# EDGE COLORINGS OF GRAPHS AVOIDING MONOCHROMATIC MATCHINGS OF A GIVEN SIZE

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*Dedicated to the memory of Professor R. H. Schelp*

Abstract. Let $k$ and $\ell$ be positive integers. With a graph $G$, we associate the quantity $c_{k,\ell}(G)$, the number of $k$-colorings of the edge set of $G$ with no monochromatic matching of size $\ell$. Consider the function $c_{k,\ell}\colon \mathbb{N} \longrightarrow \mathbb{N}$ given by $c_{k,\ell}(n) = \max\{c_{k,\ell}(G)\colon |V(G)| = n\}$, the maximum of $c_{k,\ell}(G)$ over all graphs $G$ on $n$ vertices. In this paper, we determine $c_{k,\ell}(n)$ and the corresponding extremal graphs for all large $n$ and all fixed values of $k$ and $\ell$.

## 1. Introduction

Let $F$ be a fixed graph and $k$ be a positive integer. In this paper, we study *$F$-free $k$-colorings* of a graph $G$, that is, edge colorings of $G$ with $k$ colors such that there is no monochromatic copy of $F$. (Edge colorings in this work need not be proper.) More precisely, given a graph $G$, we consider the number $c_{k,F}(G)$ of $F$-free $k$-colorings of $G$, and we study the extremal function $c_{k,F}\colon \mathbb{N} \longrightarrow \mathbb{N}$, where $c_{k,F}(n)$ maximizes $c_{k,F}(G)$ over all graphs $G$ on $n$ vertices. In other words, $c_{k,F}(n) = \max\{c_{k,F}(G)\colon |V(G)| = n\}$. For instance, if there is a single color available, we must have $c_{k,F}(n) = 1$, with equality $c_{1,F}(n) = c_{1,F}(G)$ for every graph $G$ on $n$ vertices that does not contain a copy of $F$. A graph $G$ on $n$ vertices with $c_{k,F}(G) = c_{k,F}(n)$ is called *$(k,F)$-extremal.* The study of the function $c_{k,F}$ and of the set of $(k,F)$-extremal graphs has been motivated by a conjecture of Erdős and Rothschild [3] concerning edge colorings of graphs with a given number of colors and no monochromatic cliques of a given order.

The graphs $F$ forbidden in this paper are matchings $I_\ell$ consisting of $\ell$ independent edges, where $\ell \geq 2$. For convenience, we use the notation $c_{k,I_\ell}(\cdot) = c_{k,\ell}(\cdot)$ and we say that $(k,I_\ell)$-extremal graphs are simply *$(k,\ell)$-extremal.* To state the main result in this paper, we need a preliminary definition.

**Definition 1.1.** *The following concepts will be used throughout the paper.*

(a) *Given integers $c \geq 1$ and $n \geq c+2$, let $G_{n,c} = ([n], E_{n,c})$ be the graph on the vertex set $[n] = \{1,\ldots,n\}$ such that $\{i,j\} \in E_{n,c}$ if and only if $\min\{i,j\} \leq c$, i.e., the graph $G_{n,c}$ contains exactly $c$ vertices each of which is joined by an edge to every other vertex.*

(b) *Given integers $k,\ell \geq 2$, let $c(k,\ell)$ be the quantity defined by*

$$c(k,\ell) = \begin{cases} \ell - 1 & \text{if } k \in \{2,3\} \\ \lceil (\ell-1)k/3 \rceil & \text{if } k \geq 4. \end{cases} \tag{1}$$

**Theorem 1.2.** *Let $k,\ell \geq 2$ be fixed integers. There exists $n_0 = n_0(k,\ell)$ such that, for $n \geq n_0$, we have $c_{k,\ell}(n) = c_{k,\ell}(G_{n,c(k,\ell)})$. Moreover, for $n \geq n_0$, the graph $G_{n,c(k,\ell)}$ is the unique $(k,\ell)$-extremal graph up to isomorphism.*

When $\ell = 2$, this theorem is a very special case of a hypergraph result obtained by the current authors in [6]. Erdős and Gallai [4] have shown that, for $n$ sufficiently large, the

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The first author acknowledges the support by FAPERGS (Proc. 10/0388-2) and CNPq (Proc. 484154/2010-9). The second author was partially supported by CNPq (Proc. 308509/2007-2, 484154/2010-9). This work was partially supported by the U. of São Paulo, through the MaCLinC project.

graph $G_{n,\ell-1}$ is the *extremal graph* of the forbidden graph $I_\ell$, that is, $G_{n,\ell-1}$ is the graph on $n$ vertices with the largest number of edges that does not contain a matching $I_\ell$ as a subgraph. Note that for $k \in \{2,3\}$ we have $G_{n,c(k,\ell)} = G_{n,\ell-1}$. We should mention that, for general forbidden graphs $F$, the number $\mathrm{ex}(n,F)$ of edges in an $n$-vertex extremal graph for $F$ is a parameter that appears in natural upper and lower bounds on $c_{k,F}(n)$, namely

$$k^{\mathrm{ex}(n,F)} \leq c_{k,F}(n) \leq k^{k\,\mathrm{ex}(n,F)}. \tag{2}$$

To obtain the lower bound, note that an extremal graph $G$ on $n$ vertices for $F$ has $k^{\mathrm{ex}(n,F)}$ distinct $F$-free $k$-colorings, as all $k$-colorings of the set of edges of $G$ have this property. Concerning the upper bound, for any $k$-coloring of the set of edges of a graph on $n$ vertices with $k\,\mathrm{ex}(n,F)+1$ edges, at least one color class contains at least $\mathrm{ex}(n,F)+1$ edges, and hence contains a copy of $F$. Therefore the value of $c_{k,F}(n)$ is achieved by a graph with at most $k\,\mathrm{ex}(n,F)$ edges, from which we deduce that $c_{k,F}(n) \leq k^{k\,\mathrm{ex}(n,F)}$.

Theorem 1.2 implies that, for large $n$, the lower bound in (2) is tight for $k \in \{2,3\}$, whereas it is not tight for $k \geq 4$. This phenomenon replicates, for the forbidden graphs $I_\ell$, what has been observed in other classes of forbidden graphs $F$, such as complete graphs and odd cycles (see Yuster [13] and Alon, Balogh, Keevash and Sudakov [1], which rely on the Szemerédi Regularity Lemma [12]). In the case of complete graphs, this verifies the Erdős and Rothschild Conjecture for $k \in \{2,3\}$, but disproves it for $k \geq 4$. More generally, Lemma 2.1 in [1] implies that, given a graph $F$ and $k \in \{2,3\}$, we have

$$c_{k,F}(n) \leq k^{\mathrm{ex}(n,F)+o(n^2)}.$$

In other words, the extremal graph of $F$ is never "far" from being $(k,F)$-extremal when $k \in \{2,3\}$ and $F$ is not bipartite, as the Erdős-Stone Theorem [5] ensures that $\mathrm{ex}(n,F) = \Omega(n^2)$ in this case. The work in [1] also implies that $c_{k,F}(n) > k^{\mathrm{ex}(n,F)}$ if $F$ is not bipartite and $k \geq 4$. To the best of our knowledge, this is the first time that the same behavior is observed for a class of bipartite graphs.

Naturally, one might ask whether the following holds for every fixed graph $F$: given $k \in \{2,3\}$ and $\varepsilon > 0$, there exists $n_0$ such that

$$c_{k,F}(n) \leq k^{(1+\varepsilon)\,\mathrm{ex}(n,F)}$$

for every $n \geq n_0$. However, the current authors [7] have answered this question in the negative. As a matter of fact, the extremal graph is far from optimal for 3-colorings with forbidden paths with two edges (i.e., for proper 3-edge colorings). The same occurs for 2-colorings with forbidden stars on $t \geq 3$ leaves.

In the last few years, there has been substantial progress in the study of $c_{k,F}(n)$ and of closely-related graph and hypergraph functions. For instance, in the case of graphs, Pikhurko and Yilma [11] have determined, for $n$ sufficiently large, the families of graphs $G$ on $n$ vertices such that $c_{4,K_3}(G) = c_{4,K_3}(n)$ and such that $c_{4,K_4}(G) = c_{4,K_4}(n)$. Balogh [2] has considered another question, which extends the problem of finding $c_{k,F}(n)$. For a fixed graph $F$ and a *fixed edge coloring* of $F$, he wishes to determine the $n$-vertex graph with the largest number of edge colorings avoiding copies of $F$ with the prescribed coloring. He has shown that, for any 2-coloring of the edges of a complete graph $K_r$, where $r \geq 3$, maximality with respect to this problem is only achieved by the extremal graph for $K_r$ (namely the Turán graph $T_{r-1}(n)$), as long as $n$ is sufficiently large.

Concerning hypergraphs, recent results by Lefmann, Person, Rödl and Schacht [9] established the following: if $k \in \{2,3\}$, $F$ is the Fano plane and $n$ is sufficiently large, then the number of $k$-colorings of the hyperedges of any $n$-vertex 3-uniform hypergraph $H$ on $n$ vertices is at most $k^{\mathrm{ex}(n,F)}$, where, as usual, the quantity $\mathrm{ex}(n,F)$ denotes the maximum number of hyperedges in a 3-uniform $n$-vertex hypergraph containing no Fano plane. They have also shown that equality is attained by the unique extremal hypergraph for $\mathrm{ex}(n,F)$ and that, for fixed $k \geq 4$, the inequality $c_{k,F}(n) > k^{\mathrm{ex}(n,F)}$ holds as $n$ tends to infinity. Very recently, a similar phenomenon has been proved to hold in several other instances, such as 3- and 4-uniform generalized triangles, expanded complete 2-graphs and Fan($k$)-hypergraphs. For more information, see Lefmann and Person [8] and Lefmann, Person and Schacht [10].

This paper is organized as follows. Section 2 contains the proof of Theorem 1.2 when $k = 2$ colors are used. The structure of the proof of this result for $k \geq 3$ is delineated in Section 3, leaving the proofs of technical tools to the subsequent sections.

## 2. Proving Theorem 1.2 for $k = 2$

Our objective in this section is to determine the value of $c_{2,\ell}(n)$ for every fixed $\ell$ and every sufficiently large $n$. If $n \leq 2\ell - 1$, then no graph on $n$ vertices can contain a matching $I_\ell$ of size $\ell$, hence $c_{2,\ell}(n) = 2^{\binom{n}{2}}$ and the only graph to attain this extremal value is the complete graph $K_n$. The following result, due to Erdős and Gallai [4], fully describes the extremal graphs for $I_\ell$.

**Theorem 2.1.** *Given integers $n \geq 2\ell \geq 4$, we have*

$$\mathrm{ex}(n, I_\ell) = \max\left\{\binom{2\ell-1}{2}, (\ell-1)(n-\ell+1) + \binom{\ell-1}{2}\right\}.$$

*Equality can occur only if $G$ is the union of a complete graph on $2\ell-1$ vertices with $n-2\ell+1$ isolated vertices, or if $G$ has a vertex cover $C$ of size $\ell-1$ and contains all possible edges with at least one vertex in $C$.*

In our arguments we shall make use of the following easy result.

**Lemma 2.2.** *Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$, where $|A| = \ell$. If every vertex $v \in A$ has degree at least $\ell$, then $G$ contains a matching $I_\ell$.*

Next, we shall prove the following, which is essentially the statement of Theorem 1.2 in the case when $k = 2$.

**Lemma 2.3.** *For $\ell \geq 2$, there exists $n_0 = n_0(\ell)$ such that, for $n \geq n_0$, we have*

$$c_{2,\ell}(n) = 2^{\mathrm{ex}(n,I_\ell)} = 2^{(\ell-1)(n-\ell+1)+\binom{\ell-1}{2}}.$$

*In addition to this, if $n \geq n_0$, an $n$-vertex graph $G$ satisfies $c_{2,\ell}(G) = c_{2,\ell}(n)$ if and only if $G$ is isomorphic to the graph $G_{n,\ell-1}$ (see Definition 1.1). Moreover, if a graph $G$ on $n$ vertices contains a matching $I_\ell$, then $c_{2,\ell}(G) \ll c_{2,\ell}(n)$; in fact, $c_{2,\ell}(G)$ is exponentially smaller than $c_{2,\ell}(n)$.*

*Proof.* Let $\ell \geq 2$ be fixed. Throughout the proof, we shall assume that $n$ is sufficiently large for the bounds to hold, and no special attempt is made to optimize $n_0$. In particular, we assume that $n$ satisfies

$$\begin{aligned}&\frac{5}{2}\ell - 1 < n\\ &2^{\binom{4\ell-4}{2}+4\ell-4} \cdot \ell^2 \cdot n^{4(\ell-1)\ell} \ll 2^{n-\ell+1}.\end{aligned}$$

The first condition ensures that $G_{n,\ell-1}$ is, up to isomorphism, the unique extremal graph for $I_\ell$, while the second naturally arises in a counting argument.

It is clear that, for every value of $n$ and every $I_\ell$-free $n$-vertex graph $G = (V, E)$, we have $c_{2,\ell}(G) = 2^{|E|} \leq 2^{\mathrm{ex}(n,I_\ell)}$, as no 2-coloring of an $I_\ell$-free $n$-vertex graph could possibly contain a monochromatic $I_\ell$. Moreover, equality can be attained by extremal $I_\ell$-free $n$-vertex graphs, so that $c_{2,\ell}(n) \geq 2^{\mathrm{ex}(n,I_\ell)} = 2^{(\ell-1)(n-\ell+1)+\binom{\ell-1}{2}}$ for $n$ large enough, where the equality sign holds by Theorem 2.1 and our assumption on $n$.

Now, let $G$ be an $n$-vertex graph that admits $I_\ell$-free 2-colorings of its set of edges, but contains $I_\ell$ as a subgraph. Let $I_m$ be a maximum matching in $G$ given by the edges $e_1 = \{a_1, b_1\}, \ldots, e_m = \{a_m, b_m\}$, $m \geq \ell$. We shall see that the number of $I_\ell$-free 2-colorings of $G$ is exponentially smaller than $2^{\mathrm{ex}(n,I_\ell)}$. Clearly, we must have $m \leq 2(\ell-1)$, as otherwise the matching itself would not be 2-colorable. Moreover, the maximality of $I_m$ implies that the set $W$ of vertices uncovered by the matching forms an independent set in $G$, as any edge between vertices in $W$ could be used to increase $I_m$ to a matching $I_{m+1}$.

Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$. Let $\Delta$ be one of the at most $2^{\binom{2m}{2}}$ $I_\ell$-free 2-colorings of the set of edges with both endpoints in $A \cup B$. Assume that the colors used are red and green and w.l.o.g. suppose that, with respect to $\Delta$, the edges $e_1, \ldots, e_r$ are red, and the edges $e_{r+1}, \ldots, e_m$ are green. By assumption, we have $1 \leq r \leq \ell-1$ and $1 \leq g = m-r \leq \ell-1$.

Our objective is to bound the number of possible extensions of $\Delta$ to an $I_\ell$-free 2-coloring of $G$. To achieve this, we look at the structure of the coloring $\Delta$ more closely. Given a vertex $v \in V$, we classify its neighbors as red or green according to the color of the edge joining $v$ to each neighbor.

**Claim 2.4.** *Let $\Delta'$ be an $I_\ell$-free extension of the partial coloring $\Delta$ to $G$.*

(a) *The number of vertices in the set $\{a_{r+1}, b_{r+1}, \ldots, a_m, b_m\}$ with $\ell - r$ or more red neighbors in $W$ is bounded above by $\ell - r - 1$.*
(b) *The number of vertices in the set $\{a_1, b_1, \ldots, a_r, b_r\}$ with $\ell - g$ or more green neighbors in $W$ is bounded above by $\ell - g - 1$.*

*Proof.* For a contradiction, assume instead that there is a subset $S \subset \{a_{r+1}, b_{r+1}, \ldots, a_m, b_m\}$, $|S| \geq \ell - r$, such that every vertex in $S$ has at least $\ell - r$ red neighbors in $W$. By Lemma 2.2 there must be a matching of size $\ell - r$ in $\Delta'$ formed uniquely by red edges with one endpoint in $S$ and the other in $W$. Together with the edges $e_1, \ldots, e_r$, this yields a red matching $I_\ell$, which is not possible.

Analogously, the number of vertices in the set $\{a_1, b_1, \ldots, a_r, b_r\}$ with at least $\ell - (m - r) = \ell - g$ green neighbors in $W$ with respect to some extension of $\Delta$ is bounded above by $\ell - (m-r) - 1 = \ell - g - 1$. This asserts the validity of the above claim. □

So, to extend $\Delta$ to an $I_\ell$-free 2-coloring of $G$, one may first choose $j_1$ $(0 \leq j_1 \leq \ell - g - 1)$ vertices in $\{a_1, b_1, \ldots, a_r, b_r\}$ to have at least $\ell - g$ green neighbors in $W$, and $j_2$ $(0 \leq j_2 \leq \ell - r - 1)$ vertices in $\{a_{r+1}, b_{r+1}, \ldots, a_m, b_m\}$ to have at least $\ell - r$ red neighbors in $W$. Once these vertices are chosen, there are at most $2^{(j_1+j_2)(n-2m)}$ ways of coloring the set of edges joining them to $W$. The set of edges between each of the $2m - j_1 - j_2$ remaining vertices in $\{a_1, b_1, \ldots, a_m, b_m\}$ and $W$ may be colored in at most

$$\sum_{i=0}^{\max\{\ell-r-1,\ell-g-1\}} \binom{n-2m}{i}$$

ways. Hence, the total number of extensions of $\Delta$ to an $I_\ell$-free 2-coloring of $G$ is bounded above by

$$\begin{aligned}
&\sum_{j_1=0}^{\ell-g-1} \sum_{j_2=0}^{\ell-r-1} \binom{2r}{j_1} \binom{2m-2r}{j_2} 2^{(j_1+j_2)(n-2m)} \left( \sum_{i=0}^{\max\{\ell-r-1,\ell-g-1\}} \binom{n-2m}{i} \right)^{2m-j_1-j_2} \\
\leq\ &\sum_{j_1=0}^{\ell-g-1} \sum_{j_2=0}^{\ell-r-1} \binom{2r}{j_1} \binom{2m-2r}{j_2} n^{\ell(2m-j_1-j_2)} 2^{(j_1+j_2)(n-2m)}
\end{aligned}$$

$$
\begin{aligned}
&< \ell^2 \cdot 2^{2m} \cdot n^{2m\ell} \cdot 2^{(2\ell-r-g-2)(n-2m)} \\
&= \ell^2 \cdot 2^{2m} \cdot n^{2m\ell} \cdot 2^{(2\ell-m-2)(n-2m)}.
\end{aligned}
$$

Summing over all of the at most $2^{\binom{2m}{2}}$ possible choices of $\Delta$ and observing that $\ell \leq m \leq 2(\ell-1)$, we conclude that, for $n$ sufficiently large, $c_{2,\ell}(G)$ is bounded above by

$$
\begin{aligned}
& 2^{\binom{2m}{2}} \cdot \ell^2 \cdot 2^{2m} \cdot n^{2m\ell} \cdot 2^{(2\ell-m-2)(n-2m)} \\
\leq\ & 2^{\binom{4\ell-4}{2}} \cdot 2^{4\ell-4} \cdot \ell^2 \cdot n^{2m\ell} \cdot 2^{(2\ell-m-2)(n-2m)} \\
<\ & 2^{\binom{4\ell-4}{2}+4\ell-4} \cdot \ell^2 \cdot n^{4(\ell-1)\ell} \cdot 2^{(\ell-2)(n-\ell+1)} \qquad (3)\\
\ll\ & 2^{(\ell-1)(n-\ell+1)} < 2^{\mathrm{ex}(n,I_\ell)}.
\end{aligned}
$$

Note that (3) is exponentially smaller than $2^{\mathrm{ex}(n,I_\ell)}$, which finishes the proof. □

## 3. Proof of Theorem 1.2 for $k \geq 3$

In this section, we describe the proof of Theorem 1.2 in the case $k \geq 3$. For convenience, we use the term $(k,\ell)$-*coloring* of a graph $G$ to mean an $I_\ell$-free $k$-coloring of the set of edges of $G$. A graph $G$ is $(k,\ell)$-*colorable* if it admits a $(k,\ell)$-coloring.

We frequently use the following lemma, whose simple proof can be found in [6].

**Lemma 3.1.** *Let $m \geq 2$ be an integer. All optimal solutions $s = (s_1, \ldots, s_c)$ to the maximization problem*

$$
\begin{array}{l}
\max \prod_{i=1}^{c} s_c \\
c, s_1, \ldots, s_c \text{ positive integers,} \\
s_1 + \cdots + s_c \leq m,
\end{array} \qquad (4)
$$

*have the following form.*

(a) *If $m \equiv 0 \pmod 3$, then $c = m/3$ and all the components of $s$ are equal to 3.*
(b) *If $m \equiv 1 \pmod 3$, then either $c = \lceil m/3 \rceil$, with exactly two components equal to 2 and all remaining components equal to 3, or $c = \lfloor m/3 \rfloor$, with exactly one component equal to 4 and all remaining components equal to 3.*
(c) *If $m \equiv 2 \pmod 3$, then $c = \lceil m/3 \rceil$ with exactly one component equal to 2 and all remaining components equal to 3.*

*As a consequence, the optimal value of (4) is $3^{m/3}$ if $m \equiv 0 \pmod 3$, and $4 \cdot 3^{\lfloor m/3 \rfloor - 1}$ if $m \equiv 1 \pmod 3$, and $2 \cdot 3^{\lfloor m/3 \rfloor}$ if $m \equiv 2 \pmod 3$.*

Given the number $k$ of colors and the forbidden matching $I_\ell$, the following function appears often in the remainder of this paper.

**Definition 3.2.** *Given positive integers $k$ and $\ell$, let $D(k,\ell)$ be defined by*

$$
D(k,\ell) = \begin{cases} 3^{(\ell-1)k/3} & \text{if } (\ell-1)k \equiv 0 \pmod 3 \\ 4 \cdot 3^{\lfloor (\ell-1)k/3 \rfloor - 1} & \text{if } (\ell-1)k \equiv 1 \pmod 3 \\ 2 \cdot 3^{\lfloor (\ell-1)k/3 \rfloor} & \text{if } (\ell-1)k \equiv 2 \pmod 3. \end{cases}
$$

Let $0 < \gamma = \gamma(k,\ell) < 1/3$ be such that, whenever the nonzero components of an integral vector $s = (s_1, \ldots, s_c)$ with $\sum_{i=1}^{c} s_i \leq (\ell-1)k$ and $s_i \leq k$ for every $i$ do not form an optimal solution to (4) with $m = (\ell-1)k$, we have

$$
\prod_{i=1, s_i \neq 0}^{c} s_i < D(k,\ell)^{1-3\gamma}. \qquad (5)
$$

For the proof of Theorem 1.2, we shall now establish the following result.

**Lemma 3.3.** *Let $k \geq 3$ and $\ell \geq 2$ be integers. Then there exists an integer $n_0$ such that, if $n \geq n_0$, we have $c_{k,\ell}(n) = c_{k,\ell}(G_{n,c(k,\ell)})$. Moreover, the following assertions hold:*

(i) *the graph* $G_{n,c(k,\ell)}$ *is, up to isomorphism, the unique* $(k,\ell)$*-extremal graph;*
(ii) *in particular, for* $k=3$*, we have*

$$c_{3,\ell}(n) = 3^{\mathrm{ex}(n,I_\ell)} = 3^{(\ell-1)(n-\ell-1)+\binom{\ell-1}{2}};$$

(iii) *for* $(\ell-1)k \not\equiv 1 \pmod 3$*, if a graph* $G$ *on* $n$ *vertices has a minimum vertex cover of size* $c \neq c(k,\ell)$*, then* $c_{k,\ell}(G)$ *is exponentially smaller than* $c_{k,\ell}(n)$*.*

*Proof.* Let $k \geq 3$ and $\ell \geq 2$ be fixed. We first prove the result for $(\ell-1)k \not\equiv 1 \pmod 3$. With foresight, choose $n_0$ sufficiently large so that, for every $n \geq n_0$,

$$\begin{aligned} &k^{k(\ell-1)} D(k,\ell)^{n(1-\gamma)} < D(k,\ell)^{n-c(k,\ell)},\\ &k^{\binom{2(\ell-1)k}{2}} < D(k,\ell)^{\gamma n}, \text{ and}\\ &(2\ell k)^{(\ell-1)k} \cdot e^{2\ell^2 k^2(\ln n+\ln k)} < D(k,\ell)^{\gamma n}. \end{aligned} \tag{6}$$

The constant $\gamma = \gamma(k,\ell)$ in these inequalities satisfies the condition stated in (5).

Let $G=(V,E)$ be a $(k,\ell)$-colorable graph on $n \geq n_0$ vertices. The proof is structured in terms of a minimum vertex cover $C = \{v_1,\ldots,v_c\}$ of $G$. The first step is to obtain an upper bound on the size $c=|C|$ that does not depend on $n$.

**Lemma 3.4.** *If* $G=(V,E)$ *is a* $(k,\ell)$*-colorable graph, then* $G$ *has a vertex cover of size at most* $2k(\ell-1)$*.*

*Proof.* Let $F$ be a maximum set of pairwise independent edges in $G$ and let $\Delta$ be a $(k,\ell)$-coloring of $G$. Since there are at most $k$ color classes in $\Delta$ and the intersection of $F$ with each color class has size at most $\ell-1$, we conclude that $|F| \leq (\ell-1)k$. Due to the maximality of $F$, we deduce that the endpoints of the edges in $F$ form a vertex cover of $G$, and our result follows. □

The second step of the proof is to show that, if $G$ has the largest number of $(k,\ell)$-colorings over all $n$-vertex graphs, then $G$ must have a minimum vertex cover of size $c(k,\ell)$.

**Lemma 3.5.** *For* $(\ell-1)k \not\equiv 1 \pmod 3$*, if* $n \geq n_0$ *and an* $n$*-vertex graph* $G=(V,E)$ *is* $(k,\ell)$*-extremal, then the minimum size of a vertex cover in* $G$ *is equal to* $c(k,\ell)$*.*

To conclude the proof, we show that $G_{n,c(k,\ell)}$ is the unique $(k,\ell)$-extremal graph over all graphs on $n$ vertices with minimum vertex cover of size $c(k,\ell)$.

**Lemma 3.6.** *For* $(\ell-1)k \not\equiv 1 \pmod 3$*, up to isomorphism, the unique maximum of* $c_{k,\ell}(G)$ *over all graphs* $G$ *with vertex set of size* $n \geq n_0$ *and minimum vertex cover of size* $c(k,\ell)$ *is achieved by the graph* $G_{n,c(k,\ell)}$*.*

The proofs of Lemmas 3.5 and 3.6 are given in Sections 4 and 5, respectively.

We now comment on the case $(\ell-1)k \equiv 1 \pmod 3$. The main difference is that, in this case, there are two families of optimal solutions to (4) with $m=(\ell-1)k$. Because of this, the arguments used in the proof of Lemma 3.5 only show that, for large $n$, the minimum size of the vertex cover in a $(k,\ell)$-extremal graph $G$, i.e., a graph for which $c_{k,\ell}(G)=c_{k,\ell}(n)$, lies in the set $\{c(k,\ell)-1, c(k,\ell)\}$. We may then adapt the arguments in the proof of Lemma 3.6 to demonstrate that the $(k,\ell)$-extremal graph is, up to isomorphism, either $G_{n,c(k,\ell)}$ or $G_{n,c(k,\ell)-1}$. To conclude the proof, we then show that the latter has fewer $(k,\ell)$-colorings than $G_{n,c(k,\ell)}$. Details regarding these steps may be found in Section 6. □

## 4. Proof of Lemma 3.5

We start this section with a lower bound on the number of $(k,\ell)$-colorings of the graph $G_{n,c(k,\ell)}$. Let $C=\{v_1,\ldots,v_{c(k,\ell)}\}$ be a minimum vertex cover of this graph, consider an optimal solution $s=(s_1,\ldots,s_{c(k,\ell)})$ to the optimization problem (4), and fix a way of distributing

$k(\ell-1)$ balls, with $(\ell-1)$ balls having each of the $k$ colors, into $c(k,\ell)$ distinct bins in such a way that the $i$-th bin receives $s_i$ balls, all of which have different colors. Assuming that the colors are indexed by $1,\ldots,k$, we consider the set $S$ of all edge colorings of $G_{n,c(k,\ell)}$ with the following properties: (i) if $e=\{v_i,v_j\}$, $1\leq i<j\leq c(k,\ell)$, then $e$ is colored with the color of smallest index among all colors in bin $i$ or $j$; (ii) if $e=\{v_i,w\}$, where $w\notin C$, then $e$ may be colored with any of the colors in bin $i$. It is easy to see that every coloring in $S$ is a $(k,\ell)$-coloring of $G_{n,c(k,\ell)}$. Hence we have

$$c_{k,\ell}(G_{n,c(k,\ell)}) \geq |S| = \prod_{i=1}^{c(k,\ell)} s_i^{n-c(k,\ell)} = D(k,\ell)^{n-c(k,\ell)}. \tag{7}$$

We shall see that graphs whose minimum vertex covers do not have size $c(k,\ell)$ have fewer $(k,\ell)$-colorings than $D(k,\ell)^{n-c(k,\ell)}$ when $n\geq n_0$. To this end, let $G=(V,E)$ be a $(k,\ell)$-colorable graph on $n$ vertices with minimum vertex cover $C=\{v_1,\ldots,v_c\}$. Let $F$ be the set of all edges of $G$ with both endpoints in $C$ and consider the graph $G'$ obtained from $G$ by the removal of $F$. Note that $G'$ is a bipartite graph with bipartition $(C,V\setminus C)$.

Consider a $(k,\ell)$-coloring $\Delta$ of $G$. For a vertex $v_i\in C$ and a color $\sigma\in\{1,\ldots,k\}$, we say that $\sigma$ is *substantial* for $v_i$ if at least $\ell$ edges incident with $v_i$ in $G'$ have color $\sigma$. In other words, a color is substantial for vertex $v_i$ if it appears "many" times among the edges that are incident with $v_i$, but not with any other cover element Moreover, we say that vertex $v_i$ is *$s$-influential* if there are precisely $s$ colors that are substantial for $v_i$. Finally, a color $\sigma$ is said to be *influential* if it is substantial for exactly $\ell-1$ vertices in $C$. Observe that influential colors are "maximally substantial", since a color cannot be substantial for more than $\ell-1$ vertices due to Lemma 2.2.

Given $j\in\{0,\ldots,(\ell-1)k\}$, let $\mathcal{I}_j$ be the set of all non-negative integral solutions to the equation $s_1+\cdots+s_c=j$ such that $s_i\leq k$ for every $i$. For any such vector $s=(s_1,\ldots,s_c)$, let $\Delta_s(G')$ be the set of all $(k,\ell)$-colorings of $G'$ for which vertex $v_i$ is $s_i$-influential for every $i\in\{1,\ldots,c\}$. As the sets $\Delta_s(G')$ partition the set of all $(k,\ell)$-colorings of $G'$, we have

$$c_{k,\ell}(G') = \sum_{j=0}^{(\ell-1)k}\sum_{s\in\mathcal{I}_j}\left|\Delta_s(G')\right|. \tag{8}$$

Clearly, every $(k,\ell)$-coloring of $G$ is the combination of a $(k,\ell)$-coloring of $G'$ with a coloring of the edges of $F$ with at most $k$ colors. We know that there are at most

$$k^{|F|}\leq k^{\binom{c}{2}} \tag{9}$$

colorings of the latter type, thus equation (8) becomes

$$c_{k,\ell}(G)\leq k^{\binom{c}{2}}\sum_{j=0}^{(\ell-1)k}\sum_{s\in\mathcal{I}_j}\left|\Delta_s(G')\right|. \tag{10}$$

We now concentrate on $(k,\ell)$-colorings of $G'$.

**Lemma 4.1.** *If a color $\sigma$ is influential with respect to a $(k,\ell)$-coloring $\Delta$, then every edge $e$ with color $\sigma$ is incident with a vertex for which it is substantial.*

*Proof.* Suppose that $\sigma$ is influential with respect to $\Delta$ and let $v_{i_1},\ldots,v_{i_{\ell-1}}$ be vertices in the vertex cover $C$ for which $\sigma$ is substantial. Let $e=\{x,v_j\}$ be an edge in $G'$, where $v_j\in C\setminus\{v_{i_1},\ldots,v_{i_{\ell-1}}\}$ and $x\notin C$, and assume that $e$ has color $\sigma$. Consider the bipartite subgraph $B$ of $G'$ induced by $U=\{v_{i_1},\ldots,v_{i_{\ell-1}}\}$ and by the set $W$ of all vertices $y\neq x$ that are adjacent to a vertex in $U$ through an edge with color $\sigma$. The degree of every vertex in $U$ is at least $\ell-1$ in $B$, hence $B$ contains a matching $M$ of size $\ell-1$ by Lemma 2.2. Thus

$M \cup \{e\}$ is also a matching, and every edge in this matching has color $\sigma$, contradicting the assumption that $\Delta$ is a $(k,\ell)$-coloring of $G$. $\square$

Lemma 4.1 will be used to bound from above the number of colorings in $\Delta_s(G')$. Fix a vector $s = (s_1, \ldots, s_c)$ with $s_1 + \cdots + s_c = j$, where $s_i \leq k$ for every $i$. The number of ways to choose the colors that are substantial, with their respective multiplicity, is bounded above by $\binom{(\ell-1)k}{j}$, which is the number of ways of choosing $j$ different colors from a set of $(\ell-1)k$ different colors. Once the colors are chosen, they may be distributed in at most $\frac{j!}{s_1!s_2!\cdots s_c!}$ ways among the cover vertices, where the upper bound again pretends that all the colors chosen are distinct. Now, given that the colors have been distributed, the edges in $G'$ that are incident with vertex $v_i$ may be colored in at most

$$\sum_{(a_1,\ldots,a_{k-s_i})} \left( \prod_{t=1}^{k-s_i} \binom{n-c}{a_t} \right) s_i^{n-c-\sum_{t=1}^{k-s_i} a_t} \leq \sum_{(a_1,\ldots,a_{k-s_i})} \left( \prod_{t=1}^{k-s_i} \binom{n-c}{a_t} \right) s_i^{n-c}$$

ways, if $s_i \geq 1$, where the sum is such that each $a_t$ ranges from 0 to $\ell-1$. This is because there are at most $n-c$ edges in $G'$ that are incident with $v_i$, from which $a_t$, $0 \leq a_t \leq \ell-1$, may be chosen for every color that is not substantial for vertex $v_i$. All remaining edges incident with vertex $v_i$ may be colored with any of the $s_i$ colors that are substantial for $v_i$. Note that

$$\begin{aligned}
\sum_{(a_1,\ldots,a_{k-s_i})} \left( \prod_{t=1}^{k-s_i} \binom{n-c}{a_t} \right) s_i^{n-c} &\leq \sum_{(a_1,\ldots,a_{k-s_i})} \left( \prod_{t=1}^{k-s_i} (n-c)^{a_t} \right) s_i^{n-c} \\
= \left( \sum_{(a_1,\ldots,a_{k-s_i})} (n-c)^{\sum_{t=1}^{k-s_i} a_t} \right) s_i^{n-c} &= \left( \sum_{p=0}^{\ell-1} (n-c)^p \right)^{k-s_i} s_i^{n-c} \\
= \left( \frac{(n-c)^\ell - 1}{n-c-1} \right)^{k-s_i} s_i^{n-c} &\leq n^{\ell k} \cdot s_i^{n-c}. \qquad (11)
\end{aligned}$$

If $s_i = 0$, then there are at most $(\ell-1)k$ edges incident with vertex $v_i$ in $G'$, which may be colored with at most $k$ colors in at most

$$k^{(\ell-1)k} \qquad (12)$$

ways.

Combining inequalities (11) and (12), and observing that $c$ is an upper bound on the number of vanishing components in a vector $s = (s_1, \ldots, s_c)$, we obtain

$$\begin{aligned}
\sum_{s \in \mathcal{I}_j} \left| \Delta_s(G') \right| &\leq \binom{(\ell-1)k}{j} \cdot n^{c\ell k} \cdot k^{c(\ell-1)k} \cdot \sum_{s \in \mathcal{I}_j} \frac{j!}{s_1!s_2!\cdots s_c!} \prod_{i=1, s_i \neq 0}^{c} s_i^{n-c} \\
&\leq \binom{(\ell-1)k}{j} \cdot e^{c\ell k(\ln n + \ln k)} \cdot \sum_{s \in \mathcal{I}_j} \frac{j!}{s_1!s_2!\cdots s_c!} \prod_{i=1, s_i \neq 0}^{c} s_i^{n-c}. \qquad (13)
\end{aligned}$$

Observe that, for our fixed value of $k$, the product $\prod_{i=1, s_i \neq 0}^{c} s_i^{n-c}$ is maximized when the nonzero components of $s$ are the components of a vector in the set $\mathcal{S}(k,\ell)$ of optimal solutions to (4) with $m = k(\ell-1)$, which is described in Lemma 3.1. Recall that the number $D(k,\ell)$ given in the statement of Definition 3.2 is precisely the optimal value of (4), and, whenever the nonzero components of the integral vector $s = (s_1, \ldots, s_c)$ are not an optimal solution to

(4), we have

$$\prod_{i=1, s_i \neq 0}^{c} s_i < D(k,\ell)^{1-3\gamma}. \tag{14}$$

**Lemma 4.2.** *Let $k$ and $\ell$ be positive integers and fix $n_0$ as in (6). For $n \geq n_0$, let $G$ be an $n$-vertex graph with minimum vertex cover of cardinality $c \leq 2(\ell-1)k$. Then*

$$k^{\binom{c}{2}} \sum_{j=0}^{k(\ell-1)} \sum_{s \in \mathcal{I}_j \setminus \mathcal{S}(k,\ell)} \left|\Delta_s(G')\right| \leq D(k,\ell)^{n(1-\gamma)}.$$

*In particular, if $\Delta_s(G') = \emptyset$ for every $s = (s_1, \ldots, s_c)$ whose nonzero components are the components of a vector in the set $\mathcal{S}(k,\ell)$ of optimal solutions to (4), then*

$$c_{k,\ell}(G') \leq D(k,\ell)^{n(1-\gamma)}.$$

*Proof.* Let $G$ be an $n$-vertex graph with a minimum vertex cover $C$ of cardinality $c \leq 2k(\ell-1) < 2k\ell$. Each edge with both endpoints in the vertex cover $C$ can be colored with at most $k$ colors. The inequalities (13) and (14) imply with (6) that

$$\begin{aligned}
& k^{\binom{c}{2}} \cdot \sum_{j=0}^{(\ell-1)k} \sum_{s \in \mathcal{I}_j \setminus \mathcal{S}(k,\ell)} \left|\Delta_s(G')\right| \\
\leq\ & k^{\binom{2(\ell-1)k}{2}} \cdot \sum_{j=0}^{(\ell-1)k} \binom{(\ell-1)k}{j} \cdot e^{c\ell k(\ln n + \ln k)} \cdot \sum_{s \in \mathcal{I}_j} \frac{j!}{s_1! s_2! \cdots s_c!} D(k,\ell)^{(n-c)(1-3\gamma)} \\
=\ & k^{\binom{2(\ell-1)k}{2}} \cdot e^{c\ell k(\ln n + \ln k)} \cdot D(k,\ell)^{(n-c)(1-3\gamma)} \cdot \sum_{j=0}^{(\ell-1)k} \binom{(\ell-1)k}{j} \cdot \sum_{s \in \mathcal{I}_j} \frac{j!}{s_1! s_2! \cdots s_c!} \\
\leq\ & D(k,\ell)^{\gamma n} \cdot e^{c\ell k(\ln n + \ln k)} \cdot D(k,\ell)^{(n-c)(1-3\gamma)} \cdot \sum_{j=0}^{(\ell-1)k} \binom{(\ell-1)k}{j} c^j \\
\leq\ & e^{2\ell^2 k^2 (\ln n + \ln k)} \cdot D(k,\ell)^{n(1-2\gamma)} \cdot (1+c)^{(\ell-1)k} \\
\leq\ & (2\ell k)^{(\ell-1)k} \cdot e^{2\ell^2 k^2 (\ln n + \ln k)} \cdot D(k,\ell)^{n(1-2\gamma)} \\
\leq\ & D(k,\ell)^{\gamma n} \cdot D(k,\ell)^{n(1-2\gamma)} = D(k,\ell)^{n(1-\gamma)},
\end{aligned} \tag{15}$$

as required.

When $\Delta_s(G') = \emptyset$ for every $s = (s_1, \ldots, s_c)$ whose nonzero components are the components of a vector in $\mathcal{S}(k,\ell)$, the inequality $c_{k,\ell}(G) \leq D(k,\ell)^{n(1-\gamma)}$ is an immediate consequence of the above. □

In light of Lemma 4.2, to conclude the proof of Lemma 3.5, it suffices to show the following.

**Claim 4.3.** *If $c \neq c(k,\ell)$, then $\Delta_s(G') = \emptyset$ for every $s = (s_1, \ldots, s_c)$ whose nonzero components are the components of a vector in $\mathcal{S}(k,\ell)$.*

Indeed, if this claim is true, then by Lemma 4.2 we have

$$c_{k,\ell}(G) \leq D(k,\ell)^{n(1-\gamma)} \ll D(k,\ell)^{n-c(k,\ell)} \leq c_{k,\ell}(G_{n,c(k,\ell)}),$$

which implies the statement of Lemma 3.5.

*Proof.* Claim 4.3 is certainly true for $c < c(k,\ell)$, as there are no vectors of length $c$ whose nonzero components give a solution in $\mathcal{S}(k,\ell)$. To prove the claim for $c > c(k,\ell)$, we note that Lemma 4.1 implies that, if $\Delta$ is a $(k,\ell)$-coloring of $G$ for which every color is influential, then, for every vertex $v_i$ in the minimum vertex cover $C$, there is a color that is substantial for $v_i$. Indeed, if we suppose for a contradiction that this is not true, then Lemma 4.1 tells us that there is no color available for edges incident with vertex $v_i$ in $G'$; however, such edges must exist by the minimality of $C$.

With this, we conclude that $\Delta_s(G') = \emptyset$ for every $s = (s_1, \ldots, s_c)$ with $s_1 + \cdots + s_c = (\ell-1)k$ but $s_j = 0$ for some $j$, which in turn proves our result. □

## 5. Proof of Lemma 3.6

By Lemma 3.5, for $n \geq n_0$, an $n$-vertex graph $G = (V,E)$ is $(k,\ell)$-extremal only if the size of a minimum vertex cover of $G$ is $c(k,\ell)$. Thus let $G = (V,E)$ be an $n$-vertex graph with minimum vertex cover $C = \{v_1, \ldots, v_{c(k,\ell)}\}$ for which there are vertices $v_i \in C$ and $x \neq v_i$ such that $\{v_i, x\} \notin E$. We shall show that the graph $G$ is not $(k,\ell)$-extremal. This is done in two parts. First, we prove that every $(k,\ell)$-coloring of $G$ can be extended to a $(k,\ell)$-coloring of $G \cup \{v_i, x\}$. To conclude the proof, we show that at least one of the $(k,\ell)$-colorings of $G$ can be extended in more than one way, and hence $c_{k,\ell}(G) < c_{k,\ell}(G \cup \{v_i, x\})$.

We first consider the case when vertex $v_i$ covers at most $(\ell-1)k$ edges not covered by any other element of $C$. Consider the graph $G'$ obtained from $G$ by the removal of these edges. On the one hand, since $G'$ has a vertex cover of size smaller than $c(k,\ell)$, the number of $(k,\ell)$-colorings of $G'$ is bounded above by $D(k,\ell)^{n(1-\gamma)}$. On the other hand, every $(k,\ell)$-coloring of $G$ consists of the union of a $(k,\ell)$-coloring of $G'$ and a $k$-coloring of the edges removed from $G$, so that there are at most

$$k^{k(\ell-1)} D(k,\ell)^{n(1-\gamma)} \tag{16}$$

such colorings, which is smaller than $c_{k,\ell}(G_{n,c(k,\ell)}) \geq D(k,\ell)^{n-c(k,\ell)}$ by our choice of $n_0$ in (6).

Now, suppose that vertex $v_i$ covers more than $(\ell-1)k$ edges not covered by any other element of $C$. Then, by the pigeonhole principle, for every $(k,\ell)$-coloring $\Delta$ of $G$, there is a color $\sigma$ that is substantial for vertex $v_i$. If $\Delta$ induces a matching $M$ of size $\ell-1$ with color $\sigma$, then, as $\sigma$ is substantial for $v_i$, the vertex $v_i$ must be an endpoint of some edge in $M$. In other words, the coloring $\Delta$ can be extended to a $(k,\ell)$-coloring of $G \cup \{v_i, x\}$ by assigning the color $\sigma$ to $\{v_i, x\}$.

To conclude the proof, we observe that there exists a coloring $\Delta$ of $G$ using $\lceil c(k,\ell)/(\ell-1) \rceil$ colors, namely the one that assigns the first color to all edges incident with the first $\ell-1$ cover elements, then assigns the second color to all uncolored edges incident with the next $\ell-1$ cover elements, and so on. However, note that

$$\left\lceil \frac{c(k,\ell)}{\ell-1} \right\rceil \leq \left\lceil \frac{((\ell-1)k+2)/3}{\ell-1} \right\rceil \leq \frac{((\ell-1)k+2)/3}{\ell-1} + \frac{\ell-2}{\ell-1} < k$$

if $k \geq 2$. Hence the coloring $\Delta$ can be extended to a coloring of $G \cup \{v_i, x\}$ in at least two ways, either by coloring $\{v_i, x\}$ with a color that is substantial for vertex $v_i$ or by coloring $\{v_i, x\}$ with a color that is not used by $\Delta$.

## 6. The case $(\ell-1)k \equiv 1 \pmod 3$

To finish the proof of Theorem 1.2, we still need to address the case $(\ell-1)k \equiv 1 \pmod 3$. The peculiarity of this case lies in the fact that there are two types of optimal solutions to the optimization problem (4). As we have discussed in Section 3, this implies that the arguments used in the proof of Lemma 3.5 only allow one to show that, for large $n$, the minimum size of the vertex cover in a $(k,\ell)$-extremal graph $G$, i.e., a graph for which $c_{k,\ell}(G) = c_{k,\ell}(n)$, lies in the set $\{c(k,\ell)-1, c(k,\ell)\}$. With this, it is not hard to adapt the arguments in the proof of Lemma 3.6 to show that the $(k,\ell)$-extremal graph lies in the set $\{G_{n,c(k,\ell)-1}, G_{n,c(k,\ell)}\}$. We then show that maximality is attained by $G_{n,c(k,\ell)}$.

**Lemma 6.1.** *For $(\ell-1)k \equiv 1 \pmod 3$, up to isomorphism, the maximum of $c_{k,\ell}(G)$ over all graphs $G$ with vertex set of size $n \geq n_0$ and minimum vertex cover of size either $c(k,\ell)-1$ or $c(k,\ell)$ is achieved by $G_{n,c(k,\ell)-1}$ or $G_{n,c(k,\ell)}$.*

*Proof.* Let $G$ be a graph with minimum vertex cover of size $c(k,\ell)-1$ and assume that $G \neq G_{n,c(k,\ell)-1}$. The fact that $c_{k,\ell}(G) < c_{k,\ell}(G_{n,c(k,\ell)-1})$ can be derived directly from the proof of Lemma 3.6, unless $k=4$ and $\ell=2$, in which case $c(k,\ell)-1=1$ and the conclusion is straightforward.

We now show that $c_{k,\ell}(G) < c_{k,\ell}(G_{n,c(k,\ell)})$ for every graph $G \neq G_{n,c(k,\ell)}$ with minimum vertex cover of size $c(k,\ell)$. Let $C = \{v_1, \ldots, v_{c(k,\ell)}\}$ be a minimum vertex cover of $G$ and assume that there are vertices $v_i \in C$ and $x \neq v_i$ such that $\{v_i, x\} \notin E$. We shall show that the graph $G$ is not $(k,\ell)$-extremal. Let $m$ be the number of edges whose only endpoint in $C$ is $v_i$ (the minimality of $C$ implies that $m \geq 1$). As in the proof of Lemma 3.6, we consider two cases, according to whether $m \leq (\ell-1)k$ or $m > (\ell-1)k$. In the latter case, we may again extend colorings of $G$ to colorings of $G \cup \{v_i, x\}$, and the result follows with the arguments used in the case $(\ell-1)k \not\equiv 1 \pmod 3$.

For $m \leq (\ell-1)k$, let $G'$ be obtained from $G$ by the deletion of all edges whose single endpoint in $C$ is $v_i$. In this case, we need to be more careful than in the proof of Lemma 3.6, as the upper bound (16) does not hold directly for the number of $(k,\ell)$-colorings in $G'$, since, when $(\ell-1)k \equiv 1 \pmod 3$, there is an optimal solution to (4) of length $c(k,\ell)-1$. Let $\mathcal{C}_1$ be the family of $(k,\ell)$-colorings of $G$ for which there is a color that is substantial for $v_i$, and let $\mathcal{C}_2$ contain the remaining $(k,\ell)$-colorings of $G$, so that $c_{k,\ell}(G) = |\mathcal{C}_1| + |\mathcal{C}_2|$. We shall bound the number of $(k,\ell)$-colorings in each of these two families separately.

Let $\Delta$ be a coloring in $\mathcal{C}_1$, and consider the coloring $\Delta'$ of $G'$ obtained by ignoring the colors of the deleted edges. Let $\mathcal{C}_1'$ be the set of all $(k,\ell)$-colorings of $G'$ obtained in this way. If $\sigma$ is substantial for $v_i$ with respect to $\Delta$, then it can be substantial for at most $\ell-2$ other vertices in $C$, and hence $\sigma$ cannot be influential with respect to $\Delta'$. By Lemma 4.2 applied to $G'$, we have

$$|\mathcal{C}_1'| \leq D(k,\ell)^{n(1-\gamma)}.$$

Moreover, each coloring of $\mathcal{C}_1'$ corresponds to at most $k^{k(\ell-1)}-1$ colorings of $\mathcal{C}_1$, since there is at least one coloring of the deleted edges for which no color is substantial (namely one in which there are at most $\ell-1$ edges with each of the $k$ colors). In particular, we have

$$|\mathcal{C}_1| \leq \left(k^{k(\ell-1)}-1\right)|\mathcal{C}_1'| \leq \left(k^{k(\ell-1)}-1\right) D(k,\ell)^{n(1-\gamma)}.$$

Now, let $\Delta$ be a coloring in $\mathcal{C}_2$ and let $\sigma$ be the color of an edge whose single endpoint in the vertex cover is $v_i$. Since $\sigma$ is not substantial for $v_i$, Lemma 4.1 implies that $\sigma$ cannot be influential with respect to $\Delta$, so that, by applying Lemma 4.2 to $G$, we obtain

$$|\mathcal{C}_2| \leq D(k,\ell)^{n(1-\gamma)}.$$

As a consequence, we have

$$c_{k,\ell}(G) = |\mathcal{C}_1| + |\mathcal{C}_2| \leq k^{k(\ell-1)} D(k,\ell)^{n(1-\gamma)} < c_{k,\ell}(G_{n,c(k,\ell)}),$$

as required.

□

To conclude the analysis of the case $(\ell-1)k \equiv 1 \pmod 3$, we show that maximality is attained by $G_{n,c(k,\ell)}$.

**Lemma 6.2.** *For $(\ell-1)k \equiv 1 \pmod 3$ and large $n$, we have $c_{k,\ell}(G_{n,c(k,\ell)}) > c_{k,\ell}(G_{n,c(k,\ell)-1})$.*

*Proof.* For simplicity, we use the notation $c = c(k,\ell)$. Let $C^* = \{v_1, \ldots, v_{c-1}\}$ be the minimum vertex cover of $G^* = G_{n,c-1}$ and let $v_c$ be an additional vertex in this graph. We may view $G = G_{n,c}$ as the graph obtained by the addition of all edges $\{v_c, w\}$ to $G^*$ for every vertex $w \notin C^*$.

Let $\mathcal{S}_1 = \mathcal{S}_1(k,\ell)$ and $\mathcal{S}_2 = \mathcal{S}_2(k,\ell)$ denote the sets of optimal solutions to the optimization problem (4) with one component equal to 4 and with two components equal to 2, respectively. From equation (8), combined with Lemma 4.2, we deduce that

$$\sum_{s\in\mathcal{S}_2} |\Delta_s(G)| \leq c_{k,\ell}(G) \leq \sum_{s\in\mathcal{S}_2} |\Delta_s(G)| + D(k,\ell)^{n(1-\gamma)}$$

$$\sum_{s^*\in\mathcal{S}_1} |\Delta_{s^*}(G^*)| \leq c_{k,\ell}(G^*) \leq \sum_{s^*\in\mathcal{S}_1} |\Delta_{s^*}(G^*)| + D(k,\ell)^{n(1-\gamma)} .$$

Because of equation (7), we know that, for large $n$, the two inequalities are dominated by the sums. Hence, in order to establish our result, it suffices to show that

$$\sum_{s\in\mathcal{S}_2} |\Delta_s(G)| \geq 2 \sum_{s^*\in\mathcal{S}_1} |\Delta_{s^*}(G^*)| \tag{17}$$

for $n$ sufficiently large.

Note that, for every vector $s^*$ in $\mathcal{S}_1$ and every coloring in $\Delta_{s^*}(G^*)$, there exists exactly one vertex $v_i \in C^*$ with the property that precisely four colors are substantial for $v_i$. Moreover, these are the only colors that appear in the edges for which $v_i$ is the single endpoint in $C^*$. This will be used to define an injective map $\phi$ of the colorings of $\Delta_{s^*}(G^*)$ into $\Delta_s(G)$, where $s$ is the vector in $\mathcal{S}_2$ whose components equal to 2 are precisely $s_i$ and $s_c$.

To define the mapping $\phi$, suppose that the $k$ colors are ordered from 1 to $k$, fix a coloring $\Delta^* \in \Delta_{s^*}(G^*)$ with $s_i^* = 4$, and let $\sigma_1 < \sigma_2 < \sigma_3 < \sigma_4$ be the substantial colors in $v_i$. The image of $\Delta^*$ under $\phi$ is a coloring $\Delta \in \Delta_s(G)$ with $\sigma_1, \sigma_2$ substantial for $v_i$ and $\sigma_3, \sigma_4$ substantial for $v_c$. The colors associated with all the edges that are not incident with $v_i$ or $v_c$, or the edges that join vertices in $C = C^* \cup \{v_c\}$ remain the same. An edge $\{v_i, w\}$, $w \notin C$ receives color $\sigma_1$ if it has color $\sigma_1$ or $\sigma_3$ with respect to $\Delta^*$, otherwise it is assigned $\sigma_2$. On the other hand, an edge $\{v_c, w\}$, $w \notin C$ receives color $\sigma_3$ if $\{v_i, w\}$ has color $\sigma_2$ or $\sigma_3$ with respect to $\Delta^*$, otherwise it is assigned color $\sigma_4$. It is easy to see that $\Delta$ has no monochromatic $I_\ell$.

The crucial fact about this construction is that the color of $\{v_i, w\}$ that was originally assigned by $\Delta^*$ is uniquely determined by the colors of $\{v_i, w\}$ and $\{v_c, w\}$ in $\Delta = \phi(\Delta^*)$. The injectivity of $\phi$ is an easy consequence of this fact, and hence

$$\sum_{s^*\in\mathcal{S}_1} |\Delta_{s^*}(G^*)| \leq \sum_{s\in\mathcal{S}_2} |\Delta_s(G)| .$$

Now, note that we may interchange the roles of the colors in the above mapping, that is, the colors $\sigma_1, \sigma_2$ could be assigned to $v_c$, while $\sigma_3, \sigma_4$ would be assigned to $v_i$. Since all the colors are substantial for $v_i$ with respect to the colorings in $\Delta_{s^*}(G^*)$, the colorings created with this new mapping are all distinct from the colorings created through the original mapping. By the argument above we get

$$2 \sum_{s^*\in\mathcal{S}_1} |\Delta_{s^*}(G^*)| \leq \sum_{s\in\mathcal{S}_2} |\Delta_s(G)| ,$$

which establishes (17) and thus concludes the proof. $\square$

*Acknowledgements:* The authors are indebted to an anonymous referee for valuable comments and suggestions. The authors are also grateful to NUMEC/USP, Núcleo de Modelagem Estocástica e Complexidade of the University of São Paulo, for its hospitality.

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