Edge colorings of graphs avoiding some fixed monochromatic subgraph with linear Turán number

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Abstract

Let F be a graph and k be a positive integer. With a graph G, we associate the quantity $c_{k,F}(G)$, the number of k-colorings of the edge set of G with no monochromatic copy of F. Consider the function $c_{k,F}\colon \mathbb{N} \longrightarrow \mathbb{N}$ given by $c_{k,F}(n) = \max\{c_{k,F}(G)\colon |V(G)| = n\}$, the maximum of $c_{k,F}(G)$ over all graphs G on n vertices. In this paper we study the asymptotic behavior of $c_{k,F}$ and describe the extremal graphs for the case in which F is a matching, a short path or a star.

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1 Introduction

Let F be a fixed graph and k be a positive integer. In this paper, we study *F-free k-colorings* of a graph G, that is, edge colorings of G with k colors such that there is no monochromatic copy of F. More precisely, given a graph G, we consider the number $c_{k,F}(G)$ of F-free k-colorings of G, and we study the extremal function $c_{k,F}\colon \mathbb{N} \longrightarrow \mathbb{N}$, where $c_{k,F}(n)$ maximizes $c_{k,F}(G)$ over all graphs G on n vertices; formally, $c_{k,F}(n) = \max\{c_{k,F}(G)\colon |V(G)| = n\}$. For instance, if there is a single color available, we have $c_{1,F}(n) = 1$, with equality $c_{1,F}(n) = c_{1,F}(G)$ for every graph G on n vertices that does not contain a copy of F. A graph G on n vertices with $c_{k,F}(G) = c_{k,F}(n)$ is called *extremal*.

The function $c_{k,F}(n)$ is related with the *Turán number* $\mathrm{ex}(n,F)$ of F, i.e., the maximum number of edges in a graph on n vertices that does not contain a subgraph isomorphic to F. Indeed, from this definition, it is easy to see that the inequalities

$$k^{\mathrm{ex}(n,F)} \leq c_{k,F}(n) \leq k^{k\cdot \mathrm{ex}(n,F)}$$

hold for every n. To obtain the lower bound, one may choose a *Turán graph* G for F, that is, a graph on n vertices with $\mathrm{ex}(n,F)$ edges and no copy of F. Since any k-coloring of the edges of G is trivially F-free, there are $k^{\mathrm{ex}(n,F)}$ such colorings for G. To obtain the upper bound, observe that for any k-coloring of the set of edges of a graph on n vertices with $(k\,\mathrm{ex}(n,F)+1)$ edges, at least one color class contains at least $(\mathrm{ex}(n,F)+1)$ edges, and hence contains a copy of F. Therefore the value of $c_{k,F}(n)$ is achieved by a graph with at most $k\,\mathrm{ex}(n,F)$ edges, from which we deduce that $c_{k,F}(n) \leq k^{k\cdot \mathrm{ex}(n,F)}$.

The following precise result, which was motivated by a conjecture of Erdős and Rothschild [2], has been obtained by Yuster [7] in the case of triangle-free 2-colorings (i.e., in the case $k = 2$ and $\ell = 3$), and has later been extended to general K_ℓ-free k-colorings by Alon, Balogh, Keevash, and Sudakov [1]. Here, and in the remainder of this work, we say that $f(n) \ggg g(n)$ if $\lim_{n\to\infty} f(n)/g(n) \geq \exp(cn^2)$ for some positive constant c, that is, $f(n)$ is greater than $g(n)$ by an exponential function of n^2.

Theorem 1.1 *Let $\ell \geq 3$ be fixed. For n sufficiently large and $k \in \{2,3\}$, we have*

$$c_{k,K_\ell}(n) = k^{\mathrm{ex}(n,K_\ell)} = k^{(\ell-2)n^2/(2(\ell-1))+o(n^2)},$$

with $c_{k,K_\ell}(G) = c_{k,K_\ell}(n)$ only if G is the n-vertex Turán graph for K_ℓ, while, for $k \geq 4$, we have $c_{k,K_\ell}(n) \ggg k^{\mathrm{ex}(n,K_\ell)}$.

In the case of $k = 4$ colors, Pikhurko and Yilma [6] have determined the families of graphs G on n vertices such that $c_{4,K_3}(G) = c_{4,K_3}(n)$ and such that

$c_{4,K_4}(G) = c_{4,K_4}(n)$, where n is sufficiently large. The analogue of Theorem 1.1 has been obtained for other classes of forbidden monochromatic subgraphs F: the case of edge color-critical graphs has been settled in [1] for $k \in \{2, 3\}$, implying for instance the case of odd cycles.

A common feature of the above results is the following: when the number of colors is either two or three, the corresponding Turán graph is eventually extremal. On the other hand, the Turán graph is suboptimal by an exponential function of n^2 when at least four colors are used. Actually, given a graph F, Lemma 2.1 in Alon *et al.* [1] may be used to establish that, for $k \in \{2, 3\}$, we have $c_{k,F}(n) \leq k^{\mathrm{ex}(n,F)+o(n^2)}$. In particular, when the number k of colors is two or three and the forbidden graph F is not bipartite (and hence $\mathrm{ex}(n, F) = \Omega(n^2)$), this implies that G is never too far from being extremal in the sense that, given $\varepsilon > 0$, there exists n_0 such that, for $n \geq n_0$, we have

$$c_{k,F}(n) \leq k^{(1+\varepsilon)\,\mathrm{ex}(n,F)}.$$

Again by [1], we have $c_{k,F}(n) \ggg k^{\mathrm{ex}(n,F)}$ if F is not bipartite and $k \geq 4$.

In light of this, it is natural to ask if a similar behavior holds for bipartite graphs. Here, we study the function $c_{k,F}(n)$ for some forbidden bipartite graphs F with linear Turán number. It turns out that, in sharp contrast to the non-bipartite case, the relationship between this function and the Turán number $\mathrm{ex}(n, F)$ depends heavily on the graph F. For instance, when F is a matching I_ℓ consisting of $\ell \geq 2$ independent edges, we observe the same phenomenon of the non-bipartite case: the corresponding Turán graph is extremal if the number of colors is $k \in \{2, 3\}$, while it is not extremal for $k \geq 4$. Now, if F is a path on three or four vertices, Turán graphs may be extremal for $k = 2$ colors, but they are not extremal if the number of colors is $k \geq 3$. Finally, if F is a star $S_t = K_{1,t}$ with t leaves, then the Turán graph associated with it is not extremal even when 2-colorings are considered. This behavioral variety suggests that it might be difficult to obtain general properties of $c_{k,F}(n)$ and its extremal graphs when F is bipartite.

2 Forbidden matchings

In this section, we show that, when F is a matching I_ℓ of $\ell \geq 2$ edges, the behavior of the function $c_{k,F}(n)$ resembles the non-bipartite case, that is, Turán graphs are extremal when $k \in \{2, 3\}$, but are not extremal otherwise. Furthermore, we determine the extremal graphs for all values of k and ℓ if n is sufficiently large. To state these results, we need a preliminary definition.

Definition 2.1 (a) Given integers $c \geq 1$ and $n \geq c$, let $G_{n,c} = ([n], E_{n,c})$ be the graph on the vertex set $[n] = \{1, \ldots, n\}$ such that $\{i, j\} \in E_{n,c}$ if and only if $\min\{i, j\} \leq c$.
(b) Given integers $k, \ell \geq 2$, let $c(k, \ell) = \ell - 1$ if $k \in \{2, 3\}$, and $c(k, \ell) = \lceil (\ell - 1)k/3 \rceil$ if $k \geq 4$.

Theorem 2.2 **[5]** *Let $k, \ell \geq 2$ be fixed integers. There exists $n_0 = n_0(k, \ell)$ such that, for $n \geq n_0$, we have $c_{k,I_\ell}(n) = c_{k,I_\ell}(G_{n,c(k,\ell)})$. Moreover, the graph $G_{n,c(k,\ell)}$ is the unique extremal graph up to isomorphism.*

When $\ell = 2$, this theorem is a special case of the results obtained by the current authors in [4]. Erdős and Gallai [3] have shown that, if $(\ell-1)(n-\ell+1) + \binom{\ell-1}{2} > \binom{2\ell-1}{2}$, the graph $G_{n,c(k,\ell)} = G_{n,\ell-1}$ is the Turán graph for I_ℓ, so that, for large n, the Turán graph is indeed extremal if and only if $k \in \{2, 3\}$.

3 Forbidden paths and stars

We now address colorings with forbidden monochromatic paths and stars. Given $\ell \geq 2$, let P_ℓ be the path on ℓ vertices and $S_\ell = K_{1,\ell}$ be the star with ℓ edges. As the colorings of G avoiding monochromatic copies of P_3 are precisely the proper k-edge-colorings of G, in particular we study the largest number of proper k-edge-colorings over all graphs on n vertices.

We shall see that, for F being a path or a star, the behavior of $c_{k,F}(n)$ is different from the non-bipartite case. The Turán graphs on n vertices associated with forbidding each of these graphs have been obtained by Erdős and Gallai [3] in the case of paths, and in the case of stars they seem to be folklore. In both cases, the number of edges is asymptotically linear in n.

If F is a path on three vertices, we are able to show that its Turán graph is extremal for 2-colorings, i.e., $c_{2,P_2}(n) = 2^{\lfloor n/2 \rfloor}$ for all $n \geq 2$. However, for 3-colorings, we prove the following, which determines $c_{3,P_3}(n)$ and the extremal graphs for all values of n.

Theorem 3.1 *Let $n \geq 2$ be a positive integer. For $n = 2$ we have $c_{3,P_3}(2) = 3$, and $c_{3,P_3}(G) = c_{3,P_3}(2)$ holds only for $G = K_2$. For $n = 3$ we have $c_{3,P_3}(3) = 6$, and $c_{3,P_3}(G) = c_{3,P_3}(3)$ is only achieved by $G = K_3$ or $G = P_3$. For $n \geq 4$, the function $c_{3,P_3}(n)$ is given by*

$$c_{3,P_3}(n) = \begin{cases} 18^{n/4} & \textit{if } n \equiv 0 \bmod 4, \\ 30 \cdot 18^{(n-5)/4} & \textit{if } n \equiv 1 \bmod 4, \\ 66 \cdot 18^{(n-6)/4} & \textit{if } n \equiv 2 \bmod 4, \\ 126 \cdot 18^{(n-7)/4} & \textit{if } n \equiv 3 \bmod 4. \end{cases}$$

There is a single graph G on $n \geq 4$ vertices with $c_{3,P_3}(G) = c_{3,P_3}(n)$. For $4 \leq n \leq 7$ it is a cycle on n vertices. For $n = 4n'+i$, $n' \geq 2$ and $i \in \{0,1,2,3\}$, the graph G consists of n' vertex-disjoint cycles, $(n'-1)$ of which on four vertices and one of which on $(4+i)$ vertices.

For even values of n, the Turán graph for P_3 is a matching $I_{n/2}$, which admits $3^{n/2}$ distinct proper 3-edge-colorings. For n divisible by 4, however, the above result states that the number of proper 3-edge-colorings in a vertex-disjoint collection of 4-cycles is $18^{n/4} \geq 2.05^n$, which is much larger than $3^{n/2} \leq 1.74^n$. A similar exponential gap can be verified for all values of n in the case $k = 4$, and, more generally, for every $k \geq 4$.

With regard to forbidden paths P_4 on four vertices, we obtain the following.

Theorem 3.2 *Let G be an n-vertex graph. For $n \equiv 0 \bmod 3$, it holds that $c_{2,P_4}(G) \leq 2^n$. If either $n \not\equiv 0 \bmod 3$ or G is not isomorphic to the Turán graph for P_4, then $c_{2,P_4}(G) \leq 31 \cdot 2^{n-5}$.*

As proved in [3], if $n \equiv 0 \bmod 3$, the n-vertex Turán graph for P_4 is unique and consists of $n/3$ pairwise vertex-disjoint triangles. On the other hand, the Turán graph is not unique for $n \not\equiv 0 \bmod 3$, where it is given by every combination of $g \leq \lfloor n/3 \rfloor$ pairwise vertex-disjoint triangles with a star on the remaining $(n-3g)$ vertices. In particular, the number of P_4-free 2-colorings of the Turán graph is equal to 2^n if $n \equiv 0 \bmod 3$ and to 2^{n-1} if $n \not\equiv 0 \bmod 3$. Hence Theorem 3.2 implies that, in the case $n \equiv 0 \bmod 3$, the Turán graph for P_4 is the unique extremal graph. For $n \not\equiv 0 \bmod 3$, if G is an n-vertex Turán graph for P_4, we deduce from Theorem 3.2 that the multiplicative gap between $c_{2,P_4}(G)$ and $c_{2,P_4}(n)$ is smaller than 2. However, if $n = 4$, a triangle with one attached edge admits $10 > 2^{4-1} = 8$ distinct P_4-free 2-colorings. For $n = 5$, two triangles sharing exactly one vertex admit $18 > 2^{5-1} = 16$ distinct P_4-free 2-colorings. This shows that, at least for a few small values of $n \not\equiv 0 \bmod 3$, we have $c_{2,P_4}(n) > 2^{n-1}$, and the Turán graph is not extremal.

Moreover, Turán graphs are never extremal for $k \geq 3$, since the following lower bound holds for every $k \geq 3$ and $n \geq 6$:

$$c_{k,P_4}(n) > (k(k-1))^{n-2} > k^n.$$

So far nothing is known for forbidden paths P_ℓ with $\ell \geq 5$.

Finally, when the forbidden graph is a star S_t with $t \geq 3$ rays, we obtain upper and lower bounds on $c_{k,S_t}(n)$ for every $k \geq 2$. In this case, the corresponding Turán graph fails to be extremal even when $k = 2$. Indeed, note that, if a graph G admits a k-coloring of its edges with no monochromatic star S_t, its maximum degree is at most $k(t-1)$. Now, for every integer d in the range $1 \leq d \leq k(t-1)$, let $\chi_{k,t}(d)$ be the number of k-colorings of a star S_d with no monochromatic S_t. Let $\chi_{k,t} = \max_{0 \leq d \leq (t-1)k} \chi_{k,t}(d)$ be the maximum number of S_t-free k-colorings of S_d, which is achieved by $d = (t-1)k$. We are able to show, with entropy arguments, that $c_{k,S_t}(n) \leq (\chi_{k,t})^{n/2}$. Moreover, we may estimate the number of S_t-free 2-colorings of the complete bipartite graph $K_{t,t}$ to prove that $c_{2,S_t}(K_{t,t}) \geq 2^{t^2-2}$. For n divisible by $2t$, if we consider $n/(2t)$ vertex-disjoint copies of $K_{t,t}$, this implies that $c_{2,S_t}(n) \geq 2^{\frac{t^2-2}{2t}n}$. For $t \geq 3$, this number is exponentially larger than $2^{(t-1)n/2}$. Note that the latter is precisely the number of S_t-free 2-colorings of an n-vertex $(t-1)$-regular graph, which is a Turán graph for S_t. However, the extremal graphs for S_t are not known even for small values of $t \geq 3$.

References

[1] N. Alon, J. Balogh, P. Keevash, and B. Sudakov, *The number of edge colorings with no monochromatic cliques*, J. London Math. Soc. (2) 70, 2004, 273–288.

[2] P. Erdős, *Some new applications of probability methods to combinatorial analysis and graph theory*, Proc. of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1974), 39–51. Congressus Numerantium, No. X, Utilitas Math., 1974, 39–51.

[3] P. Erdös and T. Gallai, *On maximal paths and circuits of graphs*, Acta Math. Acad. Sci. Hungar. 10, 1959, 337–356.

[4] C. Hoppen, Y. Kohayakawa, and H. Lefmann, *Hypergraphs with many Kneser colorings (Extended version)*, arXiv:1102.5543v1, 39 pp.

[5] C. Hoppen, Y. Kohayakawa, and H. Lefmann, *Edge colorings of graphs avoiding monochromatic matchings of a given size*, preprint.

[6] O. Pikhurko, and Z. B. Yilma, *The maximum number of K_3-free and K_4-free edge 4-colorings*, to appear in J. London Math. Soc., 24 pp.

[7] R. Yuster, *The number of edge colorings with no monochromatic triangle*, J. Graph Theory 21, 1996, 441–452.