Distributions of Points in d Dimensions and Large k-Point Simplices (Extended Abstract)

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Abstract. We consider a variant of Heilbronn's triangle problem by asking for fixed dimension $d \geq 2$ and for fixed integers $k \geq 3$ with $k \leq d+1$ for a distribution of n points in the d-dimensional unit-cube $[0,1]^d$ such that the minimum volume of a k-point simplex among these n points is as large as possible. Denoting by $\Delta_{k,d}(n)$ the supremum of the minimum volume of a k-point simplex among n points over all distributions of n points in $[0,1]^d$ we will show that $c_k \cdot (\log n)^{1/(d-k+2)}/n^{(k-1)/(d-k+2)} \leq \Delta_{k,d}(n) \leq c'_k/n^{(k-1)/d}$ for $3 \leq k \leq d+1$, and moreover $\Delta_{k,d}(n) \leq c''_k/n^{(k-1)/d+(k-2)/(2d(d-1))}$ for $k \geq 4$ even, and constants $c_k, c'_k, c''_k > 0$.

1 Introduction

For integers $n \geq 3$, Heilbronn's problem asks for the supremum $\Delta_2(n)$ of the minimum area of a triangle formed by three of n points over all distributions of n points in the unit-square $[0,1]^2$. For primes n, the points $P_k = 1/n \cdot (k \bmod n, k^2 \bmod n)$, $k = 0, 1, \ldots, n-1$, show that $\Delta_2(n) = \Omega(1/n^2)$. Komlós, Pintz and Szemerédi [10] improved this to $\Delta_2(n) = \Omega(\log n/n^2)$, see [5] for a deterministic polynomial time algorithm achieving this lower bound on $\Delta_2(n)$, which is currently the best known. Upper bounds were proved in a series of papers by Roth [15–18] and Schmidt [19]. The currently best known upper bound is due to Komlós, Pintz and Szemerédi [9], who proved $\Delta_2(n) = O(2^{c\sqrt{\log n}}/n^{8/7})$ for some constant $c > 0$. We remark that for n points chosen uniformly at random and independently of each other from $[0,1]^2$, the expected value of the minimum area of a triangle among these n points is $\Theta(1/n^3)$, as was shown by Jiang, Li and Vitany [8].
A variant of Heilbronn's problem considered by Barequet asks, given a fixed dimension $d \geq 2$, for the supremum $\Delta_{d+1,d}(n)$ of the minimum volume of a $(d+1)$-point simplex among n points in the d-dimensional unit-cube $[0,1]^d$ over all distributions of n points in $[0,1]^d$. He showed in [2] the lower bound $\Delta_{d+1,d}(n) = \Omega(1/n^d)$, which was improved in [11] to $\Delta_{d+1,d}(n) = \Omega(\log n/n^d)$. In [14], a deterministic polynomial time algorithm was given achieving this lower bound on $\Delta_{4,3}(n)$. Recently, Brass [6] improved the known upper bound $\Delta_{d+1,d}(n) = O(1/n)$ to $\Delta_{d+1,d}(n) = O(1/n^{(2d+1)/(2d)})$ for odd $d \geq 3$. Moreover, an on-line version of this variant was investigated in [3] for dimensions $d = 3, 4$.
Here we consider the following generalization of Heilbronn's problem: given fixed integers d, k with $3 \leq k \leq d+1$, find for any integer $n \geq k$ a distribution of n

points in the d-dimensional unit-cube $[0,1]^d$ such that the minimum volume of a k-point simplex among these n points is as large as possible. Let $\Delta_{k,d}(n)$ denote the corresponding supremum values – over all distributions of n points in $[0,1]^d$ – on the minimum volume of a k-point simplex among n points in $[0,1]^d$.
The parameter $\Delta_{3,d}(n)$, i.e. areas of triangles in $[0,1]^d$, was investigated by this author in [12], where it was shown that $c_3 \cdot (\log n)^{1/(d-1)}/n^{2/(d-1)} \leq \Delta_{3,d}(n) \leq c_3'/n^{2/d}$ for constants $c_3, c_3' > 0$. Here we prove the following bounds.

Theorem 1. *Let d, k be fixed integers with $3 \leq k \leq d+1$. Then, for constants $c_k, c_k', c_k'' > 0$, which depend on k, d only, for every integer $n \geq k$ it is*

$$c_k \cdot \frac{(\log n)^{1/(d-k+2)}}{n^{(k-1)/(d-k+2)}} \leq \Delta_{k,d}(n) \leq \frac{c_k'}{n^{(k-1)/d}} \qquad \textit{for } k \textit{ odd} \qquad (1)$$

$$c_k \cdot \frac{(\log n)^{1/(d-k+2)}}{n^{(k-1)/(d-k+2)}} \leq \Delta_{k,d}(n) \leq \frac{c_k''}{n^{(k-1)/d+(k-2)/(2d(d-1))}} \quad \textit{for } k \textit{ even.} \quad (2)$$

For $d = 2$ and $k = 3$, the lower bound (1) is just the result from [10]. For $k = d+1$, this yields the bounds from [6] and [11]. Indeed, our arguments for proving Theorem 1 yield a randomized polynomial time algorithm, which finds a distribution of n points in $[0,1]^d$ achieving these lower bounds.

2 A Lower Bound on $\Delta_{k,d}(n)$

Fist we introduce some notation which is used throughout this paper.
Let dist (P_i, P_j) be the *Euclidean distance* between the points P_i and P_j. A *simplex* given by k points $P_1, \ldots, P_k \in [0,1]^d$ is the set of all points $P_1 + \sum_{i=2}^k \lambda_i \cdot (P_i - P_1)$ with $\lambda_i \geq 0$, $i = 2, \ldots, k$, and $\sum_{i=2}^k \lambda_i \leq 1$. The $(k-1)$-dimensional *volume of a k-point simplex* determined by the points $P_1, \ldots, P_k \in [0,1]^d$, $2 \leq k \leq d+1$, is defined by vol $(P_1, \ldots, P_k) := 1/(k-1)! \cdot \prod_{j=2}^k$ dist $(P_j; < P_1, \ldots, P_{j-1} >)$, where dist $(P_j; < P_1, \ldots, P_{j-1} >)$ denotes the Euclidean distance of the point P_j from the affine space $< P_1, \ldots, P_{j-1} >$ generated by $P_1, \ldots, P_{j-1}$ with $< P_1 > := P_1$.
In our arguments we will use hypergraphs. A hypergraph $\mathcal{G} = (V, \mathcal{E})$ with vertex set V and edge set $\mathcal{E}$ is *k-uniform* if $|E| = k$ for all edges $E \in \mathcal{E}$. A subset $I \subseteq V$ of the vertex set V is *independent* if I contains no edges from $\mathcal{E}$. The largest size $|I|$ of an independent set in $\mathcal{G}$ is the *independence number* $\alpha(\mathcal{G})$. A hypergraph $\mathcal{G} = (V, \mathcal{E})$ is *linear* if $|E \cap E'| \leq 1$ for all distinct edges $E, E' \in \mathcal{E}$.
First we prove the lower bound in (1), (2) from Theorem 1, namely that

$$\Delta_{k,d}(n) \geq c_k \cdot (\log n)^{1/(d-k+2)}/n^{(k-1)/(d-k+2)} . \qquad (3)$$

Proof. Let d, k be fixed integers with $3 \leq k \leq d+1$. For arbitrary integers $n \geq k$ and a suitable constant $\alpha > 0$, we select uniformly at random and independently of each other $N := n^{1+\alpha}$ points $P_1, P_2, \ldots, P_N$ from $[0,1]^d$.
For certain values $D_j := N^{-\gamma_j}$ for some constants $\gamma_j > 0$, $j = 2, \ldots, k-1$, and some value $V_0 > 0$, where all these will be fixed later, we form a random hypergraph $\mathcal{G} = \mathcal{G}(D_2, \ldots, D_{k-1}, V_0) = (V, \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_k)$ with vertex set $V =$

$\{1, 2, \ldots, N\}$, where vertex i corresponds to the random point $P_i \in [0,1]^d$, and with j-element edges, $j = 2, \ldots, k$. For $j = 2, \ldots, k-1$, let $\{i_1, \ldots, i_j\} \in \mathcal{E}_j$ be a j-element edge if and only if vol $(P_{i_1}, \ldots, P_{i_j}) \leq D_j$. Moreover, let $\{i_1, \ldots, i_k\} \in \mathcal{E}_k$ be a k-element edge if and only if vol $(P_{i_1}, \ldots, P_{i_k}) \leq V_0$ and $\{i_1, \ldots, i_k\}$ does not contain any j-element edges $E \in \mathcal{E}_j$ for $j = 2, \ldots, k-1$. An independent set $I \subseteq V$ in this hypergraph $\mathcal{G}$ yields $|I|$ many points in $[0,1]^d$ such that each k-point simplex among these $|I|$ points has volume bigger than V_0. Our aim is to show the existence of a large independent set $I \subseteq V$ in $\mathcal{G}$. For doing so, we will use a result on the independence number of linear k-uniform hypergraphs due to Ajtai, Komlós, Pintz, Spencer and Szemerédi [1], see [7].

Theorem 2. **[1, 7]** *Let $k \geq 3$ be a fixed integer. Let $\mathcal{G} = (V, \mathcal{E})$ be a k-uniform hypergraph on $|V| = n$ vertices with average degree $t^{k-1} = k \cdot |\mathcal{E}|/|V|$. If $\mathcal{G}$ is linear, then for some constant $c_k^* > 0$ its independence number $\alpha(\mathcal{G})$ satisfies*

$$\alpha(\mathcal{G}) \geq c_k^* \cdot \frac{n}{t} \cdot \log^{\frac{1}{k-1}} t \ . \tag{4}$$

The difficulty in our arguments is, to find a certain subhypergraph of our random non-uniform hypergraph $\mathcal{G}$ to which we can apply Theorem 2. For doing so, we will select a random induced subhypergraph $\mathcal{G}^*$ of $\mathcal{G}$ by controling certain parameters of $\mathcal{G}^*$. For $j = 2, \ldots, k-1$, let $|BP_j(\mathcal{G})|$ be a random variable counting the number of 'bad j-pairs of simplices' in $\mathcal{G}$, which are among the N random points $P_1, \ldots, P_N \in [0,1]^d$ those unordered pairs of k-point simplices arising from $\mathcal{E}_k$, which share j vertices. We will show that in the random nonuniform hypergraph $\mathcal{G}$ the expected numbers $E(|\mathcal{E}_i|)$ and $E(|BP_j(\mathcal{G})|)$ of i-element edges and of 'bad j-pairs of simplices' arising from $\mathcal{E}_k$, respectively, $i, j = 2, \ldots, k-1$, are not too big. Then in a certain induced subhypergraph of $\mathcal{G}$, which will be obtained by a random selection of vertices from V, we will delete one vertex from each i-element edge $E \in \mathcal{E}_i$ and from each 'bad j-pair of simplices' arising from $\mathcal{E}_k$, $i, j = 2, \ldots, k-1$. This yields a *k-uniform linear* subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_k^*)$ of $\mathcal{G}$, thus $\mathcal{G}^*$ fulfills the assumptions of Theorem 2 and then we can apply it.

Lemma 1. *For $i = 2, \ldots, k$ with $2 \leq k \leq d+1$ and random points $P_1, \ldots, P_i \in [0,1]^d$ for constants $c_i^* > 0$ and a real $V > 0$ it is*

$$\textit{Prob } (\textit{vol } (P_1, \ldots, P_i) \leq V) \leq c_i^* \cdot V^{d-i+2} \ . \tag{5}$$

Proof. Let $P_1, \ldots, P_i$ be i random points in $[0,1]^d$. We may assume that the i points are numbered in such a way that for $2 \leq g \leq h \leq i$ it is

$$\text{dist } (P_g; < P_1, \ldots, P_{g-1} >) \geq \text{dist } (P_h; < P_1, \ldots, P_{g-1} >) \ . \tag{6}$$

The point P_1 can be anywhere in $[0,1]^d$. Given the point P_1, the probability, that its Euclidean distance from the point $P_2 \in [0,1]^d$ is within the infinitesimal range $[r_1, r_1 + dr_1]$, is at most the difference of the volumes of the d-dimensional balls with center P_1 and with radii $(r_1 + dr_1)$ and r_1, respectively, hence

$$\text{Prob } (r_1 \leq \text{ dist } (P_1, P_2) \leq r_1 + dr_1) \leq d \cdot C_d \cdot r_1^{d-1} dr_1 \ ,$$

where throughout this paper C_d denotes the value of the volume of the d-dimensional unit-ball in $\mathbb{R}^d$ with $C_1 := 2$.
Given the points P_1 and P_2 with dist $(P_1, P_2) = r_1$, the probability that the distance dist $(P_3; < P_1, P_2 >)$ of the point $P_3 \in [0,1]^d$ from the line $< P_1, P_2 >$ is within the infinitesimal range $[r_2, r_2 + dr_2]$ is at most the difference of the volumes of cylinders centered at the line $< P_1, P_2 >$ with radii $r_2 + dr_2$ and r_2, respectively, and, by assumption (6), with height $2 \cdot r_1 = 2 \cdot$ dist (P_1, P_2), thus

$$\text{Prob } (r_2 \leq \text{ dist } (P_3; < P_1, P_2 >) \leq r_2 + dr_2) \leq 2 \cdot r_1 \cdot (d-1) \cdot C_{d-1} \cdot r_2^{d-2} dr_2 .$$

In general, by condition (6), given the points $P_1, \ldots, P_g$, $g < i$, with dist $(P_f; < P_1, \ldots, P_{f-1} >) = r_{f-1}$ for $f = 2, \ldots, g$, the projection of the point P_{g+1} onto the affine space $< P_1, \ldots, P_f >$ must lie in a shape of volume at most $2^{f-1} \cdot r_1 \cdot \ldots \cdot r_{f-1}$. Hence for $g < i - 1$ we obtain

$$\begin{aligned} &\text{Prob } (r_g \leq \text{dist } (P_{g+1}; < P_1, \ldots, P_g >) \leq r_g + dr_g) \\ &\leq 2^{g-1} \cdot r_1 \cdot \ldots \cdot r_{g-1} \cdot (d - g + 1) \cdot C_{d-g+1} \cdot r_g^{d-g} dr_g \ . \end{aligned}$$

For $g = i - 1$, however, to satisfy vol $(P_1, \ldots, P_i) \leq V$, we must have $1/(i-1)! \cdot \prod_{g=2}^{i}$ dist $(P_g; < P_1, \ldots, P_{g-1} >) \leq V$, hence the projection of the point P_i onto the affine space $< P_1, \ldots, P_{i-1} >$ must lie in a shape of volume $2^{i-2} \cdot r_1 \cdot \ldots \cdot r_{i-2}$ and the point P_i has Euclidean distance at most $\frac{(i-1)! \cdot V}{r_1 \cdot \ldots \cdot r_{i-2}}$ from $< P_1, \ldots, P_{i-1} >$, which happens with probability at most

$$2^{i-2} \cdot r_1 \cdot \ldots \cdot r_{i-2} \cdot C_{d-i+2} \cdot \left(\frac{(i-1)! \cdot V}{r_1 \cdot \ldots \cdot r_{i-2}} \right)^{d-i+2} .$$

Thus for some constants $c_i^*, c_i^{**} > 0$ we infer

$$\begin{aligned} &\text{Prob (vol } (P_1, \ldots, P_i) \leq V) \\ &\leq \int_{r_{i-2}=0}^{\sqrt{d}} \cdots \int_{r_1=0}^{\sqrt{d}} 2^{i-2} \cdot C_{d-i+2} \cdot \frac{((i-1)! \cdot V)^{d-i+2}}{(r_1 \cdot \ldots \cdot r_{i-2})^{d-i+1}} \cdot \\ &\cdot \prod_{g=1}^{i-2} \left(2^{g-1} \cdot r_1 \cdot \ldots \cdot r_{g-1} \cdot (d-g+1) \cdot C_{d-g+1} \cdot r_g^{d-g} \right) dr_{i-2} \ldots dr_1 \leq \\ &\leq c_i^{**} \cdot V^{d-i+2} \cdot \int_{r_{i-2}=0}^{\sqrt{d}} \cdots \int_{r_1=0}^{\sqrt{d}} \prod_{g=1}^{i-2} r_g^{2i-2g-3} dr_{i-2} \ldots dr_1 \\ &\leq c_i^* \cdot V^{d-i+2} \qquad \text{as } 2 \cdot i - 2 \cdot g - 3 > 0. \end{aligned}$$

□

Corollary 1. *For* $i = 2, \ldots, k-1$ *with* $2 \leq k \leq d+1$ *and constants* $c_i', c_k' > 0$, *it is*

$$E(|\mathcal{E}_i|) \leq c_i' \cdot N^{i - \gamma_i (d-i+2)} \qquad \text{and} \qquad E(|\mathcal{E}_k|) \leq c_k' \cdot V_0^{d-k+2} \cdot N^k \ . \tag{7}$$

Proof. There are $\binom{N}{i}$ possibilities to choose i out of the N random points $P_1, \ldots, P_N \in [0,1]^d$, and by (5) from Lemma 1 with $V := N^{-\gamma_i}$ for $i = 2, \ldots, k-1$ and $V := V_0$ for $i = k$ the inequalities (7) follow. □

Lemma 2. *For $j = 2, \ldots, k-1$ with $3 \le k \le d+1$ and constants $c'_{2,j} > 0$ it is*

$$E(|BP_j(\mathcal{G})|) \le c'_{2,j} \cdot V_0^{2(d-k+2)} \cdot N^{2k-j+\gamma_j(d-k+2)} \ . \tag{8}$$

Proof. For $j = 2, \ldots, k-1$, we show the upper bound $O(V_0^{2(d-k+2)} \cdot N^{\gamma_j(d-k+2)})$ on the probability that $2k-j$ random points, chosen uniformly and independently of each other in $[0,1]^d$, yield a 'bad j-pair of simplices'. Since there are $\binom{N}{2k-j}$ possibilities to choose $2k-j$ out of the N random points $P_1, \ldots, P_N \in [0,1]^d$, the upper bound (8) follows. There are $\binom{2k-j}{k}$ choices for k out of $2k-j$ points and $\binom{k}{j}$ possibilities to choose the j common points, say the two simplices are determined by the points $P_1, \ldots, P_k$ and $P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_k$. By Lemma 1 we know that Prob (vol $(P_1, \ldots, P_k) \le V_0) \le c_k^* \cdot V_0^{d-k+2}$. If $\{P_1, \ldots, P_k\} \in \mathcal{E}_k$, then by construction of our hypergraph $\mathcal{G}$ we have vol $(P_1, \ldots, P_j) > N^{-\gamma_j}$, and we condition on this in the following. Given the points $P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_g$, $g = j, \ldots, k-1$, with dist $(Q_f; < P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_{f-1} >) = r_f$, $f = j+1, \ldots, g$, we infer for $g \le k-2$:

$$\begin{aligned} &\text{Prob } (r_g \le \text{dist } (Q_{g+1}; < P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_g >) \le r_g + dr_g) \\ &\le (\sqrt{d})^{g-1} \cdot (d+1-g) \cdot C_{d+1-g} \cdot r_g^{d-g} dr_g \ , \end{aligned}$$

since all points Q_{g+1}, which satisfy dist $(Q_{g+1}; < P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_g >) \le r$, are contained in a product of a $(g-1)$-dimensional shape of volume at most $(\sqrt{d})^{g-1}$ and a $(d+1-g)$-dimensional ball of radius r.

For $g = k-1$, having fixed the points $P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_{k-1} \in [0,1]^d$, to fulfill vol $(P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_k) \le V_0$, we must have

$$\frac{(j-1)!}{(k-1)!} \cdot \text{dist } (Q_k; < P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_{k-1} >) \cdot \text{vol } (P_1, \ldots, P_j) \cdot \prod_{g=j}^{k-2} r_g \le V_0 \ ,$$

and, using vol $(P_1, \ldots, P_j) > N^{-\gamma_j}$, this happens with probability at most

$$(\sqrt{d})^{k-2} \cdot C_{d-k+2} \cdot \left(\frac{(k-1)!}{(j-1)!} \cdot \frac{V_0 \cdot N^{\gamma_j}}{\prod_{g=j}^{k-2} r_g} \right)^{d-k+2} \ .$$

Putting all these probabilities together, we obtain for constants $c_{2,j}^*, c_{2,j}^{**} > 0$ the following upper bound, which finishes the proof of Lemma 2:

$$
\begin{aligned}
&\text{Prob}\,(\{P_1,\dots,P_k\},\{P_1,\dots,P_j,Q_{j+1},\dots,Q_k\} \text{ is a 'bad } j\text{-pair of simplices'})\\
&\le c_k^* \cdot V_0^{d-k+2} \cdot \int_{r_{k-2}=0}^{\sqrt{d}} \dots \int_{r_j=0}^{\sqrt{d}} d^{\frac{k-2}{2}} \cdot C_{d+2-k} \cdot \frac{(k-1)!^{d-k+2}}{(j-1)!^{d-k+2}} \cdot \\
&\quad \cdot \left(\frac{V_0 \cdot N^{\gamma_j}}{\prod_{g=j}^{k-2} r_g}\right)^{d-k+2} \cdot \prod_{g=j}^{k-2} \left(d^{\frac{g-1}{2}} \cdot (d+1-g) \cdot C_{d+1-g} \cdot r_g^{d-g}\right) dr_{k-2}\dots dr_j \le \\
&\le c_{2,j}^{**} \cdot V_0^{2(d-k+2)} \cdot N^{\gamma_j(d-k+2)} \cdot \int_{r_{k-2}=0}^{\sqrt{d}} \dots \int_{r_j=0}^{\sqrt{d}} \prod_{g=j}^{k-2} r_g^{k-g-2} dr_{k-2}\dots dr_j \\
&\le c_{2,j}^* \cdot V_0^{2(d-k+2)} \cdot N^{\gamma_j(d-k+2)} \qquad \text{as } k-g-2 \ge 0. \qquad \square
\end{aligned}
$$

Using (7) and (8) and Markov's inequality, there exist $N = n^{1+\alpha}$ points in the unit-cube $[0,1]^d$ such that the corresponding hypergraph $\mathcal{G} = (V, \mathcal{E}_2 \cup \dots \cup \mathcal{E}_k)$ on $|V| = N$ vertices satisfies for $i, j = 2, \dots, k-1$ and $3 \le k \le d+1$:

$$|\mathcal{E}_i| \le 2k \cdot c_i' \cdot N^{i-\gamma_i(d-i+2)d} \tag{9}$$
$$|\mathcal{E}_k| \le 2k \cdot c_k' \cdot V_0^{d-k+2} \cdot N^k \tag{10}$$
$$|BP_j(\mathcal{G})| \le 2k \cdot c_{2,j}' \cdot V_0^{2(d-k+2)} \cdot N^{2k-j+\gamma_j(d-k+2)}\ . \tag{11}$$

By (10) the average degree $t^{k-1} := k \cdot |\mathcal{E}_k|/|V|$ of $\mathcal{G} = (V, \mathcal{E}_2 \cup \dots \cup \mathcal{E}_k)$ among the edges from $\mathcal{E}_k$ satisfies $t^{k-1} \le 2k^2 \cdot c_k' \cdot V_0^{d-k+2} \cdot N^{k-1} =: t_0^{k-1}$. For some suitable constant $\varepsilon > 0$, we pick uniformly at random and independently of each other vertices from V with probability $p := N^\varepsilon / t_0 \le 1$. Let $V^* \subseteq V$ be the random set of the chosen vertices, and let $\mathcal{G}^* = (V^*, \mathcal{E}_2^* \cup \dots \cup \mathcal{E}_k^*)$ with $\mathcal{E}_i^* := \mathcal{E}_i \cap [V^*]^i$, $i = 2, \dots, k$, be the resulting random induced subhypergraph of $\mathcal{G}$. By (9) – (11) we infer for the expected numbers of vertices, i-element edges and 'bad j-pairs of simplices' in $\mathcal{G}^*$, $i, j = 2, \dots, k-1$, for constants $c_1, c_i, c_{2,j}, c_k > 0$:

$$
\begin{aligned}
E(|V^*|) &= p \cdot N \ge c_1 \cdot N^\varepsilon / V_0^{\frac{d-k+2}{k-1}} \\
E(|\mathcal{E}_i^*|) &= p^i \cdot |\mathcal{E}_i| \le p^i \cdot 2k \cdot c_i' \cdot N^{i-\gamma_i(d-i+2)} \le c_i \cdot N^{i\varepsilon-\gamma_i(d-i+2)} / V_0^{\frac{i(d-k+2)}{k-1}} \\
E(|\mathcal{E}_k^*|) &= p^k \cdot |\mathcal{E}_k| \le p^k \cdot 2k \cdot c_k' \cdot V_0^{d-k+2} \cdot N^k \le c_k \cdot N^{k\varepsilon} / V_0^{\frac{d-k+2}{k-1}} \\
E(|BP_j(\mathcal{G}^*)|) &= p^{2k-j} \cdot |BP_j(\mathcal{G})| \le c_{2,j} \cdot V_0^{\frac{(j-2)(d-k+2)}{k-1}} \cdot N^{(2k-j)\varepsilon+\gamma_j(d-k+2)}\ .
\end{aligned}
$$

By Chernoff's and Markov's inequality there exists an induced subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_2^* \cup \cdots \cup \mathcal{E}_k^*)$ of $\mathcal{G}$, such that for $i, j = 2, \ldots, k-1$:

$$|V^*| \geq (c_1 - o(1)) \cdot N^{\varepsilon} / V_0^{\frac{d-k+2}{k-1}} \tag{12}$$

$$|\mathcal{E}_i^*| \leq 2k \cdot c_i \cdot N^{i\varepsilon - \gamma_i(d-i+2)} / V_0^{\frac{i(d-k+2)}{k-1}} \tag{13}$$

$$|\mathcal{E}_k^*| \leq 2k \cdot c_k \cdot N^{k\varepsilon} / V_0^{\frac{d-k+2}{k-1}} \tag{14}$$

$$|BP_j(\mathcal{G}^*)| \leq 2k \cdot c_{2,j} \cdot V_0^{\frac{(j-2)(d-k+2)}{k-1}} \cdot N^{(2k-j)\varepsilon + \gamma_j(d-k+2)} . \tag{15}$$

Now we set for some suitable constant $c^* > 0$:

$$V_0 := c^* \cdot (\log n)^{\frac{1}{d-k+2}} / n^{\frac{k-1}{d-k+2}} . \tag{16}$$

Lemma 3. *For $j = 2, \ldots, k-1$ and for fixed $0 < \varepsilon < (j-1)/((2k-j-1) \cdot (1+\alpha)) - \gamma_j \cdot (d-k+2)/(2k-j-1)$ it is $|BP_j(\mathcal{G}^*)| = o(|V^*|)$.*

Proof. Using (12), (15) and (16) with $N = n^{1+\alpha}$, where $\alpha, \gamma_j > 0$ are constants, $j = 2, \ldots, k-1$, we have

$$\begin{aligned}
& |BP_j(\mathcal{G}^*)| = o(|V^*|) \\
\Longleftarrow\ & V_0^{\frac{(j-2)(d-k+2)}{k-1}} \cdot N^{(2k-j)\varepsilon + \gamma_j(d-k+2)} = o(N^{\varepsilon} / V_0^{\frac{d-k+2}{k-1}}) \\
\Longleftrightarrow\ & V_0^{\frac{(j-1)(d-k+2)}{k-1}} \cdot N^{(2k-j-1)\varepsilon + \gamma_j(d-k+2)} = o(1) \\
\Longleftrightarrow\ & n^{(1+\alpha)((2k-j-1)\varepsilon + \gamma_j(d-k+2)) - (j-1)} \cdot \log^{\frac{j-1}{k-1}} n = o(1) \\
\Longleftarrow\ & \varepsilon < \frac{j-1}{(2k-j-1) \cdot (1+\alpha)} - \frac{\gamma_j \cdot (d-k+2)}{2k-j-1} .
\end{aligned}$$

□

Lemma 4. *For $i = 2, \ldots, k-1$ and fixed $0 < \varepsilon \leq \gamma_i \cdot (d-i+2)/(i-1) - 1/(1+\alpha)$ it is $|\mathcal{E}_i^*| = o(|V^*|)$.*

Proof. By (12), (13) and (16), using $N = n^{1+\alpha}$, we infer

$$\begin{aligned}
& |\mathcal{E}_i^*| = o(|V^*|) \\
\Longleftarrow\ & N^{i\varepsilon - \gamma_i(d-i+2)} / V_0^{\frac{i(d-k+2)}{k-1}} = o(N^{\varepsilon} / V_0^{\frac{d-k+2}{k-1}}) \\
\Longleftrightarrow\ & N^{(i-1)\varepsilon - \gamma_i(d-i+2)} / V_0^{\frac{(i-1)(d-k+2)}{k-1}} = o(1) \\
\Longleftrightarrow\ & n^{(1+\alpha)((i-1)\varepsilon - \gamma_i(d-i+2)) + (i-1)} / \log^{\frac{i-1}{k-1}} n = o(1) \\
\Longleftarrow\ & \varepsilon \leq \frac{\gamma_i \cdot (d-i+2)}{i-1} - \frac{1}{1+\alpha} .
\end{aligned}$$

□

The assumptions in Lemmas 3 and 4 are satisfied for $\gamma_j := (j-1)/((d-k+5/2)(1+\alpha))$, $j = 2, \ldots, k-1$, and $\varepsilon := 1/(4kd(1+\alpha))$ and $\alpha := 1/(4kd)$, also $p = N^{\varepsilon}/t_0 \leq 1$ holds. In the induced subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_2^* \cup \cdots \cup \mathcal{E}_k^*)$ we delete one vertex from each i-element edge and from each 'bad j-pair of simplices', $i, j = 2, \ldots, k-1$. Let $V^{**} \subseteq V^*$ be the set of remaining vertices. The on V^{**} induced subhypergraph $\mathcal{G}^{**}$ of $\mathcal{G}^*$ is k-uniform, hence $\mathcal{G}^{**} = (V^{**}, \mathcal{E}_k^{**})$ with $\mathcal{E}_k^{**} := [V^{**}]^k \cap \mathcal{E}_k^*$, and fulfills $|V^{**}| = (1-o(1)) \cdot |V^*|$ by Lemmas 3 and 4. By (12) and (14) we have $|V^{**}| \geq c_1/2 \cdot N^\varepsilon / V_0^{(d-k+2)/(k-1)}$ and $|\mathcal{E}_k^{**}| \leq |\mathcal{E}_k^*| \leq 2k \cdot c_k \cdot N^{k\varepsilon}/V_0^{(d-k+2)/(k-1)}$, hence $\mathcal{G}^{**}$ has average degree $t^{k-1} = k \cdot |\mathcal{E}_k^{**}|/|V^{**}| \leq (4k^2 \cdot c_k/c_1) \cdot N^{(k-1)\varepsilon} =: t_1^{k-1}$. Now the assumptions of Theorem 2 are fulfilled by the k-uniform subhypergraph $\mathcal{G}^{**}$ of $\mathcal{G}$, as it is linear, and with (4) we obtain for constants $c_k^*, c', c_1, c_k, c^* > 0$:

$$
\begin{aligned}
\alpha(\mathcal{G}) &\geq \alpha(\mathcal{G}^{**}) \geq c_k^* \cdot \frac{|V^{**}|}{t} \cdot \log^{1/(k-1)} t \geq c_k^* \cdot \frac{|V^{**}|}{t_1} \cdot \log^{1/(k-1)} t_1 \geq \\
&\geq c_k^* \cdot \frac{c_1^{k/(k-1)} \cdot N^\varepsilon / V_0^{(d-k+2)/(k-1)}}{2 \cdot (4k^2 \cdot c_k)^{1/(k-1)} \cdot N^\varepsilon} \cdot \left(\log \left(\frac{4k^2 \cdot c_k}{c_1} \cdot N^{(k-1)\varepsilon} \right)^{\frac{1}{k-1}} \right)^{\frac{1}{k-1}} \\
&\geq c' \cdot \log^{1/(k-1)} n / V_0^{(d-k+2)/(k-1)} \qquad \text{as } N = n^{1+\alpha} \\
&\geq c' \cdot (1/c^*)^{(d-k+2)/(k-1)} \cdot \log^{1/(k-1)} n \cdot \frac{n}{\log^{1/(k-1)} n} \geq n \; ,
\end{aligned}
$$

where the last inequality follows by choosing in (16) a sufficiently small constant $c^* > 0$. Thus the hypergraph $\mathcal{G}$ contains an independent set $I \subseteq V$ with $|I| = n$. These n vertices yield n points in $[0,1]^d$, such that each k-point simplex arising from these points has volume bigger than V_0, i.e. $\Delta_{k,d}(n) = \Omega((\log n)^{1/(d-k+2)}/n^{(k-1)/(d-k+2)})$, which finishes the proof of (3). □

3 An Upper Bound on $\Delta_{k,d}(n)$

Here we show the upper bounds in Theorem 1, namely that for fixed $2 \leq k \leq d+1$ and constants $c_k', c_k'' > 0$ it is $\Delta_{k,d}(n) \leq c_k'/n^{(k-1)/d}$, moreover $\Delta_{k,d}(n) \leq c_k''/n^{(k-1)/d+(k-2)/(2d(d-1))}$ for k even.

Proof. We prove first that $\Delta_{k,d}(n) \leq c_k'/n^{(k-1)/d}$ for some constant $c_k' > 0$ and $2 \leq k \leq d+1$. Given any n points $P_1, P_2, \ldots, P_n \in [0,1]^d$, for some value $D > 0$ we construct a graph $G = G(D) = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$, where vertex i corresponds to the point $P_i \in [0,1]^d$, and edge set E with $\{i, j\} \in E$ being an edge if and only if dist $(P_i, P_j) \leq D$. An independent set $I \subseteq V$ in this graph $G = G(D)$ yields a subset $I' \subseteq \{P_1, P_2, \ldots, P_n\}$ of points in $[0,1]^d$ with Euclidean distance between any two distinct points bigger than D. Each ball $B_r(P)$ with center $P \in [0,1]^d$ and radius $r \leq 1$ satisfies vol $(B_r(P) \cap [0,1]^d) \geq$ vol $(B_r(P))/2^d$. The balls with radius $D/2$ and centers from an independent set I' have pairwise empty intersection. As each ball $B_{D/2}(P)$ has volume $C_d \cdot (D/2)^d$, we infer $|I'| \cdot C_d \cdot (D/2)^d/2^d \leq$ vol $([0,1]^d) = 1$, and hence the independence number $\alpha(G)$ of G satisfies

$$\alpha(G) \leq \frac{4^d}{C_d \cdot D^d} \; . \tag{17}$$

For $D := c/n^{1/d}$ with $c := (2 \cdot (k-1) \cdot 4^d / C_d)^{1/d}$ a constant, the average degree t of $G(D)$ satisfies $t \geq 1$ for $n \geq 2^{d+1}$, hence by Turán's theorem, $\alpha(G) \geq n/(2 \cdot t)$. With (17) this yields

$$\frac{4^d}{C_d \cdot D^d} \geq \alpha(G) \geq \frac{n}{2 \cdot t} \quad \Longrightarrow \quad t \geq \frac{C_d}{2 \cdot 4^d} \cdot n \cdot D^d \geq k-1 \, . \tag{18}$$

Hence there exists a vertex $i_1 \in V$ and $k-1$ edges $\{i_1, i_2\}, \ldots, \{i_1, i_k\} \in E$ incident at vertex i_1. By construction, each point $P_{i_j} \in [0,1]^d$, $j = 2, \ldots, k$, satisfies dist $(P_{i_1}, P_{i_j}) \leq D$, thus dist $(P_{i_j}; < P_{i_1}, P_{i_2}, \ldots, P_{i_{j-1}} >) \leq c/n^{1/d}$ for $j = 2, \ldots, k$, which implies vol $(P_{i_1}, \ldots, P_{i_k}) \leq (1/(k-1)!) \cdot c^{k-1}/n^{(k-1)/d}$, i.e. $\Delta_{k,d}(n) = O(1/n^{(k-1)/d})$.
For even $k \geq 4$ we are able to prove a better upper bound. From (18) we obtain $|E| = n \cdot t/2 \geq C_d \cdot n^2 \cdot D^d / 4^{d+1}$. Now let $c := (d \cdot 4^{d+1}/C_{d-1})^{1/d}$ and $D := 1/n^{1/d}$. We adapt an argument of Brass [6]. Each edge $\{i, j\} \in E$ determines a direction $(P_i P_j)$, which can be viewed as a vector of length 1. The minimum angular distance between these directions is at most

$$\left(\frac{d \cdot C_d}{C_{d-1} \cdot |E|} \right)^{1/(d-1)} \leq \left(\frac{d \cdot 4^{d+1}}{C_{d-1} \cdot c^d \cdot n} \right)^{1/(d-1)} \leq \frac{1}{n^{1/(d-1)}} \, .$$

Thus for some constant $c(d) > 0$ there exist $\binom{k}{2}$ directions $(P_i P_j)$, $\{i, j\} \in E$, with pairwise angular distance at most $\phi := c(d)/n^{1/(d-1)}$. The corresponding set $E^* \subseteq E$ of edges covers a subset $S \subseteq V$ of at least k vertices G. Consider a minimum subset $E^{**} \subseteq E^*$ of edges, which covers a subset $S^* \subseteq S$ of exactly k vertices. This set E^{**} contains only independent edges and stars. We pick one vertex from each independent edge $E \in E^{**}$ and the center of each star. Let $S^{**} \subseteq S^*$ be the set of chosen vertices with $|S^{**}| = s \leq k/2$.
For each vertex $v \in S^* \setminus S^{**}$ there exists an edge $\{v, w\} \in E^{**}$ for some vertex $w \in S^{**}$, hence dist $(P_v, P_w) \leq D$. Thus for each vertex $u \in S^* \setminus (S^{**} \cup \{v\})$ there is some vertex $t \in S^{**} \cup \{w\}$ such that the angular distance between the directions $(P_u P_t)$ and $(P_w P_v)$ is at most ϕ. Thus, the Euclidean distance between the point P_u and the affine space generated by the points P_r, $r \in S^{**} \cup \{v\}$, is at most D. With $D = c/n^{1/d}$ and $\sin \phi \leq \phi$ for $\phi \geq 0$, and $(s-1)! \cdot$ vol $(S^{**}) \leq (\sqrt{d})^{s-1}$ we obtain for the volume of the simplex determined by the k points P_s, $s \in S^*$, the following upper bound, which finishes the proof of Theorem 1:

$$\text{vol } (P_{s^*}; s^* \in S^*) \leq \frac{1}{(k-1)!} \cdot (\sqrt{d})^{s-1} \cdot D \cdot (D \cdot \sin \phi)^{k-s-1} \leq$$

$$\leq \frac{1}{(k-1)!} \cdot d^{(k-2)/4} \cdot D \cdot (D \cdot c(d)/n^{1/(d-1)})^{k/2-1} = \frac{d^{(k-2)/4} \cdot c(k)^{k/2-1}}{(k-1)! \cdot n^{\frac{k-1}{d} + \frac{k-2}{2d(d-1)}}} \, . \; \square$$

4 Concluding Remarks

Our arguments together with an algorithmic version of Theorem 2, see [4], yield a randomized polynomial time algorithm for obtaining a distribution of n points in $[0,1]^d$, which shows $\Delta_{k,d}(n) = \Omega((\log n)^{1/(k-1)}/n^{(k-1)/(d-k+2)})$ for fixed $3 \leq k \leq d+1$. It might be of interest to have a deterministic polynomial time algorithm achieving this lower bound, as well as investigating the case $k > d+1$, compare [13] for the case of dimension $d = 2$.

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