A RAINBOW ERDŐS-ROTHSCHILD PROBLEM

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Abstract. We consider a multicolored version of a problem introduced by Erdős and Rothschild. For a positive integer r and a graph F, we look for n-vertex graphs that admit the maximum number of r-edge-colorings with no copy of F such that all edges are assigned different colors.

1. Introduction

As usual, a graph G is said to be *F-free* if it does not contain some fixed graph F as a subgraph. The well-known Turán problem [12] for F asks for the maximum number $\mathrm{ex}(n, F)$ of edges over all F-free n-vertex graphs and for the graphs that achieve this maximum, which are called *F-extremal*. This is one of the most popular problems in extremal graph theory and there is a vast literature related with it (for more information and recent developments, we refer to Keevash [8], and the references therein). The Turán problem was generalized to a multicolored setting by Keevash, Saks, Sudakov and Verstraëte [10]. They looked for n-vertex *multigraphs* with the largest number of edges which admit an r-edge-coloring such that all color classes induce simple graphs and there is no *rainbow copy* of the forbidden graph F, that is, no copy of F such that every edge is colored differently. (We observe that the edge-colorings here are not proper unless explicitly mentioned.) Another multicolored extension of the Turán problem was introduced by Keevash, Mubayi, Sudakov and Verstraëte [9], who looked for graphs with the largest number of edges which admit a proper r-edge-coloring with no rainbow copy of F.

The problem studied in this paper has been motivated by a series of developments which followed a question raised by Erdős and Rothschild [4] regarding edge-colorings of graphs avoiding a given monochromatic subgraph. Instead of looking for n-vertex graphs with the largest number of edges that satisfy some property, Erdős and Rothschild were interested in n-vertex graphs that admit the largest number of *r-edge-colorings* such that *every color class is F-free*. In particular, they conjectured that the number of K_ℓ-free 2-colorings is maximized by the $(\ell-1)$-partite Turán graph on n-vertices. Note that F-extremal graphs are natural candidates, as any r-coloring is trivially F-free, which leads to $r^{\mathrm{ex}(n,F)}$ such colorings. Yuster [13] provided an affirmative answer to this conjecture for K_3 and any $n \geq 6$, while Alon, Balogh, Keevash and Sudakov [1] showed that, for $r \in \{2, 3\}$ and $n \geq n_0$, where n_0 is a constant depending on r and ℓ, the respective Turán graph is also optimal for the number of K_ℓ-free r-colorings. However, they also proved that Turán graphs are not optimal for any $r \geq 4$, but did not characterize the extremal graphs. A detailed account of recent progress in this problem may be found in Hoppen, Kohayakawa and Lefmann [5].

Balogh [2] added a twist to this problem by considering edge-colorings of a graph avoiding a copy of F with a *prescribed colored pattern*. Naturally, given a number r of colors and a graph F, an *r-pattern* P of F is a partition of its edge set into r (possibly empty) classes, and an edge-coloring of a graph G is said to be *(F, P)-free* if G does not contain a copy of F

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in which the partition of the edge set induced by the coloring is isomorphic to P. Regarding this problem, Balogh proved that, for $r = 2$ colors and any 2-color pattern of K_ℓ, the $(\ell-1)$-partite Turán graph on $n \geq n_0$ vertices once again yields the largest number of 2-colorings with no forbidden pattern of K_ℓ. However, he also remarked that, if we consider $r = 3$ and a rainbow-colored triangle, the complete graph on n vertices already admits $3 \cdot 2^{\binom{n}{2}} - 3$ colorings, by just choosing two of the three colors and coloring the edges of K_n arbitrarily with these two colors. This is more than $3^{n^2/4}$, which is an upper bound on the number of 3-colorings of the bipartite Turán graph.

Fix a positive integer r and a graph F, and let P be a pattern of F. Let $\mathcal{C}_{r,F,P}(G)$ be the set of all (F,P)-free r-colorings of a graph G. We write

$$c_{r,F,P}(n) = \max\{\, |\mathcal{C}_{r,F,P}(G)| \colon |V(G)| = n \,\},$$

and we say that an n-vertex graph G is $\mathcal{C}_{r,F,P}$*-extremal* if $|\mathcal{C}_{r,F,P}(G)| = c_{r,F,P}(n)$. In this paper, our main objective is to study $\mathcal{C}_{r,F,P}$-extremal graphs where P is the rainbow pattern, that is, where every edge is assigned to a different class.

Theorem 1.1. *Let F be a bipartite graph, let P be a pattern of F with $t \geq 3$ nonempty classes, and fix a positive integer $r \geq t$. Then there exists n_0 such that, for every $n \geq n_0$, we have $c_{r,F,P}(n) \leq (t-1)^{\binom{n}{2} + o(n^2)}$.*

Note that we may trivially obtain $(t-1)^{\binom{n}{2}}$ distinct (F,P)-free colorings if we color the complete graph with a fixed set of $t-1$ colors, so that Theorem 1.1 implies that the complete graph is *almost* optimal for any pattern with at least three classes in a bipartite graph. In particular, this holds for rainbow patterns of bipartite graphs with at least three edges. In fact, in the case where $F = S_t$ is a star with $t \geq 3$ edges and P is the rainbow pattern, Sanches and the current authors [7] have proved that, for all values of $r \geq 3$ and sufficiently large n, K_n is the single $\mathcal{C}_{r,F,P}$-extremal graph on n vertices. Moreover, two of the authors [6] reached the same conclusion for matchings with at least three edges; however, they have found patterns where the number of classes is less than 3 for which the complete graph is not optimal.

The structure of $\mathcal{C}_{r,F,P}$-extremal graphs seems to be rather different for rainbow patterns when F is not bipartite. For instance, in the case of rainbow patterns of complete graphs, we show that the corresponding Turán graph is almost optimal if the number of colors is large.

Theorem 1.2. *For any positive integer s, there exist r_0 such that, for all $r \geq r_0$, we have $c_{r,K_s,P}(n) \leq r^{\frac{s-2}{2(s-1)} n^2 (1+o(1))}$, where P denotes the rainbow pattern.*

The ingredients used in the proof of Theorem 1.2 allow us to derive the following more precise result for rainbow triangles.

Theorem 1.3. *For all $r \geq 3$, we have*

$$c_{r,K_3,P}(n) \leq \begin{cases} 2^{\frac{n^2}{2}(1+o(1))} & \text{for } r = 3 \\ r^{\frac{n^2}{4}(1+o(1))} & \text{for } r \geq 4. \end{cases} \tag{1}$$

In particular, this shows that K_n is almost optimal in the previous example of 3-colorings avoiding rainbow triangles, while $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is almost optimal for r-colorings with $r \geq 4$. In fact, it was recently proven [3] that K_n is indeed optimal in the case $r = 3$, and we may prove here that $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is extremal for all $r \geq 10$.

Theorem 1.4. *For all $r \geq 10$ and $n \geq 5$, the bipartite Turán graph is the single $\mathcal{C}_{r,K_3,P}$-extremal graph on n vertices.*

We believe that the bound $r \geq 10$ can be improved by performing more careful calculations. In the next section, we comment on the proofs of the above theorems. To conclude this section, we present natural questions raised by the results obtained so far: is there a pattern with at least three classes in a bipartite graph for which the complete graph is not optimal? Is there any graph F that admits an $\mathcal{C}_{r,F,P}$-extremal graph that is neither F-extremal nor complete, where P denotes the rainbow pattern?

2. Overview of the proofs

The proofs of our first three theorems follow the same general structure, which is based on a multicolored version of the Szemerédi Regularity Lemma (see [11, 1]). To bound the number of colorings with the appropriate restriction in a graph $G = (V, E)$, we first fix a coloring and find a partition $V = V_1 \cup \cdots \cup V_k$ which is simultaneously regular for the graphs induced by each color. This leads to a cluster graph with vertex set $\{1, \ldots, k\}$ where each edge $\{i, j\}$ is assigned a list consisting of all colors with respect to which the pair (V_i, V_j) is regular and has density above a certain threshold. At this point, we restrict the structure of the cluster graph and of the lists that may be assigned to it because colorings are (F, P)-free. For instance, in the cluster graphs associated with colorings as in Theorem 1.1, at most $t-1$ colors may appear in the list associated with each edge of the cluster graph, since, if we had a pair in the partition that were both regular and dense with respect to these t colors, we would find a copy of the forbidden bipartite graph F with any pattern P with t non-empty classes in the original coloring using a standard embedding argument. (Note that this justifies the factor $(t-1)^{\binom{n}{2}}$ in the upper bound, which is the number that we obtain if we color the edges of the complete graph arbitrarily with $t-1$ colors.) Once we have a result of this type, we conclude the proof by considering all feasible cluster graphs and counting the number of colorings of the original graph for which some partition would lead to each such cluster graph.

To describe the proof of Theorem 1.2, we explain how to restrict the lists in the cluster graph so as to avoid rainbow copies of K_s in the original graph.

Recall that $[r] = \{1, \ldots, r\}$ is the set of colors and let $R = (U, E)$ be a simple graph on k vertices. Let $\mathcal{L}_F^{(r)}(R) = \{L_e \subseteq [r] \,|\, e \in E\}$ be an assignment of non-empty lists $L_e \subseteq [r]$ to the edges e of R such that in R no rainbow-F can be constructed when coloring each edge with a color of the corresponding list in $\mathcal{L}_F^{(r)}(R)$. Set $\lambda_F^{(r)}(R) = \max_{\mathcal{L}_F^{(r)}(R)} \prod_{e \in E} |L_e|$ and $\lambda_{F,k}^{(r)} = \max_R\{\lambda_F^{(r)}(R)\}$, where the maximum runs over all k-vertex graphs R. Theorem 1.2 may then be obtained by combining the general argument above with the following lemma.

Lemma 2.1. *For all $s \geq 3$ there is $r_0(s)$ such that for every $r \geq r_0(s)$ and all $k = \ell(s-1)$, where ℓ is a positive integer, we have $\lambda_{K_s,k}^{(r)} \leq r^{\frac{s-2}{2(s-1)}k^2}$.*

Proof. We prove this by induction on ℓ. The result holds for $\ell = 1$, as $[(s-2)(s-1)^2]/[2(s-1)] = \binom{s-1}{2}$. Now let $\ell \geq 2$. Let $\mathcal{L}_{K_s}^{(r)}(R) = \{L_e \subseteq [r] \,|\, e \in E\}$ be an assignment of non-empty lists to the edges of R such that no rainbow copy of K_s arises. Further assume that R contains a complete graph K_{s-1}, say with vertex set $V' = \{v_1, \ldots, v_{s-1}\}$, each of whose edges e has a list L_e of size at least $\binom{s}{2}$. Let $T \subseteq V \setminus V'$ be the set of all vertices w which are adjacent to all vertices of V', and let $e_i(w) = \{w, v_i\}$, $i = 1, \ldots, s-1$, be the corresponding edges. Then, for any vertex $w \in T$ and any possible assignment of colors from each of the lists $L_{e_1(w)}, \ldots, L_{e_{s-1}(w)}$ two colors must be the same as otherwise a rainbow K_s arises. This implies that, for any vertex $w \in T$,

$$\prod_{i=1}^{s-1} \left|L_{e_i(w)}\right| \leq r^{s-3}. \tag{2}$$

Indeed, this is an application of Hall's Theorem to the bipartite graph whose vertices are given by $\{e_1(w), \ldots, e_{s-1}(w)\}$ and $[r]$, and a color is adjacent to $e_i(w)$ if it lies in $L_{e_i(w)}$. Our assumption implies that there must be $J \subseteq \{1, \dots, s-1\}$, $|J| \geq 2$, such that $\left|\bigcup_{j\in J} L_{e_j(w)}\right| \leq |J|-1$. This leads to $\prod_{i=1}^{s-1} |L_{e_i(w)}| \leq \prod_{j\in J} |L_{e_j(w)}| \cdot \prod_{i\notin J} |L_{e_i(w)}| \leq (|J|-1)^{|J|} \cdot r^{s-1-|J|}$, which is maximized by $|J| = 2$ since $|J| \leq r$. Thus $\prod_{i=1}^{s-1} |L_{e_i(w)}| \leq r^{s-3}$.

To settle this case, note that any vertex $v \in V \setminus (T \cup V')$ is adjacent to at most $(s-2)$ vertices of K_{s-1} and each of these edges has a list of size at most r. We infer, with (2) and the induction hypothesis, that

$$\begin{aligned} \lambda^{(r)}_{K_s}(R) &\leq \lambda^{(r)}_{K_s}(K_{s-1}) \cdot r^{(s-3)|T|} \cdot \left(r^{s-2}\right)^{(k-s+1-|T|)} \cdot \lambda^{(r)}_{K_s,k-s+1} \\ &\leq r^{\binom{s-1}{2}} \cdot r^{(s-3)|T|} \cdot r^{(s-2)(k-s+1-|T|)} \cdot r^{\frac{s-2}{2(s-1)}(k-s+1)^2} \leq r^{\frac{s-2}{2(s-1)}k^2}. \end{aligned}$$

To conclude the proof, assume now that the graph R does not contain a copy of K_{s-1} such that each list has at least $\binom{s}{2}$ colors. Then the number of edges $e \in E$ with lists of size at least $\binom{s}{2}$ is at most $\mathrm{ex}\,(k, K_{s-1}) \leq \frac{s-3}{2(s-2)}k^2$. We infer that $\lambda^{(r)}_{K_s}(R) \leq r^{\frac{s-3}{2(s-2)}k^2} \cdot \left(\binom{s}{2}-1\right)^{\frac{k^2}{2}-\frac{s-3}{2(s-2)}k^2} < r^{\frac{s-2}{2(s-1)}k^2}$ for $r \geq r_0(s) \geq \left(\binom{s}{2}-1\right)^{s-1}$, which finishes the proof. □

An argument of the same type may be used to establish Theorem 1.3.

Finally, the proof of Theorem 1.4 is the most involved in this paper. Even though it does not require particularly deep methods, we need to count the number of r-colorings with no rainbow triangles in a very precise manner, and consider several cases, to obtain the required result by induction on n. To give a vague idea about the structure of the proof, we consider a much simpler version, which proves the result with the restriction $r \geq 74$, in which case the result holds for all $n \geq 4$. Note, however, that for $n = 3$ and $r \geq 2$, a triangle K_3 allows $(3r^2 - 2r)$ rainbow K_3-free r-edge-colorings, while the corresponding number for the Turán graph for K_3 is only r^2. We make use of the following facts:

The proof of this weak version of Theorem 1.4 is as follows. We use the fact that, when we are given a graph H consisting of a triangle with vertex set $\{v_1, v_2, v_3\}$ and additional vertices $w_1, \dots, w_\ell$ together with some subset of the set of edges $\{w_j, v_i\}$, $i = 1, 2, 3$ and $j = 1, \dots, \ell$, the number of r-colorings of H with no rainbow triangles is bounded above by

$$6\binom{r}{2}(5r-2)^\ell + r(7r-6)^\ell. \tag{3}$$

With this, we show that the Turán graph is optimal for $n \in \{4, 5, 6\}$. In particular, we need $r \geq 74$ to ensure that $(6\binom{r}{2}(5r-2)^2 + r(7r-6)^2) \cdot r \leq r^6$ for all 5-vertex graphs.

To prove the result for all $n \geq 7$, fix an n-vertex graph G. If G is triangle-free, it certainly admits at most as many colorings as the Turán graph, so assume that G contains a triangle $W = \{v_1, v_2, v_3\}$. By the induction hypothesis, the number of r-colorings of $G \setminus W$ with no rainbow triangle is at most $r^{\lfloor (n-3)^2/4 \rfloor}$. By (3), the total number of rainbow K_3-free r-colorings of G is at most $\left(6\binom{r}{2}(5r-2)^{n-3} + r(7r-6)^{n-3}\right) \cdot r^{\lfloor (n-3)^2/4 \rfloor}$. Since $r^{\lfloor n^2/4 \rfloor} \geq r^{(n^2-1)/4}$ and $6\binom{r}{2}(5r-2)^{n-3} + r(7r-6)^{n-3} \leq 3r^2(5r)^{n-3} + r(7r)^{n-3}$, we have

$$\left[\left(3r^2(5r)^{n-3} + r(7r)^{n-3}\right) \cdot r^{\frac{(n-3)^2}{4}} < r^{\frac{n^2-1}{4}}\right] \iff \left[3r \cdot 5^{n-3} + 7^{n-3} < r^{\frac{n-1}{2}}\right],$$

which holds for $r \geq 49$ and $n \geq 7$ and establishes the claimed result.

Of course this result could be immediately improved for all $r \geq 49$ by just considering the cases $n \in \{4, 5, 6\}$ more closely. To extend this result to $r \geq 10$, in addition to counting the exact number of colorings for small graphs, we need to consider more cases in the induction hypothesis. As it turns out, we distinguish the cases where G contains a copy of K_3, but not of K_4, where G contains a copy of K_4, but not of K_5, and where G does contain K_5 as a subgraph. As a consequence, the base of induction has to deal with the first five cases directly.

We also believe that with additional effort (including precise computational assessment of the exact number of colorings with no rainbow triangles of all graphs up to a certain size), one could prove that the Turán graph is extremal for all $r \geq 4$ for $n \geq n_0(r)$.

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