_i is an unordered pair of balls, and each ball is in precisely one pair a_i. Any such pairing gives rise to a (multi)graph with degree sequence $(d_1, \ldots, d_n)$ by thinking of the ith bin as a vertex v_i, and stating that v_i and v_j are adjacent whenever balls in these bins are paired to each other. The reason why we need to mention multigraphs is that, in a pairing, two balls in the same bin could be paired (which would produce a loop), or two or more balls in one bin could be paired to balls in some other bin (which would produce multiple edges). It is not difficult to see that each (simple) graph with degree sequence $(d_1, \ldots, d_n)$ corresponds to the same number of pairings (namely $d_1! \cdots d_n!$), which is the basic idea in the *configuration model* for random regular graphs (see [7, 16] for a detailed explanation.)

In our context, since we have already chosen bn red edges to include in F_n, we may assume that, when producing F_n using pairings, we have paired $2bn$ balls, so that we need to find a pairing of the $2m - 2bn$ remaining balls. As the number of ways of doing this is the number of perfect matchings in K_{2m-2bn}, we conclude that it may be done in at most

$$\frac{(2m-2bn)!}{2^{m-bn}(m-bn)!} = (1+o(1))\sqrt{2}\left(\frac{2m-2bn}{e}\right)^{m-bn}$$

ways. The approximation used Stirling's formula.

Consequently, the number of subgraphs of K_n with degree sequence $\mathbf{d}$ that contain at least bn red edges is for n large at most

$$2\binom{t_n}{bn}\left(\frac{2m-2bn}{e}\right)^{m-bn}.$$

We know by (10) that the number of such subgraphs with no restriction on the number of red edges satisfies

$$N_{K_n}(\mathbf{d}) \sim \frac{\sqrt{2}}{\prod_{i=1}^n d_i!}\left(\frac{2m}{e}\right)^m \exp(-\lambda - \lambda^2).$$

With $d_i \le d$, $i = 1, \ldots, n$ and $\lambda = (1/(2m))\sum_i^n \binom{d_i}{2} < d$, we infer that

$$N_{K_n}(\mathbf{d}) > \frac{\sqrt{2}}{(d!)^n}\left(\frac{2m}{e}\right)^m \exp(-d - d^2)$$

for sufficiently large n. Therefore, using $\binom{n}{k} \le (en/k)^k$, we have

$$\begin{aligned}
\mathbf{P}(X_n \ge bn) &\le \frac{2\binom{t_n}{bn}\left(\frac{2m-2bn}{e}\right)^{m-bn}}{N_{K_n}(\mathbf{d})} \\
&\le \frac{2\left(\frac{et_n}{bn}\right)^{bn} \cdot \left(\frac{2m-2bn}{e}\right)^{m-bn}}{\frac{\sqrt{2}}{(d!)^n}\left(\frac{2m}{e}\right)^m \exp(-d-d^2)} \\
&\le \frac{\sqrt{2}\left(\frac{et_n}{bn}\right)^{bn} \cdot (d!)^n \cdot \exp(d+d^2)}{\left(\frac{2m}{e}\right)^{bn}} \\
&= \sqrt{2}\left(\frac{e^2 t_n}{2bmn}\right)^{bn} \cdot (d!)^n \cdot \exp(d+d^2) \\
&< \frac{1}{2},
\end{aligned}$$

by (34), as required. □

5. Final remarks and open questions

In this paper, we have found the n-vertex graph that admits the largest number of r-edge-colorings with no rainbow-$S_{\ell_1,\ldots,\ell_t}$, for all $r \ge t \ge 3$ and all $\ell_1 \ge \cdots \ge \ell_t \ge 1$. In this section, we discuss extensions of our work and open questions motivated by it.

5.1. **The case $t = 2$.** Our strategy does not apply to the case $t = 2$, as the arguments rely on the fact that K_n admits a large number of colorings when only $t - 1$ colors are used. For $t = 2$, the number of colorings will depend almost entirely on the way in which rare colors are used. Restricting to the case of $S_{2,\ell}$, we have already observed that, for $\ell = 1$, the extremal value is achieved by a matching of size $\lfloor n/2 \rfloor$, as all components are monochromatic. For $\ell \geq 2$, we believe that K_n is again extremal.

For general configurations S_{ℓ_1,ℓ_2}, several situations may occur according to the value of the parameters. For instance, if $\ell_1 = 2$, $\ell_2 = 1$ and $r \in \{2, 3\}$, the only extremal graphs are vertex-disjoint collections of cycles (that is, (Turán) extremal graphs for the star S_3), so that the number of colorings is given by r^n. To see why this is true, note that cycles may be colored arbitrarily (and hence all components of an extremal graph contain a cycle) and that, for our choice of r, an edge incident with a colored cycle can either be colored in a single way, or cannot be colored at all.

On the other hand, for $r = 2$ and $\ell_2 = 1$ but larger values of ℓ_1, we can generate feasible colorings by avoiding a monochromatic S_{ℓ_1}. In [11], Kohayakawa and two of the current authors showed that, if we take $n/((4\ell_1 - o(1))\ell_1)$ vertex disjoint copies of complete bipartite graphs $K_{(2-o(1))\ell_1,(2-o(1))\ell_1}$ and we color their edges randomly, the probability that the coloring produced does not contain a monochromatic S_{ℓ_1} is large, in a way that the number of colorings is at least

$$\left((2 - o(1))^{4\ell_1^2}\right)^{n/(4\ell_1)} = 2^{(1-o(1))n\ell_1},$$

which is much larger than $2^{\mathrm{ex}(n,S_{\ell_1})} \leq 2^{n(\ell_1-1)/2}$. On the other hand, no vertex in the extremal graph should have degree larger than or equal to $2\ell_1 - 1$, as otherwise the set of edges incident with it has to be monochromatic. In particular, neither the Turán extremal graph nor the complete graph is extremal. (Similar considerations hold for any fixed $r \geq 2$). This suggests that finding the extremal configuration may be harder in this case. Finally, since we believe K_n to be optimal for $S_{2,\ell}$, it should also be optimal for $\ell_1 \geq \ell_2 \geq 2$.

5.2. **Applying our techniques to other forbidden configurations.** We believe that our proofs can be extended in a straightforward way to slightly more general situations. To describe one such case, suppose that, in addition to $r, t \geq 2$ and $\ell \geq 1$, we have a parameter $k \geq 2$, and we are looking for edge-colorings avoiding a set of vertices $\{v_1, \ldots, v_k\}$ with the following property. For each vertex v_i, there is a set C_i of colors such that $|C_i| = t$ and each color appears at least ℓ times in edges incident with v_i, with the additional property that $C_i \cap C_j = \emptyset$ for $i \neq j$. Note that our paper dealt with the case $k = 1$.

We claim that K_n would also be the single extremal graph in this case. To see why this is true, assume for simplicity that $\ell = 1$, and note that, to produce a large number of colorings of K_n, we could fix a set of $(kt - 1)$ colors and assign them arbitrarily to the edges of K_n, leading to at least $(kt-1)^{\binom{n}{2}}$ colorings. Also observe that, just as in (1), we could write the set of colorings of a graph G as $\mathcal{C}_{k,r,t}(G) = \bigcup_{(S_1,\ldots,S_n)} \mathcal{C}_{k,r,t,(S_1,\ldots,S_n)}(G)$, where $\mathcal{C}_{k,r,t,(S_1,\ldots,S_n)}(G)$ denotes colorings for which edges incident with v_i are assigned colors in S_i.

It is not hard to see that our auxiliary lemmas can be adapted to this context in a straightforward way when we consider colorings such that $|S_i| \leq kt - 1$ for all i. The difference here is that there is no a priori restriction on the size of a set S_i. To resolve this, we show that the number of colorings for which some S_i is large is negligible, as we did for colorings in $\mathcal{S}_A$ in Lemma 2.3. Indeed, assume that $|S_v| \geq kt$ for some vertex v and consider any set $W = \{w_1, \ldots, w_{k-1}\} \subseteq V - v$. Define an auxiliary bipartite graph whose vertices are given by the union of W and $[r]$, and such that a color is adjacent to w_i if and only if it lies in S_{w_i}. Applying Hall's Theorem to this auxiliary graph, we deduce that there is some nonempty subset $J \subset W$ with the property that $|\bigcup_{w \in J} S_w| < |J|t$; otherwise, we would find a forbidden configuration 'centered' at the vertices $v, w_1, \ldots, w_{k-1}$. In particular, $|S_w| \leq |J|t - 1 < kt - 1$ for all $w \in J$. Of course, this argument may be repeated until all but at most $k-2$ vertices satisfy $|S_w| < kt - 1$, which will lead to a relatively small number of colorings in a graph with many edges. We omit the details.

5.3. **Restricted edge-colorings.** In a more general direction, we could ask about rainbow patterns, or even general patterns, in other classes of graphs. In general, fix a positive integer r and a graph F, and let P be an arbitrary pattern of F. Let $\mathcal{C}_{r,F,P}(G)$ be the set of all (F,P)-free r-colorings of a graph G. We write

$$c_{r,F,P}(n) = \max \{ |\mathcal{C}_{r,F,P}(G)| \colon |V(G)| = n \},$$

and we say that an n-vertex graph G is $\mathcal{C}_{r,F,P}$-*extremal* if $|\mathcal{C}_{r,F,P}(G)| = c_{r,F,P}(n)$.

Using a multicolored version of the Szemerédi Regularity Lemma, we may prove that K_n is 'almost extremal' whenever F is bipartite and P contains at least three nonempty classes.

Theorem 5.1. [13] *Let F be a bipartite graph, let P be a pattern of F with $t \geq 3$ nonempty classes, and fix a positive integer $r \geq t$. Then we have $c_{r,F,P}(n) \leq (t-1)^{\binom{n}{2} + o(n^2)}$.*

In particular, the results in this paper provide instances for which K_n is extremal. Two of the authors [12] reached the same conclusion for matchings with at least three edges; however, they have found patterns where the number of classes is less than three for which the complete graph is not optimal. This naturally raises the following questions:

(1) Is there a pattern with at least three classes in a bipartite graph for which the complete graph is not optimal? If this is the case, is there a rainbow pattern satisfying this condition?

Considering rainbow patterns in graphs that are not bipartite, it is known that the complete graph is extremal for rainbow triangles if $r = 3$, see [6], and that, for every s, there is r_0 such that the Turán graph for K_s is 'almost extremal' for all fixed $r \geq r_0$ provided that n is sufficiently large. This leads to other natural questions:

(2) For patterns in complete graphs, or even for all non-bipartite graphs F, are there natural conditions that ensure the existence of r_0 such that the (Turán) F-extremal graph is $\mathcal{C}_{r,F,P}$-extremal for all $r \geq r_0$? Is there any graph F that admits an $\mathcal{C}_{r,F,P}$-extremal graph that is neither F-extremal nor complete, where P denotes the rainbow pattern?

5.4. **Robustness of graph properties.** Another direction for future work concerns extensions of Lemma 3.1, which deals with the *robustness* of some graph property. We have shown that, in any graph with 'many' edges, there is an 'almost spanning' subgraph with a 'large' number of subgraphs of *any* γ-dense bounded degree sequence, which shows that K_n is robust in terms of having a large number of subgraphs with any given γ-dense bounded degree sequence. A natural question would be to weaken the density assumption, and ask about degree sequences $\mathbf{d} = (d_1, \ldots, d_n)$ such that $\sum_i d_i = o(n)$. On the one hand, we do not see how our current proof could be modified to include this case. On the other hand, note that the obvious extension of Lemma 3.1 would be false if the degree sequences were allowed to be very sparse. For instance, if $\sum_{i=1}^{n} d_i = C$ for some constant $C \geq 2$ (assume that there are c nonzero entries in $\mathbf{d}$), an opponent might partition the vertex set of K_n into n/c sets V_i of size c and remove all edges between vertices with both endpoints in each V_i to produce an n-vertex graph H. Overall, she would remove $\binom{c}{2} n/c$ edges in the graph. On the other hand, regardless of the choice of W, whose size is constrained by the theorem, we have $V_i \subset W$ for some i, and hence the conclusion of the theorem will fail if all nonzero entries are assigned to the vertices corresponding to V_i, as there are no subgraphs with this degree sequence in

$H[V_i]$. Of course, the same type of counter-example might be used for $\sum_i d_i = f(n)$ if $f(n)$ grows slowly.

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