ON GRAPHS WITH A LARGE NUMBER OF EDGE-COLORINGS AVOIDING A RAINBOW TRIANGLE

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Abstract. Inspired by previous work of Balogh [A remark on the number of edge colorings of graphs, European Journal of Combinatorics 27(4) (2006), 565–573], we show that, given $r \geq 5$ and n large, the balanced complete bipartite graph $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ is the n-vertex graph that admits the largest number of r-edge-colorings for which there is no triangle whose edges are assigned three distinct colors. Moreover, we extend this result to lower values of n when $r \geq 10$, and we provide approximate results for $r \in \{3, 4\}$.

1. Introduction

There have been considerable advances in edge-coloring problems whose origin may be traced back to a question of Erdős and Rothschild [4], who asked for n-vertex (simple) graphs that admit the largest number of r-*edge-colorings* such that *every color class is* F-*free*, where r is a positive integer and F is a fixed graph. (We remark that edge-colorings in this work are not necessarily proper.) In particular, when F is the complete graph K_{k+1} on $k+1$ vertices, they conjectured that the number of F-free 2-colorings is maximized by the n-vertex graph with the largest number of edges avoiding K_{k+1} as a subgraph. This is the k-partite Turán graph $T_k(n)$ on n-vertices, namely the balanced, complete k-partite n-vertex graph [15]. As usual, given a positive integer n and a graph F, we shall write $\mathrm{ex}(n, F)$ for the largest number of edges in an n-vertex graph that does not contain F as a subgraph.

In other words, Erdős and Rothschild intuited that, even though n-vertex graphs containing copies of K_{k+1} may have more than $\mathrm{ex}(n, K_{k+1})$ edges and may still contain a large number of colorings such that no copy of K_{k+1} is colored monochromatically, the number of such colorings is surpassed by the number $2^{\mathrm{ex}(n,K_{k+1})}$ of colorings of $T_k(n)$. Yuster [16] proved this conjecture for $F = K_3$ and $n \geq 6$ and Alon, Balogh, Keevash and Sudakov [1] proved it for $k \geq 3$ provided that n is sufficiently large. They also showed that, for large n, the graph $T_k(n)$ still admits the largest number of r-edge colorings avoiding K_{k+1} if $r = 3$, but that this is not the case for any $r \geq 4$, where the 'competition' between configurations that may be colored arbitrarily and configurations with more edges, but with restrictions on the way edges may be colored, tilts toward the latter. Determining the extremal configurations for $k \geq 2$ and $r \geq 4$ turned out to be a difficult problem, and so far the solution is only known for $r = 4$ and $k \in \{2, 3\}$ (see Pikhurko and Yilma [12]).

In this paper, we consider a colored version of this problem: given a number r of colors and a graph F, we define an r-*pattern* P of F as a partition of its edge set into at most r classes, and an edge-coloring of a graph G is said to contain (F, P) if G contains a copy of F in which the partition of the edge set induced by the coloring is isomorphic to P. Naturally, we say that an edge-coloring of G is (F, P)-*free* if it does not contain (F, P). Note that we forbid monochromatic colorings if P consists of a single class.

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Let $\mathcal{C}_{r,F,P}(G)$ be the set of (F,P)-free r-edge colorings of a graph G. For fixed r, F and P, the problem is to find the n-vertex graph (or the family of n-vertex graphs) that admits the largest number of (F,P)-free edge-colorings, which we call *$\mathcal{C}_{r,F,P}$-extremal graphs.* The work of Balogh [2] was the first to treat patterns that are not monochromatic. His results imply that, for $r=2$ colors and any pattern P of K_{k+1} that uses both colors, the graph $T_k(n)$ once again yields the largest number of 2-colorings avoiding (K_{k+1},P) for $n \geq n_0$. However, he also remarked that, if we consider $r=3$ and a triangle where all colors appear (which we call a *multicolored* or a *rainbow* triangle), the complete graph on n vertices already admits $3 \cdot 2^{\binom{n}{2}} - 3$ colorings, by just choosing two of the three colors and coloring the edges of K_n arbitrarily with these two colors. This is more than $3^{n^2/4}$, which is an upper bound on the number of 3-colorings of the bipartite Turán graph.

The current authors, along with co-authors, have considered the monochromatic version of this problem for some families of bipartite graphs F, such as matchings [6], paths and stars [7], and the problem for general patterns for matchings [8], stars [10] and complete graphs [9]. We refer the reader to [7] and [9] for more precise accounts of the state of the art for monochromatic patterns and for the general case, respectively.

In this paper, we deal with (F,P)-free edge-colorings of graphs in the particular case where $F=K_3$ and P is the rainbow pattern. For simplicity, we shall refer to this pattern as K_3^R. We prove that, for $r \geq 5$, the Turán graph $T_2(n) = K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$ is the unique extremal graph when n is sufficiently large.

Theorem 1.1. *Let $r \geq 5$ be a positive integer. There is n_0 such that every graph of order $n > n_0$ has at most $r^{\mathrm{ex}(n,K_3)}$ distinct K_3^R-free r-edge colorings. Moreover, the Turán graph $T_2(n)$ is the only graph on n vertices for which equality is achieved.*

This result is an improvement on [9, Theorem 1.2], which reached the same conclusion for $r \geq 3^{20}$. Our proof of Theorem 1.1 refines the argument used to prove [9, Theorem 1.2], which in turn was based on a strategy developed in [1] for monochromatic patterns (and applied to general patterns in [2]). Since it relies on the Regularity Lemma of Szemerédi [14], the conclusion of Theorem 1.1 holds for very large values of n_0. With different arguments, we were able to prove the above result for $n_0 = 5$ provided that $r \geq 10$. We remark that Yuster's result [16] for 2-edge colorings avoiding monochromatic triangles has also been obtained for all $n \geq 6$. To the best of our knowledge, this is the only other known result for this problem that holds for such a small value of n_0.

Theorem 1.2. *For all $n \geq 5$ and $r \geq 10$, the maximum number of K_3^R-free r-edge colorings of an n-vertex graph is $r^{\mathrm{ex}(n,K_3)}$, and this number is achieved only by the Turán graph $T_2(n)$.*

We observe that K_4 admits more K_3^R-free 10-colorings than $T_2(4)$ and that K_5 admits more K_3^R-free 9-colorings than $T_2(5)$, so that Theorem 1.2 is tight. Other comments about possible improvements to this result are in Remark 4.8.

For the cases $r \in \{3,4\}$, which have not been covered in Theorem 1.1, we have the following approximate result.

Theorem 1.3. *Given $r \in \{3,4\}$ and $\beta > 0$, there exists n_0 such that the number of K_3^R-free r-edge colorings of any graph $G=(V,E)$ on $n \geq n_0$ vertices is bounded above by*

$$2^{\frac{n^2}{2}(1+\beta)}.$$

This result implies that the complete graph K_n is not far, with respect to the number edge-colorings, from achieving the extremal value for both $r \in \{3,4\}$. This was already known for $r=3$ (see [3]), while, for $r=4$, both K_n and $T_2(n)$ are close to achieving the extremal value. One may easily find approximately $6 \cdot 2^{\binom{n}{2}} = 6 \cdot 2^{(n^2-n)/2}$ colorings of K_n

by choosing two of the four colors and using them arbitrarily, while $T_2(n)$ may be colored in exactly $4^{\mathrm{ex}(n,K_3)} \geq 2^{(n^2-1)/2}$ ways. We believe that the Turán graph should be extremal in this case, but the fact that two very different configurations achieve a similar number of colorings suggests that demonstrating this will require very precise estimates. It was also shown in [3] that, for $r = 3$, the Turán graph $T_2(n)$ is the unique extremal graph for the pattern of K_3 with two classes.

In Section 2, we present preliminary results for proving our main results. The proofs of Theorems 1.1 and 1.3 are the subject of Section 3, while Section 4 deals with Theorem 1.2.

2. Preliminaries

In this section, we fix the notation and introduce basic concepts and results used to prove our results. For simplicity, we shall assume that colors lie in sets $[r] := \{1, \dots, r\}$.

2.1. **Regularity Lemma.** To prove Theorems 1.1 and 1.3, we refine results of [9], which adapted the approach of [1] for general patterns. The strategy is based on the Szemerédi Regularity Lemma [14]. Let $G = (V, E)$ be a graph, and let A and B be two disjoint subsets of $V(G)$. If A and B are non-empty, define the density of edges between A and B by

$$d(A, B) = \frac{e(A, B)}{|A||B|},$$

where $e(A, B)$ is the number of edges with one endpoint in A and the other in B. When $A = B$, we write $e(A, A) = e(A)$. For $\varepsilon > 0$ the pair (A, B) is called *ε-regular* if, for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

An *equitable partition* of a set V is a partition of it into pairwise disjoint classes $V_1, \dots, V_m$ of almost equal size, i.e., $||V_i| - |V_j|| \leq 1$ for all i, j. An equitable partition of the set V of vertices of G into the classes $V_1, \dots, V_m$ is called *ε-regular* if at most $\varepsilon\binom{m}{2}$ of the pairs (V_i, V_j) are not ε-regular.

We now state a colored version of the Regularity Lemma (see [11]) that will be useful for our purposes.

Lemma 2.1. *For every $\varepsilon > 0$ and every integer r, there exists a constant $M = M(\varepsilon, r)$ such that the following property holds. If the edges of a graph G of order $n > M$ are r-colored $E(G) = E_1 \cup \cdots \cup E_r$, then there is a partition of the vertex set $V(G) = V_1 \cup \cdots \cup V_m$, with $1/\varepsilon \leq m \leq M$, which is ε-regular simultaneously with respect to all graphs $G_i = (V, E_i)$ for $1 \leq i \leq r$.*

We refer to a partition as in Lemma 2.1 as a *multicolored ε-regular partition*. Given such a partition and given a color $c \in [r]$, we can define the *cluster graph* associated with color c as follows. Given $\eta > 0$, the graph $H_c = H_c(\eta)$ is defined on the vertex set $[m]$ so that $\{i, j\} \in E(H_c)$ if and only if (V_i, V_j) is an ε-regular pair with edge density at least η with respect to the subgraph of G induced by the edges of color c.

We may also define the *multicolored cluster graph* H associated with this partition: the vertex set is $[m]$ and $e = \{i, j\}$ is an edge of H if $e \in E(H_c)$ for some $c \in [r]$. Each edge e is assigned the list of colors $L_e = \{c \in [r] \mid e \in E(H_c)\}$. Given a colored graph F, we say that a multicolored cluster graph H contains F if H contains a copy of F for which the color of each edge of F is contained in the list of the corresponding edge in H. More generally, if F is a graph with color pattern P, we say that H contains (F, P) if it contains some colored copy of F with pattern P.

In connection with these definitions, we shall use the following embedding result, whose proof is quite standard and follows the arguments in the proof of Theorem 2.1 in [11]. (See also Lemma 2.4 in [9].)

Lemma 2.2. *For every $\eta > 0$ and all positive integers k and r, there exist $\varepsilon = \varepsilon(r, \eta, k) > 0$ and a positive integer $n_0(r, \eta, k)$ with the following property. Suppose that G is an r-colored graph on $n > n_0$ vertices with a multicolored ε-regular partition $V = V_1 \cup \cdots \cup V_m$ which defines the multicolored cluster graph $H = H(\eta)$. Let F be a fixed k-vertex graph with a prescribed color pattern P on $t \leq r$ classes. If H contains (F, P), then the graph G also contains (F, P).*

2.2. **Stability.** Another basic tool in our paper are stability results for graphs. We shall apply the following theorem by Füredi [5], which builds upon earlier work by Erdős and Simonovits [13]. We state it for triangle-free graphs only.

Theorem 2.3. *Let $G = (V, E)$ be a triangle-free graph on m vertices. If $|E| = \mathrm{ex}(m, K_3) - t$ for some $t \geq 0$, then there exists a partition $V = V_1 \cup V_2$ with $e(V_1) + e(V_2) \leq t$.*

We recall the following bounds on the number of edges in the Turán graph $T_2(n)$:

$$\frac{n^2 - 1}{4} \quad \leq \quad \mathrm{ex}(n, K_3) \leq \frac{n^2}{4}. \tag{1}$$

For later use, we state the following fact about the size of the classes in a bipartite graph with a large number of edges.

Proposition 2.4. *$G = (V, E)$ be a bipartite graph on m vertices and with bipartition $V = V_1 \cup V_2$. If for some $t \geq 1$, the graph G contains at least $\mathrm{ex}(m, K_3) - t$ edges, then we have*

$$\left| |V_1| - \frac{m}{2} \right| < \sqrt{2t}.$$

Proof. If $|V_1| \leq m/2 - \sqrt{2t}$ or $|V_1| \geq m/2 + \sqrt{2t}$, then G contains at most

$$(m/2 - \sqrt{2t})(m/2 + \sqrt{2t}) = m^2/4 - 2t \overset{(1)}{<} \mathrm{ex}(m, K_3) - t$$

edges, which is a contradiction. □

2.3. **Entropy function.** Consider the entropy function $H\colon [0, 1] \to [0, 1]$ given by $H(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$ with $H(0) = H(1) = 0$. Note that $\lim_{x \to 0^+} H(x) = 0$.

It is well-known that, for all $0 \leq \alpha \leq 1$, we have

$$\binom{n}{\alpha n} \leq 2^{H(\alpha)n}. \tag{2}$$

We will also use the following upper bound on the entropy function for $x \leq 1/8$:

$$H(x) \quad \leq \quad -2x \log_2 x. \tag{3}$$

Indeed, it is easy to see that (3) is equivalent to $g(x) = x \ln x - (1 - x) \ln(1 - x) \leq 0$. Taking the derivative gives $g'(x) = \ln x + 2 + \ln(1 - x) \leq 0$ for $x \leq 1/8$. With $g(1/8) < 0$ inequality (3) follows.

3. The extremal graph for all $r \geq 5$

The objective of this section is to prove Theorems 1.1 and 1.3, which use the colored version of the Szemerédi Regularity Lemma given in Lemma 2.1. A particularly important ingredient in the proof of the first result is a stability result of [9] (see Lemma 4.3), which extends one of the main steps in the strategy developed by Alon, Balogh, Keevash and Sudakov [1] (see Theorem 1.1) to prove that the Turán graph $T_k(n)$ admits the largest number of r-edge colorings avoiding a monochromatic copy of K_{k+1}, where $r \in \{2, 3\}$. Before stating this stability result, we first describe the method in [1], which consists of two main steps:

(i) Proving a stability result which states that any n-vertex graph whose number of feasible r-edge colorings could beat the number of feasible r-edge colorings of $T_k(n)$ is similar to $T_k(n)$, in the sense that its vertex set admits a partition into k almost-balanced classes such that the number of edges with both ends in the same class is small.

(ii) Proving that $T_k(n)$ is extremal: starting with a counterexample on n vertices, one shows that it is possible to find a counterexample on $n-1$ or $n-2$ vertices whose 'gap' to the desired optimal solution increases. A recursive application of this step would lead to an $\sqrt{n}$-vertex graph whose number of colorings is too large to be feasible.

To prove the validity of Theorem 1.1, we show the following stability result.

Lemma 3.1. *Given an integer $r \geq 5$ and a constant $\delta > 0$, there exists n_0 with the following property. If $G = (V, E)$ is a graph on $n > n_0$ vertices which has at least $r^{\mathrm{ex}(n,K_3)}$ distinct K_3^R-free r-edge colorings, then there is a partition $V = W_1 \cup W_2$ of its vertex set such that $e(W_1) + e(W_2) \leq \delta n^2$.*

Before demonstrating this lemma, we prove Theorem 1.3, as the ideas in the proof shed light on the argument used in the proof of Lemma 3.1.

3.1. **Proof of Theorem 1.3.** Lemma 2.2 imposes constraints on the graph structure and on the sizes of the lists on the edges of a multicolored cluster graph associated with a K_3^R-free r-edge coloring of a graph G. Let $H = (W, E)$ be a graph where each edge $e \in E$ is associated with a nonempty list $L_e \subseteq [r]$ of colors with the property that no copy of the pattern K_3^R can be constructed with colors in the lists. Set $c^{(r)}(H, \{L_e\}_{e\in E}) = \prod_{e\in E} |L_e|$ and $c^{(r)}(H) = \max_{\{L_e\}_{e\in E}}\{c^{(r)}(H, \{L_e\}_{e\in E})\}$ and $c_m^{(r)} = \max_H\{c^{(r)}(H)\}$, where the maximum runs over all m-vertex graphs H.

Recall that, according to the statement of Theorem 1.3, the graph K_n is almost extremal with respect to the number of K_3^R-free r-edge colorings when $r = 3$, while $T_2(n)$ is almost extremal for $r \geq 4$. The following two results state that these two graphs are extremal in the 'reduced-graph level', that is, with respect to $c_m^{(r)}$.

Lemma 3.2. *$c_2^{(3)} = 3$, $c_3^{(3)} = 9$, $c_4^{(3)} = 81$ and $c_m^{(3)} = 2^{\binom{m}{2}}$ for all $m \geq 5$. For $m \geq 5$, the only m-vertex graph H with $c^{(3)}(H) = c_m^{(3)}$ is the complete graph K_m.*

Proof. By inspection, we see that $c_3^{(3)} = 9$ and the value is achieved by the 3-vertex graph containing exactly two edges e, e' with $L_e = L_{e'} = [3]$. The value $c_4^{(3)} = 81$ is achieved by a 4-cycle and any other 4-vertex graph H satisfies $c^{(3)}(H) < 81$. We use induction on $m \geq 5$. Let $H = (V, E)$ be an m-vertex graph, where each edge $e \in E$ is assigned a list L_e of colors. If $|L_e| \leq 2$ for every edge $e \in E$, then $c^{(3)}(H, \{L_e\}_{e\in E}) \leq 2^{|E|} \leq 2^{\binom{m}{2}}$ and we are done. Otherwise there exists an edge $e = \{u, v\} \in E$ such that $L_e = [3]$. For each vertex $w \in V \setminus \{u, v\}$ with $\{u, w\}$, $\{v, w\} \in E$ we must have $|L_{\{u,w\}}| = |L_{\{v,w\}}| = 1$, otherwise a rainbow triangle arises. Let ℓ denote the number of such vertices w. For all other vertices z at least one of the pairs $\{z, u\}$ and $\{z, v\}$ is not an edge in H, hence for at most one of $\{z, u\}$ or $\{z, v\}$ the size of the corresponding list is at most three. There are $(m-2-\ell)$ such vertices z. The subgraph H' of H induced on $V \setminus \{u, v\}$ contains $(m-2)$ vertices, hence

$$c^{(3)}(H, \{L_e\}_{e\in E}) \leq 3 \cdot 1^{2\ell} \cdot 3^{m-2-\ell} \cdot c^{(3)}(H') \leq 3^{m-1} \cdot c^{(3)}(H').$$

If $m = 5$, then $c^{(3)}(H') \leq c_3^{(3)} = 9$, and we infer that

$$c^{(3)}(H, \{L_e\}_{e\in E}) \leq 3^4 \cdot 9 = 3^6 < 2^{10} = 2^{\binom{5}{2}}.$$

Analogously, if $m = 6$, then $c^{(3)}(H') \le c_4^{(3)} = 81$, and we have

$$c^{(3)}(H, \{L_e\}_{e\in E}) \le 3^5 \cdot 81 = 3^9 < 2^{15} = 2^{\binom{6}{2}}.$$

For $m \ge 7$, the induction hypothesis implies that $c^{(3)}(H') \le 2^{\binom{m-2}{2}}$, and we infer that

$$c^{(3)}(H, \{L_e\}_{e\in E}) \le 3^{m-1} \cdot 2^{\binom{m-2}{2}} < 2^{\binom{m}{2}},$$

which follows from $3^{m-1} < 2^{2m-3}$ for $m \ge 4$.

The proof also shows that, for $m \ge 5$, equality $c^{(3)}(H) = 2^{\binom{m}{2}}$ holds for an m-vertex graph H only if $H = K_m$. □

For $r \ge 4$, the value of $c_m^{(r)}$ is achieved by the Turán graph $T_2(m)$.

Lemma 3.3. *For every $r \ge 4$ and $m \ge 2$, we have $c_m^{(r)} = r^{\mathrm{ex}(m,K_3)}$, and the only m-vertex graph H with $c^{(r)}(H) = c_m^{(r)}$ is the Turán graph $T_2(m)$.*

Proof. Let $r \ge 4$ be fixed and use induction on m. If H consists of a single edge, we have $c^{(r)}(H) = r = c_2^{(r)}$. By inspection, the graph H on three vertices having exactly two edges yields $c^{(r)}(H) = r^2$, and any other 3-vertex graph H' satisfies $c^{(r)}(H') < r^2$, thus $c_3^{(r)} = r^2$.

Now let $H = (V, E)$ be an m-vertex graph, $m \ge 4$, with lists L_e of colors for each edge $e \in E$. If H is triangle-free, then $c^{(r)}(H) \le r^{|E|} \le r^{\mathrm{ex}(m,K_3)}$. Therefore, assume that two vertices u and v of H lie in a triangle. Let $T \subseteq V \setminus \{u, v\}$ be the set of vertices w such that $\{u, v, w\}$ induces a triangle in H, where $|T| = t \ge 1$. Let Z_u (respectively, Z_v) be the set of vertices in $V \setminus (T \cup \{u, v\})$ such that $\{u, z\} \in E$ for each $z \in Z_u$ (respectively, $\{v, z\} \in E$ for each $z \in Z_v$), thus $|Z_u| + |Z_v| \le m - 2 - t$. We distinguish two cases according to the size of the list $L_{\{u,v\}}$.

Case (a): Let $3 \le \left|L_{\{u,v\}}\right| \le r$. In this case we must have $\left|L_{\{u,w\}}\right| = \left|L_{\{v,w\}}\right| = 1$ for each vertex $w \in T$, as otherwise a rainbow triangle arises. For each vertex $z \in Z_u \cup Z_v$ and each edge $e = \{u, z\}$ or $e = \{v, z\}$, if present in H, we have $|L_e| \le r$. Let H' be the on $V \setminus \{u, v\}$ induced subgraph of H. Using $t \ge 1$ and the induction hypothesis for H' we infer that

$$\begin{aligned} c^{(r)}(H, \{L_e\}_{e\in E}) &\le \left|L_{\{u,v\}}\right| \cdot \left(\prod_{w\in T} \left|L_{\{u,w\}}\right| \cdot \left|L_{\{v,w\}}\right|\right) \cdot \left(\prod_{\substack{z\in Z_u\\ \{u,z\}\in E}} \left|L_{\{u,z\}}\right|\right) \times \\ &\quad \times \left(\prod_{\substack{z\in Z_v\\ \{v,z\}\in E}} \left|L_{\{v,z\}}\right|\right) \cdot c^{(r)}(H') \\ &\le r \cdot 1^{2t} \cdot r^{m-2-t} \cdot c^{(r)}(H') \overset{(t\ge 1)}{\le} r^{m-2} \cdot c^{(r)}(H') \le r^{m-2} \cdot r^{\mathrm{ex}(m-2,K_3)} \\ &< r^{\mathrm{ex}(m,K_3)} \end{aligned}$$

for any $m \ge 4$, because $\mathrm{ex}(m, K_3) - \mathrm{ex}(m - 2, K_3) = m - 1$, and we are done in this case.

Case (b): Assume that $\left|L_{\{u,v\}}\right| \le 2$. If $\left|L_{\{u,v\}}\right| = 2$, then for each vertex $w \in T$ we must have $\left|L_{\{u,w\}}\right|, \left|L_{\{v,w\}}\right| \le 2$, as otherwise a rainbow triangle arises. If $\left|L_{\{u,v\}}\right| = 1$, then for any $t \in T$ at most one of the edges $\{u, w\}$ or $\{v, w\}$ might have a list of size at least three, but then we are in Case (a) again. Thus, $\left|L_{\{u,w\}}\right| \cdot \left|L_{\{v,w\}}\right| \le 4$ for each vertex $w \in T$. Let H' be the induced subgraph of H on the vertex set $V \setminus \{u, v\}$. By the induction hypothesis, we conclude that

$$\begin{aligned} c^{(r)}(H, \{L_e\}_{e \in E}) &\leq 2 \cdot 4^t \cdot r^{m-2-t} \cdot c^{(r)}(H') \leq \frac{2^{2t+1}}{r^t} \cdot r^{m-2} \cdot r^{\mathrm{ex}(m-2,K_3)} \\ &\overset{(r \geq 4)}{\leq} 2 \cdot r^{m-2} \cdot r^{\mathrm{ex}(m-2,K_3)} \overset{(m \geq 4)}{<} r^{\mathrm{ex}(m,K_3)}. \end{aligned}$$

The calculations above show that $c^{(r)}(H) = c_m^{(r)}$ can only be achieved by the Turán graph $H = T_2(m)$, which finishes the proof. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $r \in \{3, 4\}$ and $\beta > 0$. Our parameters will be defined implicitly in terms of $\eta > 0$, which we shall fix later. Let $\varepsilon = \varepsilon(r, \eta, 3) > 0$ and $n_0 = n_0(r, \eta, 3)$ satisfy the assumptions of Lemma 2.2, w.l.o.g. $\varepsilon < \eta/2$. Let $M = M(\varepsilon, r)$ be as in Lemma 2.1.

Let $n > \max\{n_0, M\}$ and let $G = (V, E)$ be an n-vertex graph. Fix a K_3^R-free r-edge coloring of G. By Lemma 2.1 we obtain a partition $V = V_1 \cup \cdots \cup V_m$ for which the graphs $G_i = (V, E_i)$ induced by the edges in color i are ε-regular for all $i \in [r]$. Let $H = H(\eta)$ be the multicolored cluster graph associated with this partition, whose vertex set is $[m]$.

We shall initially bound the number of r-edge colorings of G leading to the partition $V_1 \cup \cdots \cup V_m$ and the cluster graph H. Given a color $i \in [r]$, there are at most $\varepsilon\binom{m}{2}$ irregular pairs in G_i with respect to the partition $V = V_1 \cup \cdots \cup V_m$, so that at most

$$r \cdot \varepsilon \cdot \binom{m}{2} \cdot \left(\frac{n}{m}\right)^2 \leq \frac{r \cdot \varepsilon}{2} n^2 \tag{4}$$

edges lie in irregular pairs with respect to some color. By Lemma 2.1 and by the definition of an ε-regular partition, there are at most (using $m \geq 1/\varepsilon$)

$$m \cdot \left(\frac{n}{m}\right)^2 = \frac{n^2}{m} \leq \varepsilon n^2 \tag{5}$$

edges with both ends in some class V_i. Moreover, the total number of edges of any color c whose endpoints are in a pair (V_i, V_j) such that the density of edges in color c is less than η is bounded above by

$$r \cdot \eta \cdot \binom{m}{2} \cdot \left(\frac{n}{m}\right)^2 \leq \frac{r\eta}{2} \cdot n^2. \tag{6}$$

Using (4), (5) and (6) together with $\varepsilon < \eta/2$ gives at most $r\eta n^2$ such edges, which we can choose in at most $\binom{n^2/2}{r\eta n^2}$ ways. This set of edges may be colored in at most $r^{r\eta n^2}$ ways. If L_e denotes the list of colors associated with each edge $e \in E(H)$, the number of r-edge colorings of G that give rise to the partition $V = V_1 \cup \cdots \cup V_m$ and the multicolored cluster graph H is bounded above by

$$\begin{aligned} \binom{\frac{n^2}{2}}{r\eta n^2} \cdot r^{r\eta n^2} \cdot \left(\prod_{e \in E(H)} |L_e|\right)^{\left(\frac{n}{m}\right)^2} &\overset{(2)}{\leq} 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot \left(\prod_{e \in E(H)} |L_e|\right)^{\left(\frac{n}{m}\right)^2} \\ &\leq 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot \left(c_m^{(r)}\right)^{\left(\frac{n}{m}\right)^2}, \end{aligned}$$

where, for the second inequality, we used the fact that, by Lemma 2.2, H cannot contain a copy of the pattern K_3^R. Further note that, as $r \in \{3, 4\}$, Lemmas 3.2 and 3.3 imply that $c_m^{(r)} \leq 2^{m^2/2}$.

As $m \leq M(\varepsilon, r)$ is a constant, there are at most m^n partitions $V = V_1 \cup \cdots \cup V_m$, hence altogether at most $\sum_{m=1}^{M} m^n < M^{n+1}$ partitions of size at most M. For each partition, there are at most $2^{rM^2/2}$ choices for the cluster graphs $H_1, \ldots, H_r$. Thus, the number of K_3^R-free r-edge-colorings of G is bounded above by

$$M^{n+1} \cdot 2^{r\frac{M^2}{2}} \cdot 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot 2^{n^2/2}.$$

To obtain the desired result, it suffices to fix $\eta > 0$ so that

$$M^{n+1} \cdot 2^{r\frac{M^2}{2}} \cdot 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} < 2^{\beta n^2/2}$$

for $n \geq n_\beta$, which may be chosen sufficiently large. □

3.2. **Proof of Lemma 3.1.** Next we give a proof of Lemma 3.1. The general structure of the proof is the same as that of Theorem 1.3, but more careful arguments are needed.

Proof of Lemma 3.1. Fix the number $r \geq 5$ of colors and $\delta > 0$. With foresight, fix $\eta > 0$ small enough to ensure that

$$91H(2r\eta) + 92r\eta < \delta \text{ and } 20H(2r\eta) + 20r\eta < \frac{1}{82^2}, \tag{7}$$

where $H(\cdot)$ is the entropy function. As before, let $\varepsilon = \varepsilon(r, \eta, 3) > 0$ and $n_0 = n_0(r, \eta, 3)$ satisfy the assumptions of Lemma 2.2, where we choose $\varepsilon < \eta/2$, and let $M = M(\varepsilon, r)$ be as in Lemma 2.1.

Let $G = (V, E)$ be an n-vertex graph, where $n > \max\{n_0, M\}$ and fix a K_3^R-free r-edge coloring of G. By Lemma 2.1 there exists a multicolored ε-regular partition $V = V_1 \cup \cdots \cup V_m$, $1/\varepsilon \leq m \leq M$. Let $H = H(\eta)$ be the multicolored cluster graph associated with this partition. We write $E_j(H) = \{e \in E(H) : |L_e| = j\}$ and $e_j(H) = |E_j(H)|$, where $j \in [r]$.

Using the arguments of the proof of Theorem 1.3, we may easily derive the following upper bound on the number of r-edge colorings of G leading to the partition $V_1 \cup \cdots \cup V_m$ and the multicolored cluster graph H:

$$\binom{\frac{n^2}{2}}{r\eta n^2} \cdot r^{r\eta n^2} \cdot \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2} \overset{(2)}{\leq} 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2}. \tag{8}$$

There are altogether at most $\sum_{m=1}^{M} m^n < M^{n+1}$ possible partitions. Thus, summing over all partitions and all corresponding multicolored cluster graphs H, the number of K_3^R-free r-edge-colorings of G is bounded above by

$$M^{n+1} \cdot \sum_{H} 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2}. \tag{9}$$

By MAXCUT, for each multicolored cluster graph H we can choose a subset $E'(H) \subseteq E_2(H)$ of edges that is triangle-free and satisfies $|E'(H)| \geq e_2(H)/2$. We claim that

$$e_2(H)/2 + e_3(H) + \cdots + e_r(H) \leq |E'(H)| + e_3(H) + \cdots + e_r(H) \leq \text{ex}(m, K_3). \tag{10}$$

Indeed, if this were not the case, the multicolored cluster graph H would contain a triangle whose edges have lists of size at least two, and where at least one of the edges has a list of size three or more. In particular, H would contain a rainbow triangle, so that G would also contain a rainbow triangle by Lemma 2.2, a contradiction.

First assume that $e_3(H) + \cdots + e_r(H) \leq \text{ex}(m, K_3) - 10H(2r\eta)m^2 - 10r\eta m^2$ for *all* multicolored cluster graphs H. Because each multicolored partition gives rise to at most $2^{rM^2/2}$

multicolored cluster graphs, for $r \geq 5$ the upper bound (9) becomes

$$\begin{aligned}
& M^{n+1} \cdot 2^{r\frac{M^2}{2}} \cdot 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot \left(2^{e_2(H)} \cdot \prod_{j=3}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2} \\
\leq \quad & M^{n+1} \cdot 2^{r\frac{M^2}{2}} \cdot 2^{H(2r\eta)\frac{n^2}{2}} \cdot r^{r\eta n^2} \cdot \left(2^{e_2(H)} \cdot r^{e_3(H)+\cdots+e_r(H)}\right)^{\left(\frac{n}{m}\right)^2} \\
\overset{(10)}{\leq} \quad & r^{H(2r\eta)n^2+r\eta n^2} \cdot \left(2^{m^2/2-2(e_3(H)+\cdots+e_r(H))} \cdot r^{e_3(H)+\cdots+e_r(H)}\right)^{\left(\frac{n}{m}\right)^2} \\
= \quad & r^{H(2r\eta)n^2+r\eta n^2} \cdot 2^{n^2/2} \cdot \left(\frac{r}{4}\right)^{(e_3(H)+\cdots+e_r(H))\left(\frac{n}{m}\right)^2} \\
\leq \quad & r^{H(2r\eta)n^2+r\eta n^2} \cdot 2^{n^2/2} \cdot \left(\frac{r}{4}\right)^{n^2/4-10H(2r\eta)n^2-10r\eta n^2} \\
= \quad & \left(\frac{4^{10}}{r^9}\right)^{H(2r\eta)n^2+r\eta n^2} \cdot r^{n^2/4} \ll r^{\mathrm{ex}(n,K_3)}, \qquad (11)
\end{aligned}$$

as $r^{0.9} > 4$ for $r \geq 5$. Thus, for *some* multicolored cluster graph H we have $e_3(H) + \cdots + e_r(H) \geq \mathrm{ex}(m, K_3) - 10H(2r\eta)m^2 - 10r\eta m^2$, hence $e_2(H) \leq 20H(2r\eta)m^2 + 20r\eta m^2$ by (10).

Let $\widehat{H}$ be the subgraph of H with edge set $E_3(H) \cup \cdots \cup E_r(H)$. Since $\widehat{H}$ is K_3-free, by Theorem 2.3 there is a partition $U_1 \cup U_2 = [m]$, $m_i = |U_i|$, with

$$e_{\widehat{H}}(U_1) + e_{\widehat{H}}(U_2) \leq 10H(2r\eta)m^2 + 10r\eta m^2.$$

Proposition 2.4 implies that $|m_i - m/2| \leq \sqrt{20H(2r\eta) + 20r\eta} \cdot m$, for $i = 1, 2$.

Let $W_i = \cup_{j \in U_i} V_j$ for $i = 1, 2$. Then,

$$\begin{aligned}
e_G(W_1) + e_G(W_2) \quad & \leq \quad r\eta n^2 + (n/m)^2(e_{\widehat{H}}(U_1) + e_{\widehat{H}}(U_2) + e_1(H) + e_2(H)) \\
& \leq \quad r\eta n^2 + 10H(2r\eta)n^2 + 10r\eta n^2 + \frac{(e_1(H) + e_2(H))n^2}{m^2} \\
& \leq \quad 30H(2r\eta)n^2 + 31r\eta n^2 + \frac{e_1(H)n^2}{m^2}. \qquad (12)
\end{aligned}$$

To conclude the proof, we give an upper bound on $e_1(H)$.

To this end, we will estimate the number of pairs in $U_1 \times U_2$ that are missing from $E(\widehat{H})$, which we know to be at most $20H(2r\eta)m^2 + 20r\eta m^2$. Let $A_i = E_1 \cap [U_i]^2$, $i = 1, 2$. Fix an edge $\{a, b\} \in A_1$. Then, for each vertex $x \in U_2$, at least one of $\{x, a\}$ or $\{x, b\}$ is missing from $E(\widehat{H})$. Considering also edges in A_2, we get a total of at most $|A_1|m_2 + |A_2|m_1$ triples $\{a, b, x\}$ which cause some pair in $U_1 \times U_2$ to be missing from $E(\widehat{H})$. Each such missing edge is counted at most m times, hence we have at least

$$\frac{|A_1|m_2 + |A_2|m_1}{m}$$

missing edges, which, combined with $m_i \geq m/2 - \sqrt{20H(2r\eta) + 20r\eta} \cdot m$, implies that there are at least

$$(|A_1| + |A_2|) \cdot \left(\frac{1}{2} - \sqrt{20H(2r\eta) + 20r\eta}\right)$$

pairs in $U_1 \times U_2$ that are missing from $E(\widehat{H})$. However, we know that this number is at most $20H(2r\eta)m^2 + 20r\eta m^2$, therefore, by our choice of η, we have

$$|A_1| + |A_2| \overset{(7)}{\leq} 41H(2r\eta)m^2 + 41r\eta m^2.$$

Taking into account the at most $20H(2r\eta)m^2 + 20r\eta m^2$ edges between U_1 and U_2 that are missing from $E(\widehat{H})$, we infer that

$$e_1(H) \leq 61H(2r\eta)m^2 + 61r\eta m^2,$$

and (12) is at most

$$91H(2r\eta)n^2 + 92r\eta n^2 \overset{(7)}{\leq} \delta n^2,$$

by our choice of η. □

3.3. **Proof of Theorem 1.1.** Now we have all the tools we need to prove Theorem 1.1. We observe that the arguments in this proof allow us to derive exact results from a stability result in a more general situation, which we address in the long version of [9] (not yet published). For completeness, we include a proof of this fact restricted to rainbow triangles. (This proof adapts the ideas in the proof of [1, Theorem 1.1] to rainbow triangles.)

Proof of Theorem 1.1. For a contradiction, we choose n_0 appropriately (it will be large enough to guarantee that the assertion of Lemma 3.1 holds for $\delta = 10^{-16r}$) and we let $G \neq T_2(n)$ be a graph on $n > n_0^2$ vertices with at least $r^{\mathrm{ex}(n,K_3)+m}$ distinct K_3^R-free r-edge colorings, for some $m \geq 0$. We will show that G contains a vertex x such that the graph $G - x$ obtained by deleting x has at least $r^{\mathrm{ex}(n-1,K_3)+m+1}$ distinct K_3^R-free r-edge colorings, or it contains two vertices x and y such that $G - x - y$ has at least $r^{\mathrm{ex}(n-2,K_3)+m+2}$ distinct K_3^R-free r-edge colorings. Repeating this argument iteratively, we obtain a graph on n_0 vertices whose number of K_3^R-free r-edge colorings is at least $r^{\mathrm{ex}(n_0,K_3)+m+n-n_0} > r^{n_0^2}$. However, a graph on n_0 vertices has at most $n_0^2/2$ edges and hence the number of such colorings is at most $r^{n_0^2/2}$, which is the desired contradiction.

Let $\delta_2(n)$ denote the minimum vertex degree in the Turán graph $T_2(n)$. If G contains a vertex x of degree less than $\delta_2(n)$, then the edges incident with x may be colored in at most $r^{\delta_2(n)-1}$ ways. Thus $G - x$ has at least $r^{\mathrm{ex}(n-1,K_3)+m+1}$ distinct K_3^R-free r-edge colorings and we are done. Hence we will assume that all vertices of G have degree at least $\delta_2(n)$, i.e., G has at least $\mathrm{ex}(n, K_3)$ edges.

Consider a partition $V = V_1 \cup V_2$ of the vertex set of G which minimizes $e(V_1) + e(V_2)$. By our choice of n_0 in Lemma 3.1, we have $e(V_1) + e(V_2) < 10^{-16r}n^2$. The bipartite subgraph induced by the partition $V = V_1 \cup V_2$ contains at least $\mathrm{ex}(n, K_3) - 10^{-16r}n^2$ edges. By Proposition 2.4 we infer that $||V_i| - n/2| < \sqrt{2} \cdot 10^{-8r}n$ for each $i \in [2]$. Let $\mathcal{C}$ denote the set of all possible K_3^R-free r-edge colorings of the edges of G.

First consider the case when there is some vertex with many neighbors in its own class of the partition, say $x \in V_1$ satisfies $|N(x) \cap V_1| > n/(10^{3r}2)$. Our choice of the partition guarantees that in this case $|N(x) \cap V_2| > n/(10^{3r}2)$, otherwise we could reduce $e(V_1) + e(V_2)$ by moving x to the other part.

Let $\mathcal{C}_1$ be the subset of all colorings for which there are subsets $W_i \subseteq V_i$ with $|W_i| \geq n/(10^{3r}2r)$ such that all the edges from x to W_i are colored with color c_i, $i = 1, 2$, where c_1 and c_2 are distinct. Consider a coloring of G in $\mathcal{C}_1$. There are less than r^2 possible choices for the colors c_1 and c_2. Since there are at most $|W_1||W_2|$ edges between these two sets W_1 and W_2, we have at most $2^{|W_1||W_2|}$ ways to color edges between W_1 and W_2, as any color other than c_1, c_2 would create a K_3^R. Since there are at most $\mathrm{ex}(n, K_3) + 10^{-16r}n^2 - |W_1||W_2|$ other edges in this graph, and hence the number of colorings in $\mathcal{C}_1$ associated with this particular pair of sets and this pair of colors is at most

$$r^{\mathrm{ex}(n,K_3)+10^{-16r}n^2-|W_1||W_2|} \cdot 2^{|W_1||W_2|}.$$

As 2^{2n} is a generous upper bound on the number of pairs (W_1, W_2) with this property, we have

$$|\mathcal{C}_1| \leq r^2 \cdot r^{\mathrm{ex}(n,K_3)+10^{-16r}n^2} \cdot 2^{2n} \cdot \left(\frac{2}{r}\right)^{n^2/(10^{6r}4r^2)} < r^{\mathrm{ex}(n,K_3)-1}$$

for large enough n. Here we used $2/r < 1/\sqrt{r}$ for $r \geq 5$ and

$$r^{10^{-16r}} \cdot \left(\frac{2}{r}\right)^{\frac{1}{10^{6r}4r^2}} \leq r^{10^{-16r}} \cdot r^{-\frac{1}{10^{6r}8r^2}} = r^{10^{-16r}-\frac{10^{-6r}}{8r^2}} < 1.$$

Let $\mathcal{C}_2 = \mathcal{C} - \mathcal{C}_1$. By the above discussion, $\mathcal{C}_2$ contains at least $|\mathcal{C}| - |\mathcal{C}_1| \geq r^{\mathrm{ex}(n,K_3)+m-1}$ colorings of G. We consider one of them. There is no pair (W_1, W_2) as in the definition of $\mathcal{C}_1$. Let V_i^c be the set of all neighbors $y \in V_i$ of x where the edge $\{x, y\}$ has color c, for $i \in \{1, 2\}$. We say that a color c is *rare* for V_i if $|V_i^c| < n/(10^{3r}2r)$. Since $|N(x) \cap V_i| > n/(10^{3r}2)$, it is impossible that all colors are rare with respect to some V_i. Therefore there are sets $W_i^c \subseteq N(x) \cap V_i$ with $|W_1^c| \geq n/(10^{3r}2r)$ and $|W_2^c| \geq n/(10^{3r}2r)$ of the same color $c \in [r]$, as otherwise we would have a pair (W_1, W_2) as in the definition of $\mathcal{C}_1$.

For $r \geq 5$, the entropy function H satisfies

$$\begin{aligned} H(1/(10^{3r}2r))r &\overset{(3)}{\leq} \frac{2}{10^{3r}2r}\log_2(10^{3r}2r) \cdot r \leq \frac{1}{10^{3r}} \cdot \log_2(2^{10r}2r) \\ &< \frac{1}{10^{3r}} \cdot \log_2(2^{11r}) = \frac{11r}{10^{3r}} \leq \frac{1}{10^{2r}} \leq 10^{-10}. \end{aligned}$$

Thus, for $r \geq 5$, there are at most

$$\binom{|V_i|}{n/(10^{3r}2r)}^{r-1} \leq \binom{n}{n/(10^{3r}2r)}^{r-1} \leq \left(2^{H(1/(10^{3r}2r))n}\right)^r < 2^{10^{-10}\cdot n}$$

at most $(r-1)$ rare colors with respect to V_i for $i \in \{1, 2\}$. The remaining edges can only be assigned color c. Hence the number of ways to color all edges incident with x is bounded above by

$$r \cdot 2^{2\cdot 10^{-10}n} < r^{2\cdot 10^{-10}n}.$$

We already know that $|\mathcal{C}_2| \geq r^{\mathrm{ex}(n,K_3)+m-1}$, and it follows that the number of K_3^R-free r-edge colorings of $G - x$ is at least

$$r^{\mathrm{ex}(n,K_3)+m-1-2\cdot 10^{-10}n} \geq r^{\mathrm{ex}(n-1,K_3)+m+1}$$

for n sufficiently large. This completes the induction step in the first case.

Now assume that every vertex has at most $n/(10^{3r}2)$ neighbors in its own class V_i. We may suppose that G is not bipartite, or else by Turán's Theorem we would have $|E(G)| \leq \mathrm{ex}(n, K_3)$ and therefore $|\mathcal{C}| \leq r^{\mathrm{ex}(n,K_3)}$ with equality only for $G = T_2(n)$. Therefore, let $\{x, y\}$ be an edge contained inside one of the classes V_i, say $x, y \in V_1$. Let $W_2 \subseteq V_2$ be the set of common neighbors of x and y in V_2. Let c be the color assigned to the edge $\{x, y\}$. For each vertex $v \in W_2$, the absence of a K_3^R implies that there are at most $3r - 2$ choices for coloring the pair of edges $\{v, x\}, \{v, y\}$. Edges incident with vertices which do not have both x and y as neighbors may be colored in r ways. As x and y have at most $n/(10^{3r}2)$ neighbors in V_1, the number of ways to color edges incident with x or y is at most

$$r\cdot(3r-2)^{|W_2|}\cdot r^{n/2+\sqrt{2}\cdot 10^{-8r}n-|W_2|+10^{-3r}n} \leq \left(\frac{3r-2}{r}\right)^{n/2+10^{-8r}n} \cdot r^{n/2+\sqrt{2}\cdot 10^{-8r}n+10^{-3r}n} \ll r^{(7/8)n}$$

for $r \geq 5$ and n sufficiently large.

Hence the number of K_3^R-free r-edge colorings of $G - x - y$ is at least

$$r^{\mathrm{ex}(n,K_3)+m-(7/8)n} \geq r^{\mathrm{ex}(n-2,K_3)+m+2}.$$

This completes two induction steps and finishes the proof of Theorem 1.1. □

4. The extremal graph for small values of n

The objective of this section is to prove Theorem 1.2, which shows that, for $r \geq 10$, the Turán graph $T_2(n)$ is the single $\mathcal{C}_{r,K_3^R}$-extremal graph for all $n \geq 5$. This refines the conclusion of Theorem 1.1 for all $r \geq 10$.

We start with a straightforward observation, which is added here for future reference.

Lemma 4.1. *At most one new color may appear in an extension of a K_3^R-free coloring of the complete graph K_n to a K_3^R-free coloring of K_{n+1}. In particular, at most $n-1$ different colors occur in any K_3^R-free coloring of K_n.*

Proof. Clearly, if two new colors appeared on edges of such an extension, those two edges would create a rainbow triangle with an edge in the original coloring of K_n. The second statement follows easily by induction, as any coloring of K_2 uses a single color. □

We shall also need the following simple result, which bounds the number of possible extensions of a partial coloring.

Lemma 4.2. *Let the vertices v_1, v_2 and v_3 induce a triangle T, and let w be another vertex. Assume that there are r colors available.*

(a) *If $\{v_1, v_2\}$ has been assigned a color, there are $(3r-2)$ ways to extend this coloring to a K_3^R-free r-coloring of T.*
(b) *If T has been assigned a monochromatic coloring and $T+w$ induces a copy of K_4, there are $(7r-6)$ ways to extend this coloring to a K_3^R-free r-coloring of $T+w$.*
(c) *If T has been assigned a K_3^R-free r-coloring that is not monochromatic and $T+w$ induces a copy of K_4, there are $(5r-2)$ ways to extend this coloring to a K_3^R-free r-coloring of $T+w$.*

Proof. For part (a), assume that $\{v_1, v_2\}$ has been assigned color a. There are r possibilities to color both edges $\{v_1, v_3\}$ and $\{v_2, v_3\}$ with the same color. If they receive different colors, one of them must be assigned color a and the other may be colored in $r-1$ ways, which amounts to $2(r-1)$ different colorings. Hence there are $3r-2$ ways to color the edges $\{v_1, v_3\}$ and $\{v_2, v_3\}$ without producing a rainbow triangle.

For part (b), let a be the color assigned to T. There are r ways to assign the same color to all the edges $\{v_1, w\}, \{v_2, w\}, \{v_3, w\}$. Moreover, there are $3(r-1)$ ways to extend the coloring so that exactly two of the three edges $\{v_1, w\}, \{v_2, w\}, \{v_3, w\}$ are assigned a color $b \neq a$. The only other possible extensions are those where exactly two of the three edges incident with w are colored a, whose number is also $3(r-1)$. Thus the number of ways to assign colors to the edges $\{v_1, w\}, \{v_2, w\}, \{v_3, w\}$ with no K_3^R is exactly $(7r-6)$.

Finally, for part (c), assume that the triangle T has been assigned colors a and b, where the edge $\{v_1, v_2\}$ has color a and the edges $\{v_1, v_3\}$ and $\{v_2, v_3\}$ have color b. There are 2^3 ways to extend this coloring to a coloring of $T+w$ using only colors a and b. There are $(r-2)$ ways to assign the same color $c \notin \{a, b\}$ to the edges $\{v_1, w\}, \{v_2, w\}, \{v_3, w\}$. There are also $(r-2)$ colorings where $\{v_1, w\}$ and $\{v_2, w\}$ are assigned color $c \notin \{a, b\}$ and $\{v_3, w\}$ is colored b. Similarly, there are $(r-2)$ colorings where $\{v_1, w\}$ and $\{v_2, w\}$ are assigned color b and $\{v_3, w\}$ is colored $c \notin \{a, b\}$. The only other K_3^R-free r-colorings are those where $\{v_3, w\}$ has color b and $\{v_1, w\}$ (resp. $\{v_2, w\}$) is colored a and then $\{v_2, w\}$ (resp. $\{v_1, w\}$) is assigned a color $c \notin \{a, b\}$, which adds another $2(r-2)$ colorings. Thus in this case there are $8 + 3(r-2) + 2(r-2) = 5r-2$ ways to color $\{\{v_1, w\}, \{v_2, w\}, \{v_3, w\}\}$ such that the 4-clique $T+w$ contains no K_3^R. □

Using Lemma 4.2, we may easily compute the number of K_3^R-free r-colorings of some graphs.

Corollary 4.3. *We have*

$$\begin{aligned} |\mathcal{C}_{r,K_3^R}(K_3)| &= 3r^2 - 2r & (13)\\ |\mathcal{C}_{r,K_3^R}(K_4)| &= 15r^3 - 14r^2 & (14)\\ |\mathcal{C}_{r,K_3^R}(K_4 - e)| &= 9r^3 - 12r^2 + 4r. & (15) \end{aligned}$$

More generally, if H is a graph with vertex set $\{w_1, \ldots, w_\ell\}$ and G is obtained from H by the addition of a new triangle $T = \{v_1, v_2, v_3\}$ and a subset of edges of $\{\{v_i, w_j\} \colon i \in [3], j \in [\ell]\}$, then

$$|\mathcal{C}_{r,K_3^R}(G)| \leq \Big((3r^2 - 3r)(5r - 2)^\ell + r(7r - 6)^\ell\Big) \cdot |\mathcal{C}_{r,K_3^R}(H)|. \quad (16)$$

Proof. By Lemma 4.2(a), equation (13) follows immediately.

To count the number of K_3^R-free colorings of K_4, assume that the vertex set is $\{v_1, v_2, v_3, w\}$ and let T be the triangle induced by $\{v_1, v_2, v_3\}$. Each of the r monochromatic colorings of T may be extended in $(7r-6)$ ways by Lemma 4.2(b), while Lemma 4.2(c) implies that each of the $(3r^2 - 3r)$ colorings of T with exactly two colors may be extended in $(5r - 2)$ ways, thus

$$|\mathcal{C}_{r,K_3^R}(K_4)| = r(7r - 6) + (3r^2 - 3r)(5r - 2) = 15r^3 - 14r^2.$$

For $K_4 - e$, we do as before, but we observe that there are only two edges between w and the triangle T (and which form a triangle with one of the edges of T), so that any coloring of T may be extended in exactly $(3r - 2)$ ways. This leads to

$$|\mathcal{C}_{r,K_3^R}(K_4 - e)| = \left(3r^2 - 2r\right)(3r - 2) = 9r^3 - 12r^2 + 4r.$$

The previous discussion may be extended to a graph G with vertex set $\{v_1, v_2, v_3, w_1, \ldots, w_\ell\}$ and edges given by the union of the triangle $T = \{v_1, v_2, v_3\}$ and a subset of $\{\{v_i, w_j\} \colon i \in [3], j \in [\ell]\}$. Indeed, once T is colored, the set of edges connecting some vertex w_i to the triangle may be colored in at most $\max\{r, 3r - 2, 7r - 6, 5r - 2\}$ ways, depending on the number of edges involved and on the coloring of T. Since $\max\{r, 3r-2\} \leq \min\{7r-6, 5r-2\}$ for all $r \geq 1$, the number of ways to extend a coloring of H to G is at most

$$(3r^2 - 3r)(5r - 2)^\ell + r(7r - 6)^\ell.$$

This leads to the desired result if we combine this coloring with a K_3-free r-coloring of the edges of H. □

To give an idea of the proof of Theorem 1.2, we first prove a weaker version of it.

Proposition 4.4. *For $r \geq 74$ colors and $n \geq 4$, the maximum number of K_3^R-free r-colorings of any n-vertex graph is $r^{\mathrm{ex}(n,K_3)}$, and this number is achieved only by the Turán graph $T_2(n)$.*

Note, however, that for $n = 3$ and $r \geq 1$, a triangle allows $(3r^2 - 2r)$ distinct K_3^R-free r-colorings, while the corresponding number for the Turán graph $T_2(3)$ is only r^2.

Proof. We use induction on n. For $n = 4$, the 4-cycle $T_2(4)$ admits r^4 distinct K_3^R-free colorings. By Corollary 4.3, if G is a 4-vertex graph containing a triangle, there are at most $(15r^3 - 14r^2)$ distinct K_3^R-free r-colorings, which is less than r^4 for $r \geq 15$.

By (16), for any 5-vertex graph containing a triangle the number of K_3^R-free r-colorings is at most

$$\begin{aligned} &((3r^2 - 3r)(5r - 2)^2 + r(7r - 6)^2) \cdot r = (75r^3 - 86r^2 - 12r + 24) \cdot r^2\\ \overset{(r\geq 2)}{\leq}\ &(75r^3 - 86r^2) \cdot r^2 \overset{(r\geq 74)}{<} r^6 = r^{\mathrm{ex}(5,K_3)}. \end{aligned}$$

In the case $n = 6$, inequality (16) implies that the number of K_3^R-free r-colorings of any 6-vertex graph containing K_3 is at most

$$\begin{aligned} &((3r^2 - 3r)(5r-2)^3 + r(7r-6)^3) \cdot (3r^2 - 2r) < 9 \cdot 125 r^7 + 3 \cdot 7^3 r^6 \\ < \;& 2154 r^7 \overset{(r \geq 47)}{<} r^9 = r^{\mathrm{ex}(6, K_3)}. \end{aligned}$$

To conclude the induction, let G be an n-vertex graph with $n \geq 7$. If G is triangle-free, then G has at most $\mathrm{ex}(n, K_3)$ edges and we are done, so assume that G contains a triangle with vertex set T. By the induction hypothesis, the number of K_3^R-free r-colorings of $G - T$ is at most $r^{\mathrm{ex}(n-3, K_3)}$. By (16), the total number of K_3^R-free r-colorings of G is at most

$$\begin{aligned} & \left((3r^2 - 3r)(5r-2)^{n-3} + r(7r-6)^{n-3}\right) \cdot r^{\mathrm{ex}(n-3, K_3)} \\ < \;& \left(3r \cdot 5^{n-3} + 7^{n-3}\right) \cdot r^{\mathrm{ex}(n-3, K_3) + n - 2} \\ \overset{(r \geq 10, n \geq 7)}{<} \;& r \cdot 7^{n-3} \cdot r^{\mathrm{ex}(n-3, K_3) + n - 2} \overset{(r \geq 49)}{\leq} r^{\frac{n-1}{2}} \cdot r^{\mathrm{ex}(n-3, K_3) + n - 2} \leq r^{\mathrm{ex}(n, K_3)}. \end{aligned}$$

This concludes the induction. □

Note that the above argument already has a bottleneck at the base of induction (see the case $n = 5$), which suggests that the number of K_3^R-free r-colorings of small graphs should be computed more precisely. As it turns out, to get our result for all $r \geq 10$, we will need the following two results, whose proofs require some careful case analysis and are therefore postponed to the appendix.

Lemma 4.5. *The number of K_3^R-free r-edge-colorings of K_5 is*

$$105 r^4 - 120 r^3 + 136 r^2 - 120 r. \tag{17}$$

Lemma 4.6. *The following statements hold.*

(a) *The number of ways to extend any K_3^R-free r -coloring of a complete graph K_4 to a K_3^R-free r-coloring of a complete graph K_5 is at most $15r - 14$.*
(b) *The number of ways to extend any K_3^R-free r-coloring of K_5 to a K_3^R-free r-coloring of the K_6 is at most $31r - 18$.*

To prove that the Turán graph admits the largest number of K_3^R-free r-colorings whenever $r \geq 10$ and $n \geq 5$, we first need to understand what happens for smaller values of n.

Lemma 4.7. *The following statements hold for $r \geq 10$.*

(a) *For a 3-vertex graph G, we have $|\mathcal{C}_{r, K_3^R}(G)| \leq 3r^2 - 2r$, where equality is achieved if and only if $G = K_3$.*
(b) *For any 4-vertex graph G, we have*

$$|\mathcal{C}_{r, K_3^R}(G)| \leq \max\{r^4, 15r^3 - 14r^2\}.$$

Equality is achieved only by the Turán graph $T_2(4)$ for $r \geq 15$ and by the complete graph K_4 for $r \leq 13$. For $r = 14$, equality is achieved if and only if $G \in \{T_2(4), K_4\}$. If we restrict to $G \neq K_4$, the maximum is achieved by $T_2(4)$ for all $r \geq 10$.

Proof. For (a), any other 3-vertex graph $G \neq K_3$ has at most two edges and $r^2 < 3r^2 - 2r$ for all $r \geq 10$.

For (b), observe that Corollary 4.3 implies that K_4 admits the largest number of K_3^R-free colorings among all 4-vertex graphs that contain a triangle, followed by $K_4 - e$. It is obvious that $T_2(4)$ admits the largest number of such colorings for triangle-free graphs, so that the first statement follows from comparing r^4 with $15r^3 - 14r^2$. The second statement comes from comparing r^4 with $r(3r-2)^2$. □

Proof of Theorem 1.2. The result is trivially true if we restrict to triangle-free graphs, so let G be a graph that contains a triangle. By induction, we shall show that G admits fewer than $r^{\mathrm{ex}(n,K_3)}$ distinct K_3^R-free r-colorings. We distinguish three cases according to the *clique number* $\omega(G)$ of G (as usual $\omega(G)$ is the size of a largest clique in G).

Case 1: First assume that $\omega(G) = 3$, so that G is K_4-free. Let $W = \{w_1, w_2, w_3\}$ induce a triangle in G and let $X = V(G) - W = \{x_1, \ldots, x_{n-3}\}$. The fact that G is K_4-free implies that each vertex x_i has at most two neighbors in W, so that the number $e(W, X)$ of edges between W and X satisfies $e(W, X) \leq 2(n-3)$. By Lemma 4.2(a), the number of ways to color the at most two edges connecting x_i to the triangle W is bounded above by $\max\{r, 3r-2\} = 3r-2$.

First assume that $e(W, X) < 2(n-3)$. Our comments in the previous paragraph imply that, starting with a K_3^R-free r-coloring of the triangle W, there are at most $r(3r-2)^{n-4}$ ways to color the edges in $e(W, X)$ without creating a rainbow triangle. The edges in $G[X]$ may be colored in at most r and $(3r^2 - 2r)$, respectively, for $n = 5$ and $n = 6$. For $n = 5$, the desired result follows from

$$(3r^2 - 2r)(3r-2)r^2 < 9r^5 \overset{(r \geq 9)}{\leq} r^6 = r^{\mathrm{ex}(5,K_3)},$$

while for $n = 6$ we have

$$(3r^2 - 2r)^2 r(3r-2)^2 < 81r^7 \overset{(r \geq 9)}{\leq} r^9 = r^{\mathrm{ex}(6,K_3)}. \tag{18}$$

For $n \geq 7$, since G is K_4-free, the edges in $G[X]$ can be colored in at most $r^{\mathrm{ex}(n-3,K_3)}$ ways (this uses the induction hypothesis for $n \geq 8$ and Lemma 4.7(b) for $n = 7$). We conclude that the number of K_3^R-free r-colorings of G is at most

$$\begin{aligned} &(3r^2 - 2r) \cdot r \cdot (3r-2)^{n-4} \cdot r^{\mathrm{ex}(n-3,K_3)} < 3^{n-3} \cdot r^{n-1} \cdot r^{\mathrm{ex}(n-3,K_3)} \\ \overset{(r \geq 9)}{\leq} \; & r^{\frac{n-3}{2}} \cdot r^{n-1} \cdot r^{\frac{(n-3)^2}{4}} = r^{\frac{(n-1)(n+1)}{4}} \leq r^{\mathrm{ex}(n,K_3)}. \end{aligned} \tag{19}$$

When $e(W, X) = 2(n-3)$, we need to look at colorings more carefully. We start with the case $n \geq 5$ and $n \neq 6$, as some of our remarks in this case will be enough to handle the case $n = 6$. The arguments from the previous paragraph lead to an upper bound of

$$(3r^2 - 2r)(3r-2)^{n-3} r^{\mathrm{ex}(n-3,K_3)} < 3^{n-2} \cdot r^{n-1} \cdot r^{\mathrm{ex}(n-3,K_3)} < r^{\mathrm{ex}(n,K_3)}. \tag{20}$$

The inequality above holds for all even values of $n \geq 8$, and for $n = 13$, which will be sufficient for further considerations. We will consider the cases $n = 5$, $n = 7$, $n = 9$, $n = 11$ and $n \geq 14$ separately.

For $n \in \{5, 7, 9, 11\}$, we analyse the situation more carefully. If X induces an independent set, the number of colorings is bounded above by $(3r^2 - 2r)(3r-2)^{n-3} < 3^{n-2} r^{n-1} < r^{\mathrm{ex}(n,K_3)}$ for $r \geq 3$, so assume that $\{x_i, x_j\}$ is an edge of G. Without loss of generality, assume that x_i is adjacent to w_1 and w_2, while x_j is adjacent to w_2 and w_3 (the absence of K_4 implies that they cannot have the same neighbors). Again assume that we have colored all edges incident with vertices in W. If $e_i = \{x_i, w_2\}$ and $e_j = \{x_j, w_2\}$ are assigned the same color, the number of extensions to $G[X]$ is at most $r^{\mathrm{ex}(n-3,K_3)}$, while, if e_i and e_j have different colors, the edge $\{x_i, x_j\}$ must be assigned one of these two colors and the number of extensions is at most $2 \cdot r^{\mathrm{ex}(n-3,K_3)-1}$. This is due to the fact that, for every choice of color c for the edge $\{x_i, x_j\}$, the number of extensions to a coloring of $G[X]$ is the same, and summing these numbers over all possible colors c, we get at most $r^{\mathrm{ex}(n-3,K_3)}$. To conclude the proof of this part, we will use the fact that the number of ways to extend a coloring of W to the edges connecting x_i and x_j to W in such a way that e_i and e_j have the same color is at most $(r^2 + 4r - 4)$. To see why this is true, let a be the color of e_i and e_j and first assume that the edges $\{w_1, w_2\}$ and $\{w_2, w_3\}$ have the same color b. If $b \neq a$, each of the remaining two edges $\{w_1, x_i\}$ and $\{w_3, x_j\}$ can only be assigned b or a. If $b = a$, the remaining two edges may be colored arbitrarily. Thus we have $2^2(r-1)+r^2 = r^2+4r-4$ extensions in this case. Next assume that the colors of $\{w_1,w_2\}$ and $\{w_2,w_3\}$ are different, say b and c. There are $(r-2)$ ways to choose a color $a \notin \{b,c\}$ for e_i and e_j. Then each of the remaining two edges can only be colored in two ways. If $a=b$, the edge $\{w_1,x_i\}$ can be colored in r ways, while $\{w_3,x_j\}$ can only be assigned b or c. The case $a=c$ is analogous. Thus we have $4(r-2)+2r+2r = 8r-8 \leq r^2+4r-4$ extensions in this case, which establishes our claim.

There are $(3r^2-2r)$ ways to color the triangle W. Once we extend this coloring to the edges between W and X, we know that the number of ways to color $G[X]$ is larger when the edges e_i and e_j are assigned the same color. With the upper bounds obtained above, we deduce that the total number of colorings of G is bounded above by

$$(3r^2-2r)(3r-2)^{n-5}\left(r^{\lfloor\frac{(n-3)^2}{4}\rfloor}(r^2+4r-4)+2r^{\lfloor\frac{(n-3)^2}{4}\rfloor-1}[(3r-2)^2-(r^2+4r-4)]\right), \quad (21)$$

which is less than $r^{\mathrm{ex}(n,K_3)}$ for $n\in\{5,7,9,11\}$ and $r\geq 9$.

In the case $n=6$ we may proceed similarly. If X is an independent set, the inequality (18) becomes

$$(3r^2-2r)(3r-2)^3 < 3^4\cdot r^5 \overset{(r\geq 9)}{\leq} r^9 = r^{\mathrm{ex}(6,K_3)},$$

while, if there is an edge $\{x_i,x_j\}$, it may be replaced by

$$(3r^2-2r)(3r-2)\left[(3r^2-2r)(r^2+4r-4)+2(3r-2)[(3r-2)^2-(r^2+4r-4)]\right], \quad (22)$$

which is less than $r^{\mathrm{ex}(6,K_3)} = r^9$ for $r\geq 8$.

To conclude the proof of this case, we will use the upper bound (20) and show that

$$r^{\mathrm{ex}(n,K_3)} > (3r^2-2r)(3r-2)^{n-3}r^{\mathrm{ex}(n-3,K_3)} \quad (23)$$

for all $n\geq 14$. This will be done by induction on n, where the base is given by $n\in\{12,13\}$, for which this inequality has already been verified. Also recall that $r^{\mathrm{ex}(n,K_3)}$ is the maximum number of colorings of any K_4-free graph G on $n\in\{9,\ldots,13\}$ vertices.

For $n\geq 14$, we have, by the induction hypothesis,

$$\begin{aligned} r^{\mathrm{ex}(n,K_3)} &= r^{n-1+\mathrm{ex}(n-2,K_3)} > r^{n-1}(3r^2-2r)(3r-2)^{n-5}r^{\mathrm{ex}(n-5,K_3)} \\ &= r^3(3r^2-2r)(3r-2)^{n-5}r^{\mathrm{ex}(n-3,K_3)} > (3r^2-2r)(3r-2)^{n-3}r^{\mathrm{ex}(n-3,K_3)}, \end{aligned}$$

as required. In the last inequality, we used that $r^3 > (3r-2)^2$ for $r\geq 10$. This concludes the proof of our theorem in Case 1.

Case 2: Assume that $\omega(G)=4$, where a 4-clique is induced by $W=\{w_1,w_2,w_3,w_4\}$. Let $X = V(G)-W = \{x_1,\ldots,x_{n-4}\}$. As G is K_5-free, each vertex from X has at most three neighbors in W. Combining this with Lemma 4.2, we deduce that the number of ways to color the edges connecting some x_i to W is bounded above by $\max\{r,3r-2,5r-2,7r-6\} = 7r-6$. Thus, once the edges of $G[W]$ are colored, the edges connecting W with X can be colored in at most $(7r-6)^{n-4}$ ways.

For $n\in\{5,6\}$, we may first color $G[W]\cong K_4$ in $(15r^3-14r^2)$ ways (see Lemma 4.7(b)), extend any such coloring to the edges between W and X in at most $(7r-6)^{n-4}$ ways, while (for $n=6$) the edge $\{x_1,x_2\}$, if present, may be colored in r ways. With these arguments, for $n=5$ and $r\geq 10$ we have at most

$$(15r^3-14r^2)(7r-6) < r^6 = r^{ex(5,K_3)}$$

colorings while for $n=6$ our result follows from the fact that

$$(15r^3-14r^2)(7r-6)^2 r < 15\cdot 49\cdot r^6 \overset{(r\geq 10)}{<} r^9.$$

For $n = 7$, we use parts (a) and (b) of Lemma 4.7 to deduce that $G[X]$ and $G[W]$ may be colored in at most $(3r^2 - 2r)$ and $(15r^3 - 14r^2)$, respectively. The edges between X and W may be colored in at most $(7r - 6)^3$ ways. Our result follows from

$$(15r^3 - 14r^2)(7r - 6)^3(3r^2 - 2r) < r^{12}$$

for $r \geq 10$. This inequality may be verified directly for $r \in \{10, 11\}$, while for $r \geq 12$ the left-hand side is bounded above by

$$15 \cdot 7^3 \cdot 3 \cdot r^8 < 12^4 r^8 \leq r^{12}. \tag{24}$$

For $n = 8$ and $10 \leq r \leq 13$, by Lemma 4.7(b), the subgraph $G[X]$ has at most $15r^3 - 14r^2$ distinct K_3^R-free r-colorings, so that the number of such colorings in G is at most

$$(15r^3 - 14r^2)(7r - 6)^4(15r^3 - 14r^2),$$

which is less than r^{16} for $10 \leq r \leq 13$.

For all other cases where $n \geq 8$ and $r \geq 10$, the induced subgraph $G[X]$ has at most $r^{\mathrm{ex}(n-4,K_3)}$ distinct K_3^R-free r-colorings, either by Lemma 4.7(b) or by the induction hypothesis. Hence the number of K_3^R-free r-colorings of G is at most

$$(15r^3 - 14r^2)(7r - 6)^{n-4} r^{\mathrm{ex}(n-4,K_3)} \overset{(r \geq 10, n \geq 8)}{<} r^{\mathrm{ex}(n,K_3)}.$$

To see the inequality above, note that $\mathrm{ex}(n, K_3) - \mathrm{ex}(n - 4, K_3) = 2n - 4$, while

$$\begin{aligned}(15r^3 - 14r^2)(7r - 6)^{n-4} &\leq 15r^3 \cdot \left(\frac{7}{r}\right)^{n-4} \cdot r^{2n-8} \\ &\overset{(r\geq 10)}{\leq} 15 \cdot \left(\frac{7}{10}\right)^4 \cdot r^{2n-5} < 10 \cdot r^{2n-5} \overset{(r \geq 10)}{\leq} r^{2n-4}.\end{aligned}$$

Case 3: Finally, assume that $\omega(G) \geq 5$ where a 5-clique is induced by $W = \{w_1, w_2, w_3, w_4, w_5\}$. Let $X = V(G) - W = \{x_1, \ldots, x_{n-5}\}$. By Lemma 4.5 the graph $G[W]$ admits $(105r^4 - 120r^3 + 136r^2 - 120r)$ distinct K_3^R-free r-colorings. Each vertex from X has at most five neighbors in W. Combining this with Lemmas 4.2 and 4.6 we deduce that the number of ways to color the edges connecting some x_i to W is bounded above by $\max\{r, 3r - 2, 5r - 2, 7r - 6, 15r - 14, 31r - 18\} = 31r - 18$ Thus, once the edges of $G[W]$ are colored, the edges connecting W to X can be colored in at most $(31r - 18)^{n-5}$ ways.

In the case $n = 5$ the graph G is a K_5, and we have

$$|\mathcal{C}_{r,K_3^R}(K_5)| = 105r^4 - 120r^3 + 136r^2 - 120r \leq 105r^4 - 106r^3 \overset{(r\geq 10)}{<} r^6 = r^{\mathrm{ex}(5,K_3)}.$$

For $n = 8$, by Lemma 4.7(a), $G[X]$ has at most $(3r^2 - 2r)$ distinct K_3^R-free r-colorings. The number of K_3^R-free r-colorings of G is at most

$$(105r^4 - 120r^3 + 136r^2 - 120r)(31r - 18)^3(3r^2 - 2r) < 315 \cdot 31^3 \cdot r^9 \overset{(r\geq 10)}{<} r^{16}.$$

For $n = 9$ and $10 \leq r \leq 13$, by Lemma 4.7(b), $G[X]$ has at most $15r^3 - 14r^2$ distinct K_3^R-free r-colorings. The number of K_3^R-free r-colorings of G is at most

$$(105r^4 - 120r^3 + 136r^2 - 120r)(31r - 18)^4(15r^3 - 14r^2),$$

which is less than r^{20} for $10 \leq r \leq 13$.

For all other cases where $n \geq 6$, the induced subgraph $G[X]$ has at most $r^{\mathrm{ex}(n-5,K_3)}$ distinct K_3^R-free r-colorings, either by Lemma 4.7 or by the induction hypothesis. Hence the number of K_3^R-free r-colorings of G is at most

$$(105r^4 - 120r^3 + 136r^2 - 120r)(31r - 18)^{n-5} r^{\mathrm{ex}(n-5,K_3)} < r^{\mathrm{ex}(n,K_3)}.$$

To see why the last inequality holds, note that $\mathrm{ex}(n, K_3) - \mathrm{ex}(n-5, K_3) \geq (5/2)n - (13/2)$, while

$$(105r^4 - 120r^3 + 136r^2 - 120r)(31r - 18)^{n-5} \leq 105r^4 \cdot \left(\frac{31}{r^{3/2}}\right)^{n-5} \cdot r^{\frac{5}{2}n - \frac{25}{2}}$$

$$\overset{(n\geq 6)}{\leq} \quad 105 \cdot \left(\frac{31}{10^{3/2}}\right) \cdot r^{\frac{5}{2}n - \frac{17}{2}} \overset{(r \geq 10)}{<} 100 \cdot r^{\frac{5}{2}n - \frac{17}{2}} \leq r^{\frac{5}{2}n - \frac{13}{2}}.$$

In conclusion, for $r \geq 10$ and $n \geq 5$, the Turán graph $T_2(n)$ for K_3 on n vertices allows the maximum number of K_3^R-free r-edge colorings among all graphs on n vertices, and it is the only extremal graph in these cases. □

Remark 4.8. Theorem 1.2 cannot be improved to smaller values of n or r without affecting the other parameter, as K_4 admits more K_3^R-free r-colorings than $T_2(4)$ for all $r \leq 13$ and K_5 admits more K_3^R-free 9-colorings than $T_2(5)$.

Moreover, there are several steps in our proof of Theorem 4.8 where the bound $r \geq 10$ is needed. For instance, several inequalities in Case 3 do not hold for smaller values of r, which suggests that the case when the graph G satisfies $\omega(G) = 5$ should be treated separately from graphs with larger clique number.

Hence, using the current approach to obtain similar results for smaller values of r and $n \geq n_0$, where $n_0 > 5$ (but much smaller than in Theorem 1.1) would require more precise calculations (for instance, the exact number of colorings of graphs with five and six vertices) and tighter bounds, probably leading to long and tedious arguments.

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5. Appendix

In this appendix, we prove the two technical lemmas whose proofs have been omitted from Section 4. At first we restate Lemma 4.5.

Lemma 4.5. *The number of K_3^R-free r-edge-colorings of K_5 is*

$$105r^4 - 120r^3 + 136r^2 - 120r.$$

Proof. Let $V(K_5) = \{v_1, v_2, v_3, v_4, w\}$. We distinguish three cases, depending on the number of colors used for the K_4-subgraph $\{v_1, v_2, v_3, v_4\}$.

Case 1: Given one of the r monochromatic colorings of the complete graph on $\{v_1, v_2, v_3, v_4\}$, say in color a, by Lemma 4.1 only one of the remaining $(r-1)$ colors can additionally be used for any coloring of the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$. There is exactly one way to color the $K_5 = \{v_1, v_2, v_3, v_4, w\}$ with one color a. Let $b \neq a$ be another color. Then there are $2^4 - 1$ ways to assign colors a or b to the edges $\{v_i, w\}$ such that color b occurs at least once, which so far altogether gives a number of K_3^R-free r-colorings of

$$r + 15r(r-1). \tag{25}$$

In particular, when $K_4 = \{v_1, v_2, v_3, v_4\}$ has a monochromatic edge coloring using color a, there are

$$1 + 15(r-1) = 15r - 14 \tag{26}$$

ways to extend it to an edge coloring of $K_5 = \{v_1, v_2, v_3, v_4, w\}$ using only a and a color $b \neq a$.

Case 2: The number of colorings of K_4 where exactly two different colors a and b are used is $(2^6 - 2)$. Assume that we are given an a, b-coloring of the edges of the complete graph $\{v_1, v_2, v_3, v_4\}$. There are $2^4 = 16$ ways to color the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$, using only the colors a and b. By Lemma 4.1 only one of the remaining $(r-2)$ colors can additionally be used for any coloring of the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$. Therefore, given a specific a, b-coloring of the edges of the complete graph $\{v_1, v_2, v_3, v_4\}$, we will count the number of a, b, c-colorings of the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$, where the color $c \notin \{a, b\}$, is used at least once. There are five non-isomorphic color patterns $K^{(1)}, \ldots, K^{(5)}$ for coloring the edges of the complete graph $\{v_1, v_2, v_3, v_4\}$ using exactly two colors a and b. They are represented in Figure 1.

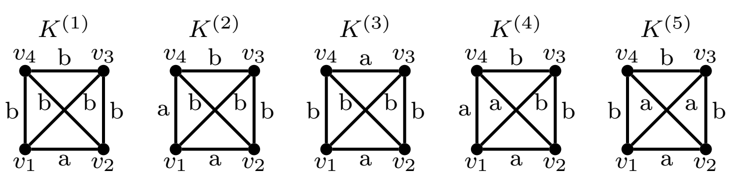

Figure 1. Five non-isomorphic patterns

Since the roles of the colors a and b can be interchanged in the color patterns, there are twelve ways to apply color pattern $K^{(1)}$ to the K_4, 24 ways for pattern $K^{(2)}$, six ways for $K^{(3)}$, eight ways for $K^{(4)}$ and twelve ways for $K^{(5)}$, which amounts to $2^6 - 2$ color patterns for the K_4 where the colors a and b occur at least once.

Case 2.1 - $K^{(1)}$: Let $\{v_1, v_2\}$ be colored a and all other edges of the complete graph $\{v_1, v_2, v_3, v_4\}$ be colored b. Let exactly one of the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$, be colored $c \neq a, b$, say, edge $\{v_1, w\}$. Since all triangles in the K_5 that contain the edge $\{v_1, w\}$ now already have two edges with two different colors, for every such triangle there is exactly one way to assign one of the colors a or b to the remaining third edge. Since there are four ways to assign color c to one of the edges $\{v_i, w\}$, we end up with four different colorings. Clearly this holds also for the patterns $K^{(2)}, \ldots, K^{(5)}$. Now let exactly two of the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$, be colored c. When $\{v_1, w\}$ and $\{v_2, w\}$ are colored c, both edges $\{v_3, w\}$ and $\{v_4, w\}$ have to be colored b. When $\{v_3, w\}$ and $\{v_4, w\}$ are colored c, both edges $\{v_1, w\}$ and $\{v_2, w\}$ have to be colored b. If any other pair of edges $\{v_i, w\}$ is colored c, this coloring cannot be extended to a K_3^R-free r-coloring using only the colors a and b. Thus there are two different colorings in this case. Now let exactly three of the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$, be colored c. If $\{v_1, w\}$, $\{v_2, w\}$ and $\{v_3, w\}$ are colored c, edge $\{v_4, w\}$ has to be colored b. If $\{v_1, w\}$, $\{v_2, w\}$ and $\{v_4, w\}$ are colored c, edge $\{v_3, w\}$ has to be colored b. These two are the only K_3^R-free r-colorings in this case. If we color all the four edges $\{v_i, w\}$, $i = 1, 2, 3, 4$, with color c, simultaneously, we never end up with a rainbow triangle, for any of the five patterns. Hence there are 16 ways to extend $K^{(1)}$ to a K_3^R-free r-coloring of K_5 using only the colors a and b and, after choosing one color $c \notin \{a, b\}$ out of the remaining $(r-2)$ colors, there are $4 + 2 + 2 + 1 = 9$ ways to extend $K^{(1)}$ to a K_3^R-free r-coloring of K_5 using only the colors a, b and c in such a way that color c is assigned to at least one edge.

Case 2.2 - $K^{(2)}$: Let the edges $\{v_1, v_2\}$ and $\{v_2, v_4\}$ be colored a and all other edges of the complete graph $\{v_1, v_2, v_3, v_4\}$ be colored b. If the edges $\{v_1, w\}$ and $\{v_4, w\}$ are colored c, then $\{v_2, w\}$ has to be colored a and $\{v_3, w\}$ has to be colored b. This is the only way to extend $K^{(2)}$ to a coloring of the $K_5 = \{v_1, v_2, v_3, v_4, w\}$ where exactly two of the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$, are colored c. If we assign color c to exactly three of the edges $\{v_i, w\}$, the only way that results in a K_3^R-free r-coloring of the K_5 is using color c on $\{v_1, w\}$, $\{v_2, w\}$ and $\{v_4, w\}$ and using color b on edge $\{v_3, w\}$. Hence there are 16 ways to extend $K^{(2)}$ to a K_3^R-free r-coloring of K_5 using only the colors a and b and, after choosing one color $c \notin \{a, b\}$ out of the remaining $(r-2)$ colors, there are $4 + 1 + 1 + 1 = 7$ ways to extend $K^{(2)}$ to a K_3^R-free r-coloring of K_5 using only the colors a, b and c in such a way that color c is assigned to at least one edge.

Case 2.3 - $K^{(3)}$: Let the edges $\{v_1, v_2\}$ and $\{v_3, v_4\}$ be colored a and all other edges of the complete graph $\{v_1, v_2, v_3, v_4\}$ be colored b. A K_3^R-free r-coloring where we color exactly two of the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$, with color c can be achieved in only two ways. Either we assign color c to $\{v_1, w\}$ and $\{v_2, w\}$, which results in assigning color b to $\{v_3, w\}$ and to $\{v_4, w\}$, or we color $\{v_3, w\}$ and $\{v_4, w\}$ with c, which results in assigning color b to $\{v_1, w\}$ and $\{v_2, w\}$. There is no K_3^R-free r-coloring where exactly three of the edges $\{v_i, w\}$ are colored c. Hence there are 16 ways to extend $K^{(3)}$ to a K_3^R-free r-coloring of K_5 using only the colors a and b and, after choosing one color $c \notin \{a, b\}$ out of the remaining $(r-2)$ colors, there are $4 + 2 + 0 + 1 = 7$ ways to extend $K^{(3)}$ to a K_3^R-free r-coloring of K_5 using only the colors a, b and c in such a way that color c is assigned to at least one edge.

Case 2.4 - $K^{(4)}$: Let the edges $\{v_1, v_2\}$, $\{v_1, v_4\}$ and $\{v_2, v_4\}$ be colored a and all other edges of the complete graph $\{v_1, v_2, v_3, v_4\}$ be colored b. A K_3^R-free r-coloring where we color exactly two of the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$, with color c can be achieved only by either assigning color c to the edges $\{v_1, w\}$ and $\{v_4, w\}$, to the edges $\{v_1, w\}$ and $\{v_2, w\}$ or to the edges $\{v_2, w\}$ and $\{v_4, w\}$. In each of these equivalent cases there is only one way to color the remaining edges. There is only one way to assign color c to three of the edges $\{v_i, w\}$, namely, the case where $\{v_1, w\}$, $\{v_2, w\}$ and $\{v_4, w\}$ are colored c and $\{v_4, w\}$ is colored b. Hence there are 16 ways to extend $K^{(4)}$ to a K_3^R-free r-coloring of K_5 using only the colors

a and b and, after choosing one color $c \notin \{a,b\}$ out of the remaining $(r-2)$ colors, there are $4+3+1+1=9$ ways to extend $K^{(4)}$ to a K_3^R-free r-coloring of K_5 using only the colors a, b and c in such a way that color c is assigned to at least one edge.

Case 2.5 - $K^{(5)}$: Let the edges $\{v_3,v_1\}$, $\{v_1,v_2\}$ and $\{v_2,v_4\}$ be colored a and all other edges of the complete graph $\{v_1,v_2,v_3,v_4\}$ be colored b. For this color pattern there are no K_3^R-free r-colorings of the $K_5 = \{v_1,v_2,v_3,v_4,w\}$ using only colors a, b and c where exactly two or three of the edges $\{v_i,w\}$, $i=1,2,3,4$, are colored c. Therefore, there are 16 ways to extend $K^{(5)}$ to a a K_3^R-free r-coloring of K_5 using only the colors a and b and, after choosing one color $c \notin \{a,b\}$ out of the remaining $(r-2)$ colors, there are $4+0+0+1=5$ ways to extend $K^{(5)}$ to a K_3^R-free r-coloring of K_5 using only the colors a, b and c in such a way that color c is assigned to at least one edge.

Note that from the five cases above it follows that when $K_4 = \{v_1,v_2,v_3,v_4\}$ has an edge-coloring using the two colors a and b, there are at most

$$16 + 9(r-2) = 9r - 2 \tag{27}$$

ways to extend it to a K_3^R-free edge coloring of $K_5 = \{v_1,v_2,v_3,v_4,w\}$ using only colors from $\{a,b\}$ or a color $c \notin \{a,b\}$. In case the color c has also been fixed before and cannot be chosen, this expression becomes

$$16 + 9 = 25. \tag{28}$$

Given r colors, there are $\binom{r}{2}$ ways to choose colors a and b and we have additional $(r-2)$ choices for color c. Thus, the total number of K_3^R-free r-colorings of the $K_5 = \{v_1,v_2,v_3,v_4,w\}$ where the induced K_4 on $\{v_1,v_2,v_3,v_4\}$ is colored with one of the patterns $K^{(1)},\ldots,K^{(5)}$ is

$$\begin{aligned} & \binom{r}{2}(12\cdot 16 + 24\cdot 16 + 6\cdot 16 + 8\cdot 16 + 12\cdot 16) \\ + & \binom{r}{2}(r-2)(12\cdot 9 + 24\cdot 7 + 6\cdot 7 + 8\cdot 9 + 12\cdot 5) \\ = & \; 225r^3 - 179r^2 - 46r. \end{aligned} \tag{29}$$

Case 3: Assume that we are given an a,b,c-coloring of the edges of the complete graph $\{v_1,v_2,v_3,v_4\}$ that does not contain a rainbow triangle. There are only two color patterns of this kind. Pattern $K^{(1)}$ consists of an a-colored star S_3, a b-colored star S_2 and the remaining edge is colored c. Pattern $K^{(2)}$ has an edge-monochromatic C_4 while each of the remaining two colors is assigned to one of the remaining two edges. By interchanging the roles of colors a, b and c, there are 72 ways to apply pattern $K^{(1)}$ to $\{v_1,v_2,v_3,v_4\}$ and 18 ways for $K^{(2)}$. By Lemma 4.1 only one of the remaining $(r-3)$ colors can additionally be used for any coloring of the edges $\{v_i,w\}$, $i=1,2,3,4$. Let d be a color different from a,b and c.

Case 3.1 - $K^{(1)}$: Let the edges $\{v_1,v_2\}$, $\{v_1,v_3\}$ and $\{v_1,v_4\}$ be colored a, let $\{v_2,v_3\}$ and $\{v_3,v_4\}$ be colored b and let $\{v_2,v_4\}$ be colored c. There are eight ways to color the edges $\{v_i,w\}$ using only the colors a and b, since two of the edges are incident with $\{v_2,v_4\}$ and therefore must be both colored a or b. Now we count the colorings of the edges $\{v_i,w\}$ with colors a, b or c where color c occurs exactly once. If $\{v_2,w\}$ or $\{v_4,w\}$ are colored c, there are in both cases two ways to extend the coloring to a K_3^R-free r-coloring for the K_5 with vertex set $\{v_1,v_2,v_3,v_4,w\}$. If $\{v_1,w\}$ or $\{v_3,w\}$ are colored c, then there is only one way to color the three remaining edges. Thus we have six K_3^R-free r-colorings of the K_5 where exactly one of the edges $\{v_i,w\}$ is colored c. When exactly two of the edges $\{v_i,w\}$ are assigned color c, we can extend it to a coloring of the K_5 only when the two c-edges form a triangle with one of the b-colored edges or with the only c-colored edge of the $K_4 = \{v_1,v_2,v_3,v_4\}$. The extension is in each case unique, which gives us three K_3^R-free r-colorings. Assigning c to the edges $\{v_2,w\}$,

$\{v_3, w\}$ and $\{v_4, w\}$ leads to $\{v_1, w\}$ being colored a and there is no other way to assign c to three of the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$. Once again, coloring all edges incident with vertex w with color c gives also a K_3^R-free r-coloring of the K_5. Hence there are $8 + 6 + 3 + 1 + 1 = 19$ ways to extend $K^{(1)}$ to a K_3^R-free r-coloring of the K_5 using only the colors a, b and c. Now we count the a, b, c, d-colorings where at least one of the edges $\{v_i, w\}$ is colored d. If exactly one of those edges is colored d, there is exactly one way to extend that coloring to a K_3^R-free r-coloring of the K_5. A K_3^R-free r-coloring where we color exactly two of the edges $\{v_i, w\}$, $i = 1, 2, 3, 4$, with color d can be achieved only by assigning color d to the edges $\{v_2, w\}$ and $\{v_4, w\}$ and in this case there is only one way to color $\{v_1, w\}$ and $\{v_3\}$. Assigning color d to three of the edges $\{v_i, w\}$ leads to a coloring of the K_5 only when we use d for $\{v_2, w\}$, $\{v_3, w\}$ and $\{v_4, w\}$. The remaining edge $\{v_1, w\}$ has to be colored a. We also may use color d on all four edges $\{v_i, w\}$ simultaneously. Thus the number of a, b, c, d-colorings of the edges $\{v_i, w\}$ for a given pattern $K^{(1)}$ for $K_4 = \{v_1, v_2, v_3, v_4\}$ where color d is used at least once amounts to $4 + 1 + 1 + 1 = 7$. Hence the pattern can be extended to a K_3^R-free r-coloring of K_5 in 19 ways using only colors a, b and c, and after choosing one color $d \notin \{a, b, c\}$ out of the remaining $(r - 3)$ colors, there are 7 ways to extend pattern $K^{(1)}$ to a K_3^R-free r-coloring of K_5 using only colors a, b, c and d in such a way that color d is assigned to at least one edge.
Case 3.2 - $K^{(2)}$: Assign color b to edge $\{v_1, v_4\}$, assign color c to edge $\{v_2, v_3\}$ and let all the remaining edges be colored a. Similar to pattern $K^{(1)}$, there are eight ways of coloring the edges $\{v_i, w\}$ using only the colors a and b. Again we consider the number of a, b, c-colorings of the edges $\{v_i, w\}$ where the color c appears at least once. If the edge $\{v_2, w\}$ (resp. $\{v_3, w\}$) is the only edge assigned color c, the edge $\{v_3, w\}$ (resp. $\{v_2, w\}$) can be colored a or b while the remaining two edges must be colored a. If $\{v_1, w\}$ or $\{v_4, w\}$ is colored c, there is only one way to color the remaining edges. Thus we have six colorings if c appears exactly once. When exactly two of the edges $\{v_i, w\}$ must have color c, we end up with one K_3^R-free r-coloring for c-colored edges $\{v_2, w\}$ and $\{v_3, w\}$ and with another one for c-colored edges $\{v_1, w\}$ and $\{v_4, w\}$, and there is no K_3^R-free r-coloring for all other selections of pairs from $\{v_i, w\}$. Coloring exactly three of the edges $\{v_i, w\}$ with color c leads to two K_3^R-free r-colorings, namely one for assigning c to $\{v_1, w\}$, $\{v_2, w\}$ and $\{v_4, w\}$ and one for using c on $\{v_1, w\}$, $\{v_3, w\}$ and $\{v_4, w\}$. Together with the case where all the edges incident with vertex w are colored c, so far we have $8 + 6 + 2 + 2 + 1 = 19$ K_3^R-free r-colorings of the K_5. Counting a, b, c, d-colorings of the edges incident with w, when coloring one of these edges with color d, there is exactly one way to color the remaining edges using only the colors a, b and c. When using d exactly twice, we end up only with K_3^R-free r-colorings when d is either assigned to $\{v_2, w\}$ and $\{v_3, w\}$ or to the edges $\{v_1, w\}$ and $\{v_4, w\}$. There are no colorings where d is used three times, but it certainly can be assigned to all the four edges $\{v_i, w\}$. Thus the number of such a, b, c, d-colorings of the edges incident with w is $4 + 2 + 0 + 1 = 7$. Therefore, the pattern can be extended to a K_3^R-free r-coloring of K_5 in 19 ways using only colors a, b and c, and after choosing one color $d \notin \{a, b, c\}$ out of the remaining $(r - 3)$ colors, there are 7 ways to extend pattern $K^{(2)}$ to a K_3^R-free r-coloring of K_5 using only colors a, b, c and d in such a way that color d is assigned to at least one edge.

Note that from the discussion above it follows that when $K_4 = \{v_1, v_2, v_3, v_4\}$ has an edge-coloring using the three colors a, b and c, there are at most

$$19 + 7(r - 3) = 7r - 2 \tag{30}$$

ways to extend it to a K_3^R-free edge coloring of $K_5 = \{v_1, v_2, v_3, v_4, w\}$ using only colors from $\{a, b, c\}$ or a color $d \notin \{a, b, c\}$. In case the color d has also been fixed before and cannot be chosen, this expression becomes

$$19 + 7 = 26. \tag{31}$$

Given r colors, there are $\binom{r}{3}$ ways to choose colors a, b and c and we have additional $(r-3)$ choices for color d. Thus, the total number of K_3^R-free r-colorings of the $K_5 = \{v_1, v_2, v_3, v_4, w\}$ where the induced K_4 on $\{v_1, v_2, v_3, v_4\}$ is colored with one of the patterns $K^{(1)}$ or $K^{(2)}$ is

$$\binom{r}{3}(72 \cdot 19 + 18 \cdot 19) + \binom{r}{3}(r-3)(72 \cdot 7 + 18 \cdot 7) \tag{32}$$

$$= \quad 105r^4 - 345r^3 + 300r^2 - 60r. \tag{33}$$

We obtain the total number of K_3^R-free r-colorings of K_5 by summing (25), (29) and (33), which is $105r^4 - 120r^3 + 136r^2 - 120r$. □

From the proof of Lemma 4.5, we may derive a proof of Lemma 4.6, which is restated for convenience.

Lemma 4.6. *The following statements hold.*

(a) *The number of ways to extend any K_3^R-free r-coloring of a complete graph K_4 to a K_3^R-free r-coloring of a complete graph K_5 is at most $15r - 14$.*
(b) *The number of ways to extend any K_3^R-free r-coloring of K_5 to a K_3^R-free r-coloring of the K_6 is at most $31r - 18$.*

Proof. Part (a) may be derived from the proof of Lemma 4.5 by looking at the number of extensions of a coloring of K_4 to K_5 in each case. It turns out that any monochromatic coloring of K_4 may be extended in $15r - 14$ ways, which is more than the number of extensions in any other case.

For part (b), let $\{v_1, v_2, v_3, v_4, v_5\}$ and $\{v_1, v_2, v_3, v_4, v_5, w\}$ be vertex sets of complete graphs K_5 and K_6, respectively. Assume we are given some K_3^R-free r-coloring of K_5. When we know that at least one of the edges incident with w will be colored by a new color $\hat{c}$ that is not assigned to any of the edges of K_5, the total number of extensions of this kind for the coloring of K_5 is at most $2^5 - 1 = 31$, regardless of the color pattern for K_5. The reason for this is that for every edge $\{v_i, w\}$ its color is either $\hat{c}$ or, if it is not colored $\hat{c}$ itself, some other $\hat{c}$-colored edge $\{v_j, w\}$ forces the color of $\{v_i, w\}$ to be the same color that is already assigned to the edge $\{v_i, v_j\}$ in K_5. The number of extensions of this type is at most $31(r - \ell)$, where ℓ is the number of colors occurring in K_5. We distinguish according to the coloring of K_5.

Given a monochromatic coloring of K_5 by color a, there is one way to assign color a to all edges incident with vertex w. If we use one of the $(r-1)$ colors $b \neq a$ for at least one of the edges incident with w and assign color a to all remaining edges, there are 31 ways to a, b-color the edges $\{v_i, w\}$, $i = 1, \ldots, 5$. By Lemma 4.1 there are no other ways to extend the coloring of K_5. Therefore, the number of ways to extend the monochromatic coloring of K_5 to a K_3^R-free r-coloring of K_6 is

$$1 + 31(r-1) \quad = \quad 31r - 30. \tag{34}$$

Given a coloring of K_5 where two different colors a and b occur. The number of ways to extend it to an a, b-coloring of K_6 is $2^5 = 32$. By our discussion at the beginning of this proof, choosing a color $c \notin \{a, b\}$ and using it on at least one of the edges $\{v_i, w\}$ gives at most $31(r-2)$ extensions. Thus the number of ways to extend any a, b-coloring of K_5 to a K_3^R-free r-coloring of K_6 is at most

$$32 + 31(r-2) \quad = \quad 31r - 30. \tag{35}$$

Given a K_3^R-free r-coloring of K_5 where three different colors a, b and c occur. Consider first the number of ways to extend it to a K_3^R-free r-coloring of K_6 using only the colors a, b and c. Note, however, that by Lemma 4.1 in every K_4-subgraph of K_5 at least two different colors must occur. Selecting a K_4-subgraph of K_5, say, the one induced on $\{v_1, v_2, v_3, v_4\}$, by (28) and (30) its coloring can be extended to a K_3^R-free r-coloring for the K_5 on $\{v_1, v_2, v_3, v_4, w\}$ in at most $\max\{19, 25\} = 25$ ways using only the colors a, b or c. Then there are at most 3 ways to color the remaining edge $\{v_5, w\}$ by assigning to it one of the colors a, b or c. This gives at most $25 \cdot 3 = 75$ colorings. By our discussion at the beginning of this proof, choosing a color $d \notin \{a, b, c\}$ and using it on at least one of the edges $\{v_i, w\}$ gives at most $31(r-3)$ extensions. Thus the number of ways to extend any K_3^R-free coloring of K_5 using exactly three colors a, b and c to a K_3^R-free r-coloring of K_6 is at most

$$75 + 31(r-3) \quad = \quad 31r - 18. \tag{36}$$

Given an K_3^R-free r-coloring of K_5 where four different colors a, b, c and d occur. Again, consider first the number of ways to extend it to a K_3^R-free r-coloring of K_6 using only the colors a, b, c and d. By Lemma 4.1 in every K_4-subgraph of K_5 three different colors must occur. Selecting a K_4-subgraph, say, the one induced on $\{v_1, v_2, v_3, v_4\}$, by (31) its coloring can be extended to a K_3^R-free r-coloring for the K_5 on $\{v_1, v_2, v_3, v_4, w\}$ in at most 26 ways. Then there are at most 4 ways to color the remaining edge $\{v_5, w\}$. This gives at most $26 \cdot 4 = 104$ colorings. By our discussion above, choosing a color $e \notin \{a, b, c, d\}$ and using it on at least one of the edges $\{v_i, w\}$ gives at most $31(r-4)$ extensions. Thus the number of ways to extend any K_3^R-free coloring of K_5 using exactly four colors to a K_3^R-free r-coloring of K_6 is at most

$$104 + 31(r-4) \quad = \quad 31r - 20. \tag{37}$$

The maximum of (34), (35), (36), (37) is $31r - 18$, as desired. □

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