Large Triangles in the d-Dimensional Unit-Cube (Extended Abstract)

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Abstract. We consider a variant of Heilbronn's triangle problem by asking for fixed dimension $d \geq 2$ for a distribution of n points in the d-dimensional unit-cube $[0,1]^d$ such that the minimum (2-dimensional) area of a triangle among these n points is maximal. Denoting this maximum value by $\Delta_d^{off-line}(n)$ and $\Delta_d^{on-line}(n)$ for the off-line and the on-line situation, respectively, we will show that $c_1 \cdot (\log n)^{1/(d-1)}/n^{2/(d-1)} \leq \Delta_d^{off-line}(n) \leq C_1/n^{2/d}$ and $c_2/n^{2/(d-1)} \leq \Delta_d^{on-line}(n) \leq C_2/n^{2/d}$ for constants $c_1, c_2, C_1, C_2 > 0$ which depend on d only.

1 Introduction

Given any integer $n \geq 3$, originally Heilbronn's problem asks for the maximum value $\Delta_2(n)$ such that there exists a configuration of n points in the two-dimensional unit-square $[0,1]^2$ where the minimum area of a triangle formed by three of these n points is equal to $\Delta_2(n)$. For primes n, the points $P_k = 1/n \cdot (k \bmod n, k^2 \bmod n)$, $k = 0, 1, \ldots, n-1$, easily show that $\Delta_2(n) = \Omega(1/n^2)$. Komlós, Pintz and Szemerédi [11] improved this to $\Delta_2(n) = \Omega(\log n/n^2)$ and in [5] the authors provide a deterministic polynomial time algorithm achieving this lower bound, which is currently the best known. From the other side, improving earlier results of Roth [14–18] and Schmidt [19], Komlós, Pintz and Szemerédi [10] proved the upper bound $\Delta_2(n) = O(2^{c\sqrt{\log n}}/n^{8/7})$ for some constant $c > 0$. Recently, for n randomly chosen points in the unit-square $[0,1]^2$, the expected value of the minimum area of a triangle among these n points was determined to $\Theta(1/n^3)$ by Jiang, Li and Vitany [9].

A variant of Heilbronn's problem considered by Barequet [2] asks, given a fixed integer $d \geq 2$, for the maximum value $\Delta_d^*(n)$ such that there exists a distribution of n points in the d-dimensional unit-cube $[0,1]^d$ where the minimum volume of a simplex formed by some $(d+1)$ of these n points is equal to $\Delta_d^*(n)$. He showed in [2] the lower bound $\Delta_d^*(n) = \Omega(1/n^d)$, which was improved in [12] to $\Delta_d^*(n) = \Omega(\log n/n^d)$. In [13] a deterministic polynomial time algorithm was given achieving the lower bound $\Delta_3^*(n) = \Omega(\log n/n^3)$. Recently, Brass [6] showed the upper bound $\Delta_d^*(n) = O(1/n^{(2d+1)/(2d)})$ for odd $d \geq 3$, while for even $d \geq 4$ only $\Delta_d^*(n) = O(1/n)$ is known. Moreover, an on-line version of this variant was investigated by Barequet [3] for dimensions $d = 3$ and $d = 4$, where he showed the lower bounds $\Omega(1/n^{10/3})$ and $\Omega(1/n^{127/24})$, respectively.

Here we will investigate the following extension of Heilbronn's problem to higher dimensions: for fixed integers $d \geq 2$ and any given integer $n \geq 3$ find a set of n points in the d-dimensional unit-cube $[0,1]^d$ such that the minimum area of a triangle determined by three of these n points is maximal. We consider the off-line as well as the on-line version of our problem. In the off-line situation the number n of points is given in advance, while in the on-line case the points are positioned in $[0,1]^d$ one after the other and at some time this process stops. Let the corresponding maximum values on the minimum triangle areas be denoted by $\Delta_d^{off-line}(n)$ and $\Delta_d^{on-line}(n)$, respectively.

Theorem 1. *Let $d \geq 2$ be a fixed integer. Then, for constants $c_1, c_2, C_1, C_2 > 0$, which depend on d only, for every integer $n \geq 3$ it is*

$$c_1 \cdot \frac{(\log n)^{1/(d-1)}}{n^{2/(d-1)}} \leq \Delta_d^{off-line}(n) \leq \frac{C_1}{n^{2/d}} \tag{1}$$

$$\frac{c_2}{n^{2/(d-1)}} \leq \Delta_d^{on-line}(n) \leq \frac{C_2}{n^{2/d}} \ . \tag{2}$$

The lower bounds (1) and (2) differ only by a factor of $\Theta((\log n)^{1/(d-1)})$. In contrast to this, the lower bounds in the on-line situation considered by Barequet [3], i.e. maximizing the minimum volume of simplices among n points in $[0,1]^d$, differ by a factor of $\Theta(n^{1/3} \cdot \log n)$ for dimension $d = 3$ and by a factor of $\Theta(n^{31/24} \cdot \log n)$ for dimension $d = 4$ from the currently best known lower bound $\Delta_d^*(n) = \Omega(\log n / n^d)$ for any fixed integer $d \geq 2$ in the off-line situation.
In the following we will split the statement of Theorem 1 into a few lemmas.

2 The Off-Line Case

A line through points $P_i, P_j \in [0,1]^d$ is denoted by P_iP_j. Let dist (P_i, P_j) be the Euclidean distance between the points P_i and P_j. The area of a triangle determined by three points $P_i, P_j, P_k \in [0,1]^d$ is denoted by area (P_i, P_j, P_k), where area $(P_i, P_j, P_k) :=$ dist $(P_i, P_j) \cdot h/2$ and h is the Euclidean distance from point P_k to the line P_iP_j.
First we will prove the lower bound in (1) from Theorem 1, namely

Lemma 1. *Let $d \geq 2$ be a fixed integer. Then, for some constant $c_1 = c_1(d) > 0$, for every integer $n \geq 3$ it is*

$$\Delta_d^{off-line}(n) \geq c_1 \cdot \frac{(\log n)^{1/(d-1)}}{n^{2/(d-1)}} \ . \tag{3}$$

Proof. Let $d \geq 2$ be a fixed integer. For arbitrary integers $n \geq 3$ and a constant $\alpha > 0$, which will be specified later, we select uniformly at random and independently of each other $N = n^{1+\alpha}$ points $P_1, P_2, \ldots, P_N$ in the d-dimensional unit-cube $[0,1]^d$. For fixed $i < j < k$ we will estimate the probability that area $(P_i, P_j, P_k) \leq A$ for some value $A > 0$ which will be specified later. The point P_i can be anywhere in $[0,1]^d$. Given point P_i, the probability, that point

$P_j \in [0,1]^d$ has a Euclidean distance from P_i within the infinitesimal range $[r, r+dr]$, is at most the difference of the volumes of the d-dimensional balls with center P_i and with radii $(r+dr)$ and r, respectively. Hence we obtain

$$\text{Prob } (r \leq \text{ dist } (P_i, P_j) \leq r + dr) \leq d \cdot C_d \cdot r^{d-1} dr \ , \tag{4}$$

where throughout this paper C_d is equal to the volume of the d-dimensional unit-ball in $\mathbb{R}^d$. Given the points P_i and P_j with dist $(P_i, P_j) = r$, the third point $P_k \in [0,1]^d$ satisfies area $(P_i, P_j, P_k) \leq A$, if P_k is contained in the intersection $C_{i,j} \cap [0,1]^d$ with $C_{i,j}$ being a d-dimensional cylinder, centered at the line $P_i P_j$ with radius $2 \cdot A/r$ and height at most $\sqrt{d}$. The d-dimensional volume vol $(C_{i,j} \cap [0,1]^d)$ is at most $C_{d-1} \cdot (2A/r)^{d-1} \cdot \sqrt{d}$, and we infer for some constant $C'_d > 0$:

$$\begin{aligned} &\text{Prob (area } (P_i, P_j, P_k) \leq A) \\ \leq &\int_0^{\sqrt{d}} \text{vol } (C_{i,j} \cap [0,1]^d) \cdot d \cdot C_d \cdot r^{d-1} dr \\ \leq &\int_0^{\sqrt{d}} C_{d-1} \cdot \left(\frac{2 \cdot A}{r} \right)^{d-1} \cdot \sqrt{d} \cdot d \cdot C_d \cdot r^{d-1} dr \\ = &\, C_{d-1} \cdot C_d \cdot 2^{d-1} \cdot d^{3/2} \cdot A^{d-1} \cdot \int_0^{\sqrt{d}} dr = C'_d \cdot A^{d-1} \ . \end{aligned} \tag{5}$$

Definition 1. *A hypergraph $\mathcal{G} = (V, \mathcal{E})$ with vertex set V and edge set $\mathcal{E}$ is* k-uniform *if $|E| = k$ for all edges $E \in \mathcal{E}$.*
A hypergraph $\mathcal{G} = (V, \mathcal{E})$ is linear *if $|E \cap E'| \leq 1$ for all distinct edges $E, E' \in \mathcal{E}$.*
A subset $I \subseteq V$ of the vertex set is independent *if I contains no edges from $\mathcal{E}$.*
The largest size $|I|$ of an independent set in $\mathcal{G}$ is the independence number $\alpha(\mathcal{G})$.

We form a random 3-uniform hypergraph $\mathcal{G} = (V, \mathcal{E}_3)$ with vertex set $V = \{1, 2, \ldots, N\}$, where vertex i corresponds to the random point $P_i \in [0,1]^d$, and with edges $\{i, j, k\} \in \mathcal{E}_3$ if and only if area $(P_i, P_j, P_k) \leq A$. The expected number $E(|\mathcal{E}_3|)$ of edges satisfies by (5) for some constant $c_d > 0$:

$$E(|\mathcal{E}_3|) \leq \binom{N}{3} \cdot C'_d \cdot A^{d-1} \leq c_d \cdot A^{d-1} \cdot N^3 \ . \tag{6}$$

We will use the following result on the independence number of linear k-uniform hypergraphs due to Ajtai, Komlós, Pintz, Spencer and Szemerédi [1], see also [7], compare Fundia [8] and [4] for deterministic algorithmic versions of Theorem 2:

Theorem 2. [1, 7] *Let $k \geq 3$ be a fixed integer. Let $\mathcal{G} = (V, \mathcal{E})$ be a k-uniform hypergraph on $|V| = n$ vertices with average degree $t^{k-1} = k \cdot |\mathcal{E}|/|V|$. If $\mathcal{G}$ is linear, then its independence number $\alpha(\mathcal{G})$ satisfies for some constant $c^*_k > 0$:*

$$\alpha(\mathcal{G}) \geq c^*_k \cdot \frac{n}{t} \cdot (\log t)^{\frac{1}{k-1}} \ . \tag{7}$$

To apply Theorem 2, we estimate in the random hypergraph $\mathcal{G} = (V, \mathcal{E}_3)$ the expected number $E(BP_{D_0}(\mathcal{G}))$ of 'bad pairs of small triangles' in $\mathcal{G}$, which are among the random points $P_1, \ldots, P_N \in [0,1]^d$ those pairs of triangles sharing an edge and both with area at most A, where all sides are of length at least D_0. Then we will delete one vertex from each 'pair of points with Euclidean distance at most D_0' and from each 'bad pair of small triangles', which yields a linear subhypergraph $\mathcal{G}^{**}$ of $\mathcal{G}$ and $\mathcal{G}^{**}$ fulfills the assumptions of Theorem 2.

Let $P_{D_0}(\mathcal{G})$ be a random variable counting the number of pairs of distinct points with Euclidean distance at most D_0 among the N randomly chosen points. For fixed integers i, j, $1 \leq i < j \leq N$, we have

$$\text{Prob } (\text{dist } (P_i, P_j) \leq D_0) \leq C_d \cdot D_0^d ,$$

as point P_i can be anywhere in $[0,1]^d$ and, given point P_i, the probability that dist $(P_i, P_j) \leq D_0$ is bounded from above by the volume of the d-dimensional ball with radius D_0 and center P_i. For $D_0 := 1/N^\beta$, where $\beta > 0$ is a constant, the expected number $E(P_{D_0}(\mathcal{G}))$ of pairs of distinct points with Euclidean distance at most D_0 among the N points satisfies for some constant $c'_d > 0$:

$$E(P_{D_0}(\mathcal{G})) \leq \binom{N}{2} \cdot C_d \cdot D_0^d \leq c'_d \cdot N^{2-\beta d} . \tag{8}$$

For distinct points P_i, P_j, P_k, P_l, $1 \leq i < j < k < l \leq N$, there are $\binom{4}{2}$ choices for the joint side of the two triangles, given by the points P_i and P_j, say. Given point P_i, by (4) we have Prob $(r \leq \text{dist } (P_i, P_j) \leq r + dr) \leq d \cdot C_d \cdot r^{d-1} dr$. Given points P_i and P_j with dist $(P_i, P_j) = r$, the probability that the triangle formed by P_i, P_j, P_k, or by P_i, P_j, P_l, has area at most A, is at most the volume of the cylinder, which is centered at the line P_iP_j with height $\sqrt{d}$ and radius $2 \cdot A/r$. Thus, for $d \geq 3$ we have for some constant $C''_d > 0$:

$$\begin{aligned}
&\text{Prob } (P_i, P_j, P_k, P_l \text{ form a bad pair of small triangles })\\
&\leq \binom{4}{2} \cdot \int_{D_0}^{\sqrt{d}} \left(C_{d-1} \cdot \sqrt{d} \cdot \left(\frac{2 \cdot A}{r} \right)^{d-1} \right)^2 \cdot d \cdot C_d \cdot r^{d-1} dr\\
&= C''_d \cdot A^{2d-2} \cdot \int_{D_0}^{\sqrt{d}} \frac{dr}{r^{d-1}}\\
&= \frac{C''_d}{d-2} \cdot A^{2d-2} \cdot \left(\frac{1}{D_0^{d-2}} - \frac{1}{d^{(d-2)/2}} \right) \leq C''_d \cdot A^{2d-2} \cdot N^{\beta(d-2)} .
\end{aligned} \tag{9}$$

For dimension $d = 2$ the expression (9) is bounded from above by $C''_2 \cdot A^2 \cdot \log N$ for a constant $C''_2 > 0$. Thus, for each $d \geq 2$ we have

$$\begin{aligned}
&\text{Prob } (P_i, P_j, P_k, P_l \quad \text{form a bad pair of small triangles })\\
&\leq C''_d \cdot A^{2d-2} \cdot N^{\beta(d-2)} \cdot \log N ,
\end{aligned}$$

and we obtain for the expected number $E(BP_{D_0}(\mathcal{G}))$ of such bad pairs of small triangles among the N points for a constant $c''_d > 0$ that

$$E(BP_{D_0}(\mathcal{G})) \leq \binom{N}{4} \cdot C''_d \cdot A^{2d-2} \cdot N^{\beta(d-2)} \cdot \log N$$
$$\leq c''_d \cdot A^{2d-2} \cdot N^{4+\beta(d-2)} \cdot \log N \; . \tag{10}$$

Using (6), (8) and (10) and Markov's inequality there exist $N = n^{1+\alpha}$ points in the unit-cube $[0,1]^d$ such that the corresponding 3-uniform hypergraph $\mathcal{G} = (V, \mathcal{E}_3)$ on $|V| = N$ vertices satisfies

$$|\mathcal{E}_3| \leq 3 \cdot c_d \cdot A^{d-1} \cdot N^3 \tag{11}$$
$$P_{D_0}(\mathcal{G}) \leq 3 \cdot c'_d \cdot N^{2-\beta d} \tag{12}$$
$$BP_{D_0}(\mathcal{G}) \leq 3 \cdot c''_d \cdot A^{2d-2} \cdot N^{4+\beta(d-2)} \cdot \log N \; . \tag{13}$$

By (11) the average degree $t^2 = 3 \cdot |\mathcal{E}_3|/|V|$ of $\mathcal{G} = (V, \mathcal{E}_3)$ satisfies $t^2 \leq t_0^2 := 9 \cdot c_d \cdot A^{d-1} \cdot N^3/N = 9 \cdot c_d \cdot A^{d-1} \cdot N^2$. For a suitable constant $\varepsilon > 0$, we pick with probability $p := N^\varepsilon/t_0 \leq 1$ uniformly at random and independently of each other vertices from V. Let $V^* \subseteq V$ be the random set of the picked vertices, and let $\mathcal{G}^* = (V^*, \mathcal{E}_3^*)$ with $\mathcal{E}_3^* = \mathcal{E}_3 \cap [V^*]^3$ be the resulting random induced subhypergraph of $\mathcal{G}$. Using (11), (12), (13) we infer for the expected numbers of vertices, edges, pairs of points with Euclidean distance at most D_0 and bad pairs of small triangles in $\mathcal{G}^*$ for some constants $c_1, c_2, c_3, c_4 > 0$:

$$E(|V^*|) = p \cdot N \geq c_1 \cdot N^\varepsilon / A^{\frac{d-1}{2}}$$
$$E(|\mathcal{E}_3^*|) = p^3 \cdot |\mathcal{E}_3| \leq p^3 \cdot 3 \cdot c_d \cdot A^{d-1} \cdot N^3 \leq c_2 \cdot N^{3\varepsilon}/A^{\frac{d-1}{2}}$$
$$E(P_{D_0}(\mathcal{G}^*)) = p^2 \cdot P_{D_0}(\mathcal{G}) \leq p^2 \cdot 3 \cdot c'_d \cdot N^{2-\beta d} \leq c_3 \cdot N^{2\varepsilon - \beta d}/A^{d-1}$$
$$E(BP_{D_0}(\mathcal{G}^*)) = p^4 \cdot BP_{D_0}(\mathcal{G}) \leq p^4 \cdot 3 \cdot c''_d \cdot A^{2d-2} \cdot N^{4+\beta(d-2)} \cdot \log N \leq$$
$$\leq c_4 \cdot N^{4\varepsilon + \beta(d-2)} \cdot \log N \; .$$

By Chernoff's and Markov's inequality there exists an induced subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_3^*)$ of $\mathcal{G}$ such that the following hold:

$$|V^*| \geq (c_1 - o(1)) \cdot N^\varepsilon / A^{\frac{d-1}{2}} \tag{14}$$
$$|\mathcal{E}_3^*| \leq 4 \cdot c_2 \cdot N^{3\varepsilon}/A^{\frac{d-1}{2}} \tag{15}$$
$$P_{D_0}(\mathcal{G}^*) \leq 4 \cdot c_3 \cdot N^{2\varepsilon - \beta d}/A^{d-1} \tag{16}$$
$$BP_{D_0}(\mathcal{G}^*) \leq 4 \cdot c_4 \cdot N^{4\varepsilon + \beta(d-2)} \cdot \log N \; . \tag{17}$$

Now we fix $\alpha := 1/d$ and $\beta := 1/d$ and for some suitable constant $c^* > 0$ we set

$$A := c^* \cdot \frac{(\log n)^{\frac{1}{d-1}}}{n^{\frac{2}{d-1}}} \; . \tag{18}$$

Lemma 2. *For* $0 < \varepsilon \leq (d+2)/(3 \cdot d \cdot (d+1))$ *it is*

$$BP_{D_0}(\mathcal{G}^*) = o(|V^*|) \; .$$

Proof. Using (14), (17), (18) and $N = n^{1+\alpha}$ we have

$$\begin{aligned}
&BP_{D_0}(\mathcal{G}^*) = o(|V^*|) \\
\iff& N^{4\varepsilon+\beta(d-2)} \cdot \log N = o(N^\varepsilon / A^{\frac{d-1}{2}}) \\
\iff& N^{3\varepsilon+\beta(d-2)} \cdot \log N \cdot A^{\frac{d-1}{2}} = o(1) \\
\iff& n^{(1+\alpha)(3\varepsilon+\beta(d-2))-1} \cdot (\log n)^{\frac{1}{2}} = o(1) \\
\Longleftarrow& (1+\alpha) \cdot (3\varepsilon + \beta(d-2)) < 1 \\
\iff& \varepsilon < \frac{d+2}{3 \cdot d \cdot (d+1)} \qquad \text{as } \alpha = \beta = 1/d.
\end{aligned}$$

□

Lemma 3. *For* $0 < \varepsilon < 1/(d+1)$ *it is*

$$P_{D_0}(\mathcal{G}^*) = o(|V^*|) \; .$$

Proof. By (14), (16), (18), using $N = n^{1+\alpha}$, we infer

$$\begin{aligned}
&P_{D_0}(\mathcal{G}^*) = o(|V^*|) \\
\iff& N^{2\varepsilon-\beta d} / A^{d-1} = o(N^\varepsilon / A^{\frac{d-1}{2}}) \\
\iff& N^{\varepsilon-\beta d} / A^{\frac{d-1}{2}} = o(1) \\
\iff& n^{(\varepsilon-\beta d)(1+\alpha)+1} / (\log n)^{1/2} = o(1) \\
\Longleftarrow& (\varepsilon - \beta d)(1+\alpha) < -1 \\
\iff& \varepsilon < \frac{1}{d+1} \qquad \text{as } \alpha = \beta = 1/d.
\end{aligned}$$

□

We fix $\varepsilon := 1/(3d)$, hence $p \leq 1$ for $n > n_0$. In the subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_3^*)$ we delete one vertex form each bad pair of small triangles with all side-lengths at least D_0 and from each pair of vertices where the corresponding points have Euclidean distance at most D_0. The resulting induced subhypergraph $\mathcal{G}^{**} = (V^{**}, \mathcal{E}_3^{**})$ with $\mathcal{E}_3^{**} = [V^{**}]^3 \cap \mathcal{E}_3^*$ fulfills by Lemmas 2 and 3 and by (14), (15):

$$|V^{**}| \geq (c_1 - o(1)) \cdot N^\varepsilon / A^{\frac{d-1}{2}} \tag{19}$$

$$|\mathcal{E}_3^{**}| \leq |\mathcal{E}_3^*| \leq 4 \cdot c_2 \cdot N^{3\varepsilon} / A^{\frac{d-1}{2}} \tag{20}$$

and the points corresponding to the vertices of the subhypergraph $\mathcal{G}^{**}$ do not form any bad pairs of small triangles anymore, i.e. $\mathcal{G}^{**}$ is a linear hypergraph. By (19) and (20) the hypergraph $\mathcal{G}^{**}$ has average degree

$$t^2 \leq t_1^2 := \frac{12 \cdot c_2 \cdot N^{3\varepsilon} / A^{\frac{d-1}{2}}}{(c_1 - o(1)) \cdot N^\varepsilon / A^{\frac{d-1}{2}}} = \frac{(12 + o(1)) \cdot c_2}{c_1} \cdot N^{2\varepsilon} \; . \tag{21}$$

The assumptions of Theorem 2 are fulfilled by the 3-uniform subhypergraph $\mathcal{G}^{**}$ of $\mathcal{G}$ and we infer for $t \geq 2$ with (7), (19) and (21) for its independence number

$$\begin{aligned}\alpha(\mathcal{G}) &\geq \alpha(\mathcal{G}^{**}) \geq c_3^* \cdot \frac{|V^{**}|}{t} \cdot (\log t)^{1/2} \geq c_3^* \cdot \frac{|V^{**}|}{t_1} \cdot (\log t_1)^{1/2} \geq \\ &\geq \frac{c_3^* \cdot (c_1^{3/2} - o(1)) \cdot N^{\varepsilon}}{((12 + o(1)) \cdot c_2)^{1/2} \cdot N^{\varepsilon} \cdot A^{(d-1)/2}} \cdot \left(\log \left(\frac{(12 + o(1)) \cdot c_2}{c_1} \right)^{1/2} \cdot N^{\varepsilon} \right)^{1/2} \\ &\geq c' \cdot (\log n)^{1/2} / A^{(d-1)/2} \qquad\qquad \text{as } N = n^{1+\alpha} \\ &\geq c' \cdot (1/c^*)^{\frac{d-1}{2}} \cdot \frac{n}{(\log n)^{1/2}} \cdot (\log n)^{1/2} \geq n\end{aligned}$$

for some sufficiently small constant $c^* > 0$. Thus the hypergraph $\mathcal{G}$ contains an independent set $I \subseteq V$ with $|I| = n$. These n vertices represent n points in $[0,1]^d$, where every triangle among these n points has area at least A, i.e. $\Delta_d(n)^{off-line} = \Omega((\log n)^{1/(d-1)}/n^{2/(d-1)})$. □

3 The On-Line Case

In this section we consider the on-line situation and we will show the lower bound in (2) from Theorem 1:

Lemma 4. *Let $d \geq 2$ be a fixed integer. Then, for some constant $c_2 = c_2(d) > 0$, for every integer $n \geq 3$ it is*

$$\Delta_d^{on-line}(n) \geq \frac{c_2}{n^{2/(d-1)}} \; .$$

Proof. Successively we will construct an arbitrary long sequence $P_1, P_2, \ldots$ of points in the unit-cube $[0,1]^d$ such that for suitable constants $a, b, \alpha, \beta > 0$, which will be fixed later, for every n the set $S_n = \{P_1, P_2, \ldots, P_n\}$ satisfies

(i) dist $(P_i, P_j) > a/n^{\alpha}$ for all $1 \leq i < j \leq n$, and
(ii) area $(P_i, P_j, P_k) > b/n^{\beta}$ for all $1 \leq i < j < k \leq n$.

Assume that a set $S_{n-1} = \{P_1, P_2, \ldots, P_{n-1}\} \subset [0,1]^d$ of $(n-1)$ points with (i') dist $(P_i, P_j) > a/(n-1)^{\alpha}$ for all $1 \leq i < j \leq n-1$, and (ii') area $(P_i, P_j, P_k) > b/(n-1)^{\beta}$ for all $1 \leq i < j < k \leq n-1$ is already constructed.
To have some space in $[0,1]^d$ available for choosing a new point $P_n \in [0,1]^d$ such that (i) is fulfilled, this new point P_n is not allowed to lie within any of the d-dimensional balls $B_r(P_i)$ of radius $r = a/n^{\alpha}$ with center P_i, $i = 1, 2, \ldots, n-1$. Adding the volumes of these balls yields

$$\sum_{i=1}^{n-1} \text{vol } (B_r(P_i)) < n \cdot C_d \cdot r^d = a^d \cdot C_d \cdot n^{1-\alpha d} \; .$$

For $\alpha := 1/d$ and $a^d \cdot C_d < 1/2$ we have $\sum_{i=1}^{n-1} \text{vol } (B_r(P_i)) < 1/2$.

We will show next that the regions, where condition (ii) is violated, also have volume less than $1/2$. The regions, where (ii) is violated by points $P_n \in [0,1]^d$, are given by $C_{i,j} \cap [0,1]^d$, $1 \le i < j \le n-1$, where $C_{i,j}$ is a d-dimensional cylinder centered at the line P_iP_j. These sets $C_{i,j} \cap [0,1]^d$ are contained in cylinders of height $\sqrt{d}$ and radius $2 \cdot b/(n^\beta \cdot \text{dist }(P_i, P_j))$. We sum up their volumes:

$$\begin{aligned}
&\sum_{1 \le i < j \le n-1} \text{vol }(C_{i,j} \cap [0,1]^d) \\
&\le \sum_{1 \le i < j \le n-1} \sqrt{d} \cdot C_{d-1} \cdot \left(\frac{2 \cdot b}{n^\beta \cdot \text{dist }(P_i, P_j)} \right)^{d-1} \\
&= \frac{(2 \cdot b)^{d-1} \cdot \sqrt{d} \cdot C_{d-1}}{2 \cdot n^{\beta(d-1)}} \cdot \sum_{i=1}^{n-1} \sum_{j=1; j \ne i}^{n-1} \left(\frac{1}{\text{dist }(P_i, P_j)} \right)^{d-1} .
\end{aligned} \tag{22}$$

We fix some point P_i, $i = 1, 2, \ldots, n-1$. To give an upper bound on the last sum, we will use a packing argument, compare [2]. Consider the balls $B_{r_t}(P_i)$ with center P_i and radius $r_t := a \cdot t/n^\alpha$, $t = 0, 1, \ldots$ with $t \le \sqrt{d} \cdot n^\alpha/a$. Clearly vol $(B_{r_0}(P_i)) = 0$, and for some constant $C_d^* > 0$ for $t = 1, 2, \ldots$ we have

$$\text{vol }(B_{r_t}(P_i) \setminus B_{r_{t-1}}(P_i)) \le C_d^* \cdot \frac{t^{d-1}}{n^{\alpha d}} . \tag{23}$$

Notice that for every ball $B_r(P_j)$ with radius $r = \Theta(n^{-\alpha})$ and center $P_j \in B_{r_t}(P_i) \setminus B_{r_{t-1}}(P_i)$ with $i \ne j$ we have vol $(B_r(P_j) \cap (B_{r_t}(P_i) \setminus B_{r_{t-1}}(P_i)) = \Theta(n^{-\alpha d})$. By inequalities (i') we have $n_1 = 1$ and by (23) each shell $B_{r_t}(P_i) \setminus B_{r_{t-1}}(P_i)$, $t = 2, 3, \ldots$, contains at most $n_t \le C'_d \cdot t^{d-1}$ points from the set S_{n-1} for some constant $C'_d > 0$. Using the inequality $1 + x \le e^x$, $x \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\sum_{j=1; j \ne i}^{n-1} \left(\frac{1}{\text{dist }(P_i, P_j)} \right)^{d-1} \le \sum_{t=2}^{\sqrt{d} \cdot n^\alpha/a} n_t \cdot \left(\frac{1}{a \cdot (t-1)/n^\alpha} \right)^{d-1} \le \\
&\le \sum_{t=2}^{\sqrt{d} \cdot n^\alpha/a} C'_d \cdot t^{d-1} \cdot \left(\frac{1}{a \cdot (t-1)/n^\alpha} \right)^{d-1} \le \sum_{t=2}^{\sqrt{d} \cdot n^\alpha/a} \frac{C'_d}{a^{d-1}} \cdot e^{\frac{d-1}{t-1}} \cdot n^{\alpha(d-1)} \le \\
&\le C''_d \cdot n^{\alpha d}
\end{aligned} \tag{24}$$

for some constant $C''_d > 0$. We set $\beta := 2/(d-1)$ and, using $\alpha = 1/d$ and (24), inequality (22) becomes for a sufficiently small constant $b > 0$:

$$\begin{aligned}
&\sum_{1 \le i < j \le n-1} \text{vol }(C_{i,j} \cap [0,1]^d) \\
&\le \frac{(2 \cdot b)^{d-1} \cdot \sqrt{d} \cdot C_{d-1}}{n^{\beta(d-1)}} \cdot \sum_{i=1}^{n-1} \sum_{j=1; j \ne i}^{n-1} \left(\frac{1}{\text{dist }(P_i, P_j)} \right)^{d-1} \\
&\le \frac{(2 \cdot b)^{d-1} \cdot \sqrt{d} \cdot C_{d-1}}{n^{\beta(d-1)}} \cdot \sum_{i=1}^{n-1} C''_d \cdot n^{\alpha d} \\
&\le (2 \cdot b)^{d-1} \cdot \sqrt{d} \cdot C_{d-1} \cdot C''_d \cdot n^{1 + \alpha d - \beta(d-1)} < 1/2 .
\end{aligned}$$

The forbidden regions have volume less than 1, hence there exists a point $P_n \in [0,1]^d$ such that (i) and (ii) are satisfied, thus $\Delta_d^{on-line}(n) = \Omega(1/n^{2/(d-1)})$. □

4 An Upper Bound

Here we will show the upper bound $O(1/n^{2/d})$ from Theorem 1 on the smallest area of a triangle among any n points in the d-dimensional unit-cube $[0,1]^d$.

Lemma 5. *Let $d \geq 2$ be a fixed integer. Then, for some constant $C_1 > 0$, for every integer $n \geq 3$ it is*

$$\Delta_d^{on-line}(n) \leq \Delta_d^{off-line}(n) \leq \frac{C_1}{n^{2/d}} .$$

Proof. Given any n points $P_1, P_2, \ldots, P_n \in [0,1]^d$, for some value $D_0 > 0$ we form a graph $G_{D_0} = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$, where vertex i corresponds to point $P_i \in [0,1]^d$, and edges $\{i, j\} \in E$ if and only if dist $(P_i, P_j) \leq D_0$. An independent set $I \subseteq V$ in this graph G_{D_0} yields a subset $I' \subseteq \{P_1, P_2, \ldots, P_n\}$ of points with Euclidean distance between any two distinct points bigger than D_0. Each ball $B_r(P_j)$ with center $P_j \in [0,1]^d$ and radius $r \leq 1$ satisfies vol $(B_r(P_j) \cap [0,1]^d) \geq$ vol $(B_r(P_j))/2^d$. The balls with radius $D_0/2$ and centers from the set I' have pairwise empty intersection, thus

$$\alpha(G_{D_0}) \cdot 2^{-d} \cdot C_d \cdot (D_0/2)^d \leq \text{ vol } ([0,1]^d) = 1 . \tag{25}$$

By Turán's theorem [20], for any graph $G = (V, E)$ we have the lower bound $\alpha(G) \geq n/(2 \cdot t)$ on the independence number $\alpha(G)$, where $t := 2 \cdot |E|/|V|$ is the average degree of G. This with (25) implies

$$\frac{4^d}{C_d \cdot D_0^d} \geq \alpha(G_{D_0}) \geq \frac{n}{2 \cdot t} \quad \Longrightarrow \quad t \geq \frac{C_d}{2 \cdot 4^d} \cdot n \cdot D_0^d .$$

Let $D_0 := c/n^{1/d}$ where $c > 0$ is a constant with $c^d > 2 \cdot 4^d / C_d$. Then $t > 1$ and there exist two edges $\{i, j\}, \{i, k\} \in E$ incident at vertex $i \in V$. Then the two points P_j and P_k have Euclidean distance at most D_0 from point P_i, and hence area $(P_i, P_j, P_k) \leq D_0^2/2 = 1/2 \cdot c^2/n^{2/d}$, i.e. $\Delta_d^{off-line}(n) = O(1/n^{2/d})$. □

5 Conclusion

Certainly it is of interest to improve the bounds given in this paper. Also, for the off-line case it is desirable to get a deterministic polynomial time algorithm achieving the bound $\Delta_d(n)^{off-line} = \Omega((\log n)^{1/(d-1)}/n^{2/(d-1)})$. In view of the results in [9] it is also of interest to determine the expected value of the minimum triangle area with respect to the uniform distribution of n points in $[0,1]^d$.

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