Large Triangles in the d-Dimensional Unit Cube ⋆

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Abstract

We consider a variant of Heilbronn's triangle problem by asking for a distribution of n points in the d-dimensional unit cube $[0,1]^d$ such that the minimum (2-dimensional) area of a triangle among these n points is as large as possible. Denoting by $\Delta_d^{\text{off-line}}(n)$ and $\Delta_d^{\text{on-line}}(n)$ the supremum of the minimum area of a triangle among n points over all distributions of n points in $[0,1]^d$ for the off-line and the on-line situation, respectively, for fixed dimension $d \geq 2$ we show that $c_1 \cdot (\log n)^{1/(d-1)}/n^{2/(d-1)} \leq \Delta_d^{\text{off-line}}(n) \leq c_1'/n^{2/d}$ and $c_2/n^{2/(d-1)} \leq \Delta_d^{\text{on-line}}(n) \leq c_2'/n^{2/d}$ for constants $c_1, c_2, c_1', c_2' > 0$ which depend on d only. Moreover, we provide a deterministic polynomial time algorithm that achieves the lower bound $\Omega((\log n)^{1/(d-1)}/n^{2/(d-1)})$ on the minimum area of a triangle among n points in $[0,1]^d$ in the off-line case.

Key words: Heilbronn problem, independent sets, hypergraphs.

1 Introduction

Given any integer $n \geq 3$, originally Heilbronn's problem asks for the supremum $\Delta_2(n)$ of the minimum area of a triangle formed by three of n points over all distributions of n points in the unit square $[0,1]^2$. For primes n, the points $P_k = 1/n \cdot (k \bmod n, k^2 \bmod n)$, $k = 0, 1, \ldots, n-1$, easily show that $\Delta_2(n) = \Omega(1/n^2)$, as was pointed out by Erdős, see [14]. Komlós, Pintz and

⋆ A preliminary version of this paper has appeared in 'Proceedings 10th Annual International Computing and Combinatorics Conference COCOON'04', eds. K.-Y. Chwa and J. I. Munro, LNCS 3106, (Springer Verlag, Berlin) 2004, 43–52.

Szemerédi [11] improved this to $\Delta_2(n) = \Omega(\log n/n^2)$, which is currently the best known lower bound, and Bertram-Kretzberg, Hofmeister and this author [5] provided a deterministic polynomial time algorithm achieving this lower bound. From the other side, improving earlier results of Roth [14–18] and Schmidt [19], Komlós, Pintz and Szemerédi [10] proved for some constant $c > 0$ the upper bound $\Delta_2(n) = O(2^{c\sqrt{\log n}}/n^{8/7})$. We remark that for n points chosen uniformly at random and independently of each other in the unit square $[0,1]^2$, the expected value of the minimum area of a triangle among these n points is $\Theta(1/n^3)$, as was shown recently by Jiang, Li and Vitány [9].

For fixed integers $d \geq 2$, a variant of Heilbronn's problem considered by Barequet [2], asks for the supremum $\Delta_d^*(n)$ of the minimum volume of a simplex formed by some $(d+1)$ of n points over all distributions of n points in the d-dimensional unit cube $[0,1]^d$. He showed in [2] the lower bound $\Delta_d^*(n) = \Omega(1/n^d)$, which was improved in [12] to $\Delta_d^*(n) = \Omega(\log n/n^d)$. In [13] for dimension $d = 3$ a deterministic polynomial time algorithm was given achieving the lower bound $\Delta_3^*(n) = \Omega(\log n/n^3)$. Recently, Brass [6] showed the upper bound $\Delta_d^*(n) = O(1/n^{1+1/(2d)})$ for odd $d \geq 3$, while for even $d \geq 4$ only $\Delta_d^*(n) = O(1/n)$ is known. Moreover, an on-line version of this variant was considered by Barequet [3] for dimensions $d = 3$ and $d = 4$, where he showed the lower bounds $\Omega(1/n^{10/3})$ and $\Omega(1/n^{127/24})$, respectively.

Here we investigate the following extension of Heilbronn's triangle problem to higher dimensions: for fixed integers $d \geq 2$ and any given integer $n \geq 3$, find n points in the d-dimensional unit cube $[0,1]^d$ such that the minimum area of a triangle determined by three of these n points is as large as possible. We consider the off-line as well as the on-line version of our problem. In the off-line situation the number n of points is given in advance, while in the on-line case the points are positioned one after the other in the unit cube $[0,1]^d$ and at some time suddenly this process stops. Let the corresponding supremum values – over all distributions of n points in $[0,1]^d$ – on the minimum areas of a triangle among n points in $[0,1]^d$ be denoted by $\Delta_d^{\text{off-line}}(n)$ and $\Delta_d^{\text{on-line}}(n)$, respectively.

Theorem 1 *Let $d \geq 2$ be a fixed integer. Then, for constants $c_1, c_2, c_1', c_2' > 0$, which depend on d only, for every integer $n \geq 3$, it is*

$$c_1 \cdot \frac{(\log n)^{1/(d-1)}}{n^{2/(d-1)}} \leq \Delta_d^{\text{off-line}}(n) \leq \frac{c_1'}{n^{2/d}} \tag{1}$$

$$\frac{c_2}{n^{2/(d-1)}} \leq \Delta_d^{\text{on-line}}(n) \leq \frac{c_2'}{n^{2/d}}\,. \tag{2}$$

The lower bound (1) extends the results of Komlós, Pintz and Szemerédi [11], who considered the case of dimension $d = 2$. The lower bounds (1) and (2) in the off-line and the on-line situation, respectively, differ only by a factor of $\Theta((\log n)^{1/(d-1)})$. In contrast, the lower bounds in the on-line situation considered by Barequet [3], i.e., maximizing the minimum volume of simplices among n points in $[0,1]^d$, differ by a factor of $\Theta(n^{1/3} \cdot \log n)$ for dimension $d = 3$ and by a factor of $\Theta(n^{31/24} \cdot \log n)$ for dimension $d = 4$ from the currently best known lower bound $\Delta_d^*(n) = \Omega(\log n/n^d)$; see [12] for the off-line situation for any fixed dimension $d \geq 2$.

Moreover, we provide a deterministic polynomial time algorithm – to some extent by derandomizing the arguments in the proof of Theorem 1 – which achieves the lower bound (1) in the off-line situation:

Theorem 2 *Let $d \geq 2$ be a fixed integer. For each integer $n \geq 3$ one can find deterministically in time $O(n^{5-2/d+\gamma})$ for any fixed $\gamma > 0$ some n points in the d-dimensional unit cube $[0,1]^d$ such that each triangle among these n points has area $\Omega((\log n)^{1/(d-1)}/n^{2/(d-1)})$.*

Theorem 2 extends the results from [5], where the case of dimension $d = 2$ was considered. Our considerations here also yield better estimates on the running time of an algorithm, which is similar to the one analyzed in [5].

2 The Off-Line Case

A *line* through the points $P_i, P_j \in [0,1]^d$ is denoted by P_iP_j. Let dist (P_i, P_j) be the *Euclidean distance* between the points P_i and P_j. The *area* of a triangle determined by three points $P_i, P_j, P_k \in [0,1]^d$ is denoted by area (P_i, P_j, P_k), where area $(P_i, P_j, P_k) :=$ dist $(P_i, P_j) \cdot h/2$ and h is the Euclidean distance of the point P_k from the line P_iP_j. For a subset $S \subseteq [0,1]^d$ let vol (S) be its d-dimensional *volume*. Throughout this paper, let C_d denote the volume of the d-dimensional unit ball in $\mathbb{R}^d$.

2.1 A Lower Bound on $\Delta_d^{\text{off-line}}(n)$

First we prove the lower bound in (1) from Theorem 1, namely that for fixed $d \geq 2$ and for some constant $c_1 = c_1(d) > 0$, it is

$$\Delta_d^{\text{off-line}}(n) \geq c_1 \cdot \frac{(\log n)^{1/(d-1)}}{n^{2/(d-1)}} . \tag{3}$$

PROOF. Let $d \geq 2$ be a fixed integer. For arbitrary integers $n \geq 3$ and a suitable constant $\beta > 0$, we select uniformly at random and independently of each other $N = n^{1+\beta}$ points $P_1, \ldots, P_N$ in the d-dimensional unit cube $[0,1]^d$. For fixed $1 \leq i < j < k \leq N$, we give an upper bound on the probability that area $(P_i, P_j, P_k) \leq A$ for some value $A > 0$, which will be specified later. The point P_i can be anywhere in $[0,1]^d$. Given the point P_i, the probability that the point $P_j \in [0,1]^d$ has from P_i a Euclidean distance within the infinitesimal range $[r, r + \mathrm{d}r]$, is at most the difference of the volumes of the d-dimensional balls with center P_i and with radii $(r + \mathrm{d}r)$ and r, respectively, hence

$$\text{Prob } (r \leq \text{ dist } (P_i, P_j) \leq r + \mathrm{d}r) \leq d \cdot C_d \cdot r^{d-1} \,\mathrm{d}r \,, \tag{4}$$

where C_d is equal to the volume of the d-dimensional unit ball in $\mathbb{R}^d$. Given the points P_i and P_j with dist $(P_i, P_j) = r$, the third point $P_k \in [0,1]^d$ satisfies area $(P_i, P_j, P_k) \leq A$, if P_k is contained in the set $C_{i,j} \cap [0,1]^d$, where $C_{i,j}$ is a d-dimensional cylinder, centered at the line P_iP_j with radius $2 \cdot A/r$ and height $\sqrt{d}$. The d-dimensional volume vol $(C_{i,j} \cap [0,1]^d)$ is at most $C_{d-1} \cdot (2 \cdot A/r)^{d-1} \cdot \sqrt{d}$, and we infer for some constant $c_3 > 0$:

$$\begin{aligned}
&\text{Prob } (\text{area } (P_i, P_j, P_k) \leq A) \\
\leq &\int_0^{\sqrt{d}} \text{vol } (C_{i,j} \cap [0,1]^d) \cdot d \cdot C_d \cdot r^{d-1} \,\mathrm{d}r \\
\leq &\int_0^{\sqrt{d}} C_{d-1} \cdot \left(\frac{2 \cdot A}{r}\right)^{d-1} \cdot \sqrt{d} \cdot d \cdot C_d \cdot r^{d-1} \,\mathrm{d}r \\
= &C_{d-1} \cdot C_d \cdot 2^{d-1} \cdot d^{3/2} \cdot A^{d-1} \cdot \int_0^{\sqrt{d}} \mathrm{d}r = c_3 \cdot A^{d-1}.
\end{aligned} \tag{5}$$

In our further considerations we use hypergraphs.

Definition 3 *A hypergraph $\mathcal{G} = (V, \mathcal{E})$ with vertex-set V and edge-set $\mathcal{E}$ is* k-uniform *if $|E| = k$ for all edges $E \in \mathcal{E}$. A hypergraph $\mathcal{G} = (V, \mathcal{E})$ is* linear *if $|E \cap E'| \leq 1$ for all distinct edges $E, E' \in \mathcal{E}$. A subset $I \subseteq V$ of the vertex-set is* independent *if I contains no edges from $\mathcal{E}$. The largest size $|I|$ of an independent set in $\mathcal{G}$ is the* independence number *$\alpha(\mathcal{G})$.*

We form a random 3-uniform hypergraph $\mathcal{G} = \mathcal{G}(A) = (V, \mathcal{E}_3)$ with vertex-set $V = \{1, \ldots, N\}$, where vertex i corresponds to the random point $P_i \in [0,1]^d$, $i = 1, \ldots, n$, and with with edge-set $\mathcal{E}_3$, where $\{i, j, k\} \in \mathcal{E}_3$ if and only if area $(P_i, P_j, P_k) \leq A$. The expected number $E(|\mathcal{E}_3|)$ of edges in $\mathcal{G}$ satisfies by (5) for some constant $c_3 > 0$:

$$E(|\mathcal{E}_3|) \leq \binom{N}{3} \cdot c_3' \cdot A^{d-1} \leq c_3 \cdot A^{d-1} \cdot N^3. \tag{6}$$

We use the following result on the independence number of linear k-uniform hypergraphs due to Ajtai, Komlós, Pintz, Spencer and Szemerédi [1], see [7].

Theorem 4 [1,7] *Let $k \geq 3$ be a fixed integer. Let $\mathcal{G} = (V, \mathcal{E})$ be a k-uniform hypergraph on $|V| = N$ vertices with average degree $t^{k-1} = k \cdot |\mathcal{E}|/|V|$. If $\mathcal{G}$ is linear, then its independence number $\alpha(\mathcal{G})$ satisfies for some constant $c_k^* > 0$:*

$$\alpha(\mathcal{G}) \geq c_k^* \cdot \frac{N}{t} \cdot (\log t)^{\frac{1}{k-1}}. \tag{7}$$

Let $D := N^{-\gamma}$ for a suitable constant γ with $0 < \gamma < 1$, which will be fixed later. Let $BP_D(\mathcal{G})$ be a random variable, which counts the number of 'bad pairs of triangles' in $\mathcal{G}$, which are among the N random points $P_1, \ldots, P_N \in [0,1]^d$ those unordered pairs of triangles sharing an edge, where both triangles have area at most A and all sides of the two triangles have length at least D. Let $P_D(\mathcal{G})$ be another random variable, which counts the number of unordered pairs of distinct points with Euclidean distance at most D among the N randomly chosen points $P_1, \ldots, P_N \in [0,1]^d$.

To apply Theorem 4, we estimate in the random hypergraph $\mathcal{G} = \mathcal{G}(A) = (V, \mathcal{E}_3)$ the expected number $E(BP_D(\mathcal{G}))$ of 'bad pairs of triangles' and the expected number $E(P_D(\mathcal{G}))$ of unordered pairs of distinct points with Euclidean distance at most D among the N random points $P_1, \ldots, P_N \in [0,1]^d$. We show that both numbers $E(BP_D(\mathcal{G}))$ and $E(P_D(\mathcal{G}))$ are $o(N)$ for some choice of the parameters A, D and N. Then in the hypergraph $\mathcal{G}$ on N vertices we delete one vertex (point) from each unordered pair of vertices (points) with Euclidean distance at most D and from each 'bad pair of triangles', which yields a linear induced subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_3^*)$ of $\mathcal{G} = \mathcal{G}(A)$ on $(1 - o(1)) \cdot N$ vertices, thus $\mathcal{G}^*$ fulfills the assumptions of Theorem 4. An independent set in this subhypergraph $\mathcal{G}^*$ yields a subset I of points in the unit cube $[0,1]^d$, such that the area of each triangle arising in I is bigger than A.

First we give an upper bound on the expected number $E(P_D(\mathcal{G}))$ of unordered pairs of distinct points with Euclidean distance at most D among the N random points $P_1, \ldots, P_N \in [0,1]^d$. For fixed integers i, j, $1 \leq i < j \leq N$, we have

$$\text{Prob } (\text{dist } (P_i, P_j) \leq D) \leq C_d \cdot D^d,$$

since the point P_i can be anywhere in $[0,1]^d$ and, given the point P_i with dist $(P_i, P_j) \leq D$, the point P_j is contained in the d-dimensional ball with radius D and center P_i, i.e., Prob (dist $(P_i, P_j) \leq D) \leq C_d \cdot D^d$. Thus the expected number $E(P_D(\mathcal{G}))$ of unordered pairs of distinct points with Euclidean distance at most D among the N points in $[0,1]^d$ satisfies for some constant $c_2 > 0$:

$$E(P_D(\mathcal{G})) \leq \binom{N}{2} \cdot C_d \cdot D^d \leq c_2 \cdot N^2 \cdot D^d. \tag{8}$$

Next, for fixed $1 \leq i < j < k < l \leq N$, we give an upper bound on the probability that the random points P_i, P_j, P_k, P_l yield a 'bad pair of triangles', in which case the Euclidean distance between any two points from each of the two triangles is at least D. There are $\binom{4}{2}$ choices for the joint side of the two triangles, given by the points P_i and P_j, say. Given the point $P_i \in [0,1]^d$, by (4) we have Prob $(r \leq \text{dist}\,(P_i, P_j) \leq r + \mathrm{d}r) \leq d \cdot C_d \cdot r^{d-1}\,\mathrm{d}r$. Given the points $P_i, P_j \in [0,1]^d$ with dist $(P_i, P_j) = r \geq D$, the probability that both triangles P_i, P_j, P_k and P_i, P_j, P_l have area at most A, is at most the square of the volume of the cylinder, which is centered at the line P_iP_j with height $\sqrt{d}$ and radius $2 \cdot A/r$. Thus, for dimension $d \geq 3$ we obtain for some constant $c_4 > 0$:

$$\begin{aligned}
&\text{Prob } (P_i, P_j, P_k, P_l \text{ form a 'bad pair of triangles')} \\
&\leq \binom{4}{2} \cdot \int_D^{\sqrt{d}} \left(C_{d-1} \cdot \sqrt{d} \cdot \left(\frac{2 \cdot A}{r} \right)^{d-1} \right)^2 \cdot d \cdot C_d \cdot r^{d-1}\,\mathrm{d}r \\
&= c_4 \cdot A^{2d-2} \cdot \int_D^{\sqrt{d}} \frac{\mathrm{d}r}{r^{d-1}} \qquad (9)\\
&= \frac{c_4}{d-2} \cdot A^{2d-2} \cdot \left(\frac{1}{D^{d-2}} - \frac{1}{d^{(d-2)/2}} \right) \leq c_4 \cdot A^{2d-2}/D^{d-2}.
\end{aligned}$$

For $d = 2$, with $D = N^{-\gamma}$ for some constant γ with $0 < \gamma < 1$, the expression (9) is bounded from above by $c_4 \cdot A^2 \cdot \log N$. Thus, for fixed dimension $d \geq 2$ we infer

$$\begin{aligned}
&\text{Prob } (P_i, P_j, P_k, P_l \quad \text{form a 'bad pair of triangles')} \\
&\leq c_4 \cdot A^{2d-2} \cdot \log N/D^{d-2},
\end{aligned}$$

and we obtain for a constant $c_4 > 0$ for the expected number $E(BP_D(\mathcal{G}))$ of 'bad pairs of triangles' among the N random points:

$$\begin{aligned}
E(BP_D(\mathcal{G})) &\leq \binom{N}{4} \cdot c_4 \cdot A^{2d-2} \cdot \log N/D^{d-2} \\
&\leq c_4 \cdot A^{2d-2} \cdot N^4 \cdot \log N/D^{d-2}. \qquad (10)
\end{aligned}$$

Using (6), (8), (10) and Markov's inequality there exist N points in the unit cube $[0,1]^d$ such that the corresponding 3-uniform hypergraph $\mathcal{G} = \mathcal{G}(A) = (V, \mathcal{E}_3)$ on $|V| = N$ vertices satisfies

$$\begin{aligned}
|\mathcal{E}_3| &\leq 3 \cdot c_3 \cdot A^{d-1} \cdot N^3 \qquad (11)\\
P_D(\mathcal{G}) &\leq 3 \cdot c_2 \cdot N^2 \cdot D^d \qquad (12)\\
BP_D(\mathcal{G}) &\leq 3 \cdot c_4 \cdot A^{2d-2} \cdot N^4 \cdot \log N/D^{d-2}. \qquad (13)
\end{aligned}$$

For some suitable constant $c^* > 0$, which will be specified later, we set

$$A := c^* \cdot \frac{(\log n)^{\frac{1}{d-1}}}{n^{\frac{2}{d-1}}} \, . \tag{14}$$

Recall that $N = n^{1+\beta}$ and $D = N^{-\gamma}$ for some constants $\beta, \gamma > 0$ with $\gamma < 1$.

Lemma 5 *For fixed $\beta, \gamma > 0$ with $\gamma < 4/((d-1) \cdot (1+\beta)) - 3/(d-1)$, it is*

$$BP_D(\mathcal{G}) = o(|V|) \, .$$

PROOF. Using (13) and (14) with $|V| = N = n^{1+\beta}$ and $D = N^{-\gamma}$, where $\beta, \gamma > 0$ are constants, we have

$$\begin{aligned}
& BP_D(\mathcal{G}) = o(|V|) \\
\Longleftarrow\, & A^{2d-2} \cdot N^4 \cdot \log N / D^{d-2} = o(N) \\
\Longleftrightarrow\, & N^{3+\gamma(d-2)} \cdot \log^2 n \cdot \log N / n^4 = o(1) \\
\Longleftrightarrow\, & n^{(1+\beta)(3+\gamma(d-2))-4} \cdot \log^3 n = o(1) \\
\Longleftarrow\, & (1+\beta) \cdot (3 + \gamma \cdot (d-2)) < 4 \, ,
\end{aligned}$$

which holds for $0 < \gamma < 4/((d-1) \cdot (1+\beta)) - 3/(d-1)$. $\square$

Lemma 6 *For fixed $\gamma > 1/d$, it is*

$$P_D(\mathcal{G}) = o(|V|) \, .$$

PROOF. By (12) and (14), using $|V| = N$ and $D = N^{-\gamma}$ for some constant $\gamma > 0$, we infer

$$\begin{aligned}
& P_D(\mathcal{G}) = o(|V|) \\
\Longleftarrow\, & N^2 \cdot D^d = o(N) \\
\Longleftrightarrow\, & N \cdot D^d = o(1) \\
\Longleftrightarrow\, & N^{1-\gamma d} = o(1) \\
\Longleftrightarrow\, & \gamma > 1/d \, ,
\end{aligned}$$

which holds for $\gamma > 1/d$. $\square$

Now we fix $\beta := 1/(8d)$ and $\gamma := 1/(d-1/2)$. Then all assumptions in Lemmas 5 and 6 are fulfilled. In the hypergraph $\mathcal{G} = (V, \mathcal{E}_3)$ we delete one vertex from each 'bad pair of triangles', i.e., all corresponding triangles have side-lengths at least D, and from each unordered pair of vertices where the corresponding points have Euclidean distance at most D. Let $V^* \subseteq V$ be the set of the remaining vertices. The resulting induced subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_3^*)$ with $\mathcal{E}_3^* := [V^*]^3 \cap \mathcal{E}_3^*$ fulfills $|V^*| = (1 - o(1)) \cdot |V|$ by Lemmas 5 and 6 and with (11) we obtain

$$|V^*| \geq N/2 \qquad \text{and} \qquad |\mathcal{E}_3^*| \leq 3 \cdot c_3 \cdot A^{d-1} \cdot N^3. \tag{15}$$

By (15) the hypergraph $\mathcal{G}^*$ has average degree

$$t^2 = \frac{3 \cdot |\mathcal{E}_3^*|}{|V^*|} \leq \frac{9 \cdot c_3 \cdot A^{d-1} \cdot N^3}{N/2} = 18 \cdot c_3 \cdot A^{d-1} \cdot N^2 =: t_1^2 \, . \tag{16}$$

The induced subhypergraph $\mathcal{G}^*$ does not contain any two distinct edges $E, E' \in \mathcal{E}_3^*$ with $|E \cap E'| = 2$, i.e., $\mathcal{G}^*$ is a linear hypergraph. The assumptions of Theorem 4 are fulfilled by the 3-uniform subhypergraph $\mathcal{G}^*$ of $\mathcal{G}$. Notice that the term $1/t \cdot \log^{1/2} t$ is decreasing for $t > e^{1/4}$. Thus for $t \geq 2$ we obtain for constants $c_3^*, c, c_1, c^* > 0$ with (7), (14), (15), (16) the following lower bound on its independence number:

$$\begin{aligned}
\alpha(\mathcal{G}) \geq \alpha(\mathcal{G}^*) &\geq c_3^* \cdot \frac{|V^*|}{t} \cdot \log^{1/2} t \geq c_3^* \cdot \frac{|V^*|}{t_1} \cdot \log^{1/2} t_1 \geq \\
&\geq c_3^* \cdot \frac{N/2}{(18 \cdot c_3 \cdot A^{d-1})^{1/2} \cdot N} \cdot \log^{1/2} \left((18 \cdot c_3 \cdot A^{d-1})^{1/2} \cdot N \right) \\
&\geq c \cdot \log^{1/2} n / A^{(d-1)/2} \qquad\qquad \text{as } N = n^{1+\beta} \\
&\geq c \cdot (1/c^*)^{(d-1)/2} \cdot \frac{n}{\log^{1/2} n} \cdot \log^{1/2} n \geq n
\end{aligned}$$

by choosing in (14) a small enough constant $c^* > 0$. For $t < 2$, we have $|\mathcal{E}_3^*| < 4 \cdot |V^*|/3$, hence by omitting from $\mathcal{G}^*$ successively for each pair of edges with at least one joint vertex, one of these joint vertices, we obtain $\alpha(\mathcal{G}) \geq \alpha(\mathcal{G}^*) \geq N/9 \geq n$. Thus the hypergraph $\mathcal{G}$ contains an independent set $I \subseteq V$ with $|I| = n$. These n vertices yield n points in $[0,1]^d$, where each triangle arising from these n points has area at least A, i.e., $\Delta_d^{\text{off-line}}(n) = \Omega((\log n)^{1/(d-1)}/n^{2/(d-1)})$, which finishes the proof of (3). $\square$

2.2 A Deterministic Algorithm

Next we prove Theorem 2. To provide for fixed integers $d \geq 2$ a deterministic polynomial time algorithm, which finds n points in the d-dimensional unit cube $[0,1]^d$ that achieve the lower bound $\Delta_d^{\text{off-line}}(n) = \Omega((\log n)^{1/(d-1)}/n^{2/(d-1)})$ on the minimum area of a triangle among these n points, we discretize the d-dimensional unit cube $[0,1]^d$ by considering the standard d-dimensional $T \times \cdots \times T$-grid, where $T = n^\beta$ for some constant $\beta > 0$. With this discretization we now must take into account collinear triples of grid-points in the $T \times \cdots \times T$-grid, which have area equal to zero.

PROOF. We proceed similarly as we did in Section 2.1 for proving the lower bound (1) in Theorem 1, but with some crucial differences due to the occurring collinear triples of grid-points. Set $D := T^\gamma \geq 1$ for some suitable constant $0 < \gamma < 1$. For some real number $A \geq 1/2$, which will be specified later, we form a hypergraph $\mathcal{G} = \mathcal{G}(A, D) = (V, \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_3^0)$ which contains 2-element edges and two types of 3-element edges. The vertex-set V consists of the T^d grid-points $P_1, \ldots, P_{T^d}$ from the $T \times \cdots \times T$-grid with $T = n^\beta$ for some suitable constant $\beta > 0$. The edge-sets $\mathcal{E}_2$, $\mathcal{E}_3$ and $\mathcal{E}_3^0$ are defined as follows. For distinct grid-points $P, Q \in V$ let $\{P, Q\} \in \mathcal{E}_2$ if and only if the Euclidean distance between P and Q fulfills dist $(P, Q) \leq D$. For distinct grid-points $P, Q, R \in V$ let $\{P, Q, R\} \in \mathcal{E}_3^0$ if and only if P, Q, R are on a line. Moreover, for distinct grid-points $P, Q, R \in V$ let $\{P, Q, R\} \in \mathcal{E}_3$ if and only if P, Q, R are not collinear, area $(P, Q, R) \leq A$ and pairwise the points P, Q, R have Euclidean distance bigger than D.

We are looking for a large independent set in this hypergraph $\mathcal{G} = \mathcal{G}(A, D) = (V, \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_3^0)$, as an independent set $I \subseteq V$ in $\mathcal{G}$ corresponds to $|I|$ many grid-points in the $T \times \cdots \times T$-grid, such that the area of each triangle arising among these $|I|$ grid-points is bigger than A.

We use the following algorithmic version of Theorem 4 of Bertram-Kretzberg and this author [4], see also Fundia [8].

Theorem 7 *Let $\mathcal{G} = (V, \mathcal{E})$ be a k-uniform, linear hypergraph with average degree $t^{k-1} := k \cdot |\mathcal{E}|/|V|$. Then one can find for any $\delta > 0$ in time $O(|V| + |\mathcal{E}| + |V|^3/t^{3-\delta})$ an independent set $I \subseteq V$ with $|I| = \Omega((|V|/t) \cdot (\log t)^{1/(k-1)})$.*

To find a suitable induced subhypergraph of $\mathcal{G} = \mathcal{G}(A, D) = (V, \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_3^0)$ to which Theorem 7 can be applied, we first count in the hypergraph $\mathcal{G} = \mathcal{G}(A, D) = (V, \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_3^0)$ carefully the numbers $|\mathcal{E}_2|$, $|\mathcal{E}_3|$ and $|\mathcal{E}_3^0|$ of 2- and both types of 3-element edges, respectively, and the numbers $s_2(\mathcal{G}; \mathcal{E}_3)$ of unordered pairs $\{E, E'\}$ of distinct edges $E, E' \in \mathcal{E}_3$, which have two vertices in common, i.e., $|E \cap E'| = 2$. Then in a certain induced subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_2^* \cup \mathcal{E}_3^* \cup \mathcal{E}_3^{0*})$ of $\mathcal{G}$ we destroy in one step all 2-element edges from $\mathcal{E}_2^*$, all edges from $\mathcal{E}_3^{0*}$ and all unordered pairs $\{E, E'\}$ of distinct edges $E, E' \in \mathcal{E}_3$ with $|E \cap E'| = 2$. The resulting induced subhypergraph $\mathcal{G}^{**}$ of $\mathcal{G}^*$ contains only edges from $\mathcal{E}_3^*$, hence is 3-uniform and also linear, and at this stage we apply to $\mathcal{G}^{**}$ the algorithm from Theorem 7.

For nonnegative integers $a_1, \ldots, a_d$, which are not all equal to zero, we denote by gcd $(a_1, \ldots, a_d) \geq 0$ the *greatest common divisor* of $a_1, \ldots, a_d$. Notice that for distinct grid-points $P = (p_1, \ldots, p_d)$ and $Q = (q_1, \ldots, q_d)$ in the $T \times \cdots \times T$-

grid there are exactly gcd $(|q_1-p_1|,\ldots,|q_d-p_d|)-1$ grid-points on the segment $[P,Q]$ excluding the endpoints P and Q.

Lemma 8 *For some constant $c_{3,0}>0$, the number $|\mathcal{E}_3^0|$ of collinear triples of grid-points in the d-dimensional $T\times\cdots\times T$-grid fulfills*

$$|\mathcal{E}_3^0| \leq c_{3,0}\cdot T^{2d}\cdot \log T\,. \tag{17}$$

Notice that for a constant $c_3^0>0$ one can obtain easily the upper bound $|\mathcal{E}_3^0|\leq c_3^0\cdot T^{2d+1}$ – which would suffice for our purposes –, as one can choose two grid-points P and R from the $T\times\cdots\times T$-grid in at most T^{2d} ways and at most T grid-points from the segment $[P,R]$. However, in order to get a better estimate on the running time of our algorithm, we use the upper bound (17) on the number of collinear triples of grid-points in the $T\times\cdots\times T$-grid.

PROOF. Let P,Q,R be a collinear triple in the d-dimensional $T\times\cdots\times T$-grid, where the grid-point Q lies on the segment $[P,R]$. For distinct grid-points $P=(p_1,\ldots,p_d)$ and $R=(r_1,\ldots,r_d)$ let $a_i:=r_i-p_i$, $i=1,\ldots,d$. With $0\leq p_i,r_i\leq T$ for $i=1,\ldots,d$ we have $|a_i|\leq T$. By symmetry, which we take into account by spending an additional constant factor $c>0$, we may assume that $a_d>0$. There are T^d choices for the grid-point P. Given the grid-point P and given $a_1,\ldots,a_d$, on the segment $[P,R]$ there are exactly gcd $(|a_1|,\ldots,|a_d|)-1$ grid-points Q excluding the endpoints P and R. Observe that for a divisor $t\geq 1$ there are at most $(2\cdot T+1)/t$ integers i, $-T\leq i\leq T$, which are divisible by t. Hence for constants $c,c',c_{3,0}>0$ the number of collinear triples in the d-dimensional $T\times\cdots\times T$-grid is at most

$$\begin{aligned}
&c\cdot T^d\cdot\sum_{a_d=1}^{T}\sum_{a_{d-1}=-T}^{T}\cdots\sum_{a_1=-T}^{T}\ \text{gcd}\,(|a_1|,\ldots,|a_d|)\\
&\leq c\cdot T^d\cdot\sum_{t=1}^{T} t\cdot\frac{T}{t}\cdot\left(\frac{2\cdot T+1}{t}\right)^{d-1}\leq c'\cdot T^{2d}\cdot\sum_{t=1}^{T}\frac{1}{t^{d-1}}\leq\\
&\leq c_3^0\cdot T^{2d} \qquad\qquad\qquad \text{for } d\geq 3.
\end{aligned}\tag{18}$$

For dimension $d=2$, we obtain from (18) the upper bound $c_{3,0}\cdot T^4\cdot\log T$, hence we have $|\mathcal{E}_3^0|\leq c_3^0\cdot T^{2d}\cdot\log T$ for any fixed dimension $d\geq 2$. □

Lemma 9 *For some constant $c_3>0$, the number $|\mathcal{E}_3|$ of nondegenerate triangles P,Q,R with area $(P,Q,R)\leq A$ in the d-dimensional $T\times\cdots\times T$-grid satisfies*

$$|\mathcal{E}_3|\leq c_3\cdot A^{d-1}\cdot T^{d+2}. \tag{19}$$

PROOF. Let P, Q, R yield a nondegenerate triangle in the d-dimensional $T \times \cdots \times T$-grid with area $(P, Q, R) \leq A$. A grid-point $P = (p_1, \ldots, p_d)$ can be chosen in T^d ways. Given the grid-point $P = (p_1, \ldots, p_d)$, any other grid-point $R = (r_1, \ldots, r_d)$ is determined by a vector $a = (a_1, \ldots, a_d)^\top$ with $a_i := r_i - p_i$ for $i = 1, \ldots, d$, hence $|a_i| \leq T$. Each grid-point Q with Euclidean distance at most $2 \cdot A / (\sum_{i=1}^d a_i^2)^{1/2}$ from the line PR determines a triangle P, Q, R with area at most A. By symmetry, which we take into account by a constant factor $c > 0$, we may assume that $0 \leq |a_1|, \ldots, |a_{d-1}| \leq |a_d| \leq T$ where $a_d \neq 0$. The Euclidean distance of a grid-point $Q = (p_1 + q_1, \ldots, p_d + q_d)$ with $q := (q_1, \ldots, q_d)^\top$ from the line PR is given by $(\langle q, q \rangle - \langle a, q \rangle^2 / \langle a, a \rangle)^{1/2}$, where $\langle a, b \rangle$ denotes the standard scalar product of the vectors a and b. Hence, if area $(P, Q, R) \leq A$, then for each $j \neq d$ we obtain

$$\begin{aligned}
&\langle q, q \rangle - \frac{\langle a, q \rangle^2}{\langle a, a \rangle} \leq \frac{4 \cdot A^2}{\langle a, a \rangle} \\
\iff &\sum_{1 \leq i < j \leq d} (a_i \cdot q_j - a_j \cdot q_i)^2 \leq 4 \cdot A^2 \\
\implies &|a_d \cdot q_j - a_j \cdot q_d| \leq 2 \cdot A\,.
\end{aligned} \tag{20}$$

Since the triangle P, Q, R is nondegenerate, we have $(a_i \cdot q_j - a_j \cdot q_i)^2 \geq 1$ for at least one pair (i, j) with $i \neq j$, thus area $(P, Q, R) \geq 1/2$. Therefore, given q_d there are at most $(4 \cdot A + 1)/|a_d| \leq 6 \cdot A/|a_d|$ choices for each q_j, $j \neq d$, altogether at most $(6 \cdot A/|a_d|)^{d-1}$ choices for all q_j, $j \neq d$. Since Q is a grid-point in the $T \times \cdots \times T$-grid, there are at most T choices for q_d. We infer that, given the grid-points P and R, the number of nondegenerate triangles P, Q, R with area $(P, Q, R) \leq A$ in the $T \times \cdots \times T$-grid is at most

$$T \cdot \left(\frac{6 \cdot A}{|a_d|} \right)^{d-1}.$$

Summing over all these choices of the grid-points P and R, for some constants $c, c_3 > 0$ we obtain the following upper bound on the number of nondegenerate triangles P, Q, R with area $(P, Q, R) \leq A$ in the $T \times \cdots \times T$-grid

$$\begin{aligned}
&c \cdot T^d \cdot \sum_{0 \leq |a_1|, \ldots, |a_{d-1}| \leq |a_d| \leq T; a_d \neq 0} T \cdot \left(\frac{6 \cdot A}{|a_d|} \right)^{d-1} \\
&\leq c \cdot 2 \cdot 6^{d-1} \cdot A^{d-1} \cdot T^{d+1} \cdot \sum_{a_d = 1}^{T} \frac{(2 \cdot a_d + 1)^{d-1}}{a_d^{d-1}} \\
&\leq c_3 \cdot A^{d-1} \cdot T^{d+2},
\end{aligned}$$

which proves (19). $\square$

Lemma 10 *The number $|\mathcal{E}_2|$ of unordered pairs $\{P,Q\}$ of distinct grid-points in the d-dimensional $T\times\cdots\times T$-grid with dist $(P,Q)\le D$ satisfies for some constant $c_2>0$:*

$$|\mathcal{E}_2| \le c_2 \cdot T^d \cdot D^d. \tag{21}$$

PROOF. There are T^d choices for a grid-point $P=(p_1,\ldots,p_d)$ in the d-dimensional $T\times\cdots\times T$-grid. Given the grid-point $P=(p_1,\ldots,p_d)$, any other grid-point $Q=(q_1,\ldots,q_d)$ with dist $(P,Q)\le D$ fulfills $|p_i-q_i|\le D$, $i=1,\ldots,d$, hence with $D\ge 1$ there are at most $2\cdot D+1\le 3\cdot D$ choices for each q_i, and the upper bound (21) follows. $\square$

Lemma 11 *For some constant $c_4>0$, the number $s_2(\mathcal{G};\mathcal{E}_3)$ of unordered pairs $\{E,E'\}$ of distinct edges $E,E'\in\mathcal{E}_3$ with $|E\cap E'|=2$ in the hypergraph $\mathcal{G}=\mathcal{G}(A,D)=(V,\mathcal{E}_2\cup\mathcal{E}_3\cup\mathcal{E}_3^0)$ fulfills*

$$s_2(\mathcal{G};\mathcal{E}_3) \le c_4\cdot A^{2d-2}\cdot T^{d+2}\cdot \log T/D^{d-2}. \tag{22}$$

PROOF. As in the proof of Lemma 9 we may assume that each nondegenerate triangle in the d-dimensional $T\times\cdots\times T$-grid has area at least $1/2$. For the grid-points P,Q,R,S in the $T\times\cdots\times T$-grid let P,Q,R and P,R,S be two nondegenerate triangles with $1/2\le$ area $(P,Q,R)\le A$ and $1/2\le$ area $(P,R,S)\le A$, where in each triangle P,Q,R and P,R,S pairwise the grid-points have Euclidean distance bigger than D. A grid-point P can be chosen in T^d ways. Given the grid-point $P=(p_1,\ldots,p_d)$, any other grid-point $R=(r_1,\ldots,r_d)$ is determined by a vector $a=(a_1,\ldots,a_d)^\top$ with $a_i:=r_i-p_i$, $i=1,\ldots,d$, hence $|a_i|\le T$. Each grid-point Q in the $T\times\cdots\times T$-grid with distance at most $2\cdot A/(\sum_{i=1}^d a_i^2)^{1/2}$ from the line PR determines a triangle P,Q,R with area $(P,Q,R)\le A$. By symmetry, which we take into account by a constant factor $c>0$, we may assume that $0\le|a_1|,\ldots,|a_{d-1}|\le|a_d|\le T$ with $a_d\ne 0$. The Euclidean distance of a grid-point $Q=(p_1+q_1,\ldots,p_d+q_d)$ with $q=(q_1,\ldots,q_d)^\top$ from the line PR is given by $(\langle q,q\rangle-\langle a,q\rangle^2/\langle a,a\rangle)^{1/2}$, hence, if area $(P,Q,R)\le A$, for each $j\ne d$ we have by (20):

$$|a_d\cdot q_j - a_j\cdot q_d| \le 2\cdot A\,.$$

Since $A\ge 1/2$, given q_d there are at most $(4\cdot A+1)/|a_d|\le 6\cdot A/|a_d|$ choices for each q_j, $j\ne d$. Altogether, given q_d, there are at most $(6\cdot A/|a_d|)^{d-1}$ choices for all q_j, $j\ne d$. Given the grid-points P and R, by varying over the at most T possible values for q_d, the number of nondegenerate triangles P,Q,R with area $(P,Q,R)\le A$ in the d-dimensional $T\times\cdots\times T$-grid is at most

$$T\cdot\left(\frac{6\cdot A}{|a_d|}\right)^{d-1},$$

hence there are at most $(T \cdot (6 \cdot A/|a_d|)^{d-1})^2$ choices for the grid-points Q and S.

Since we have by assumption dist $(P, R) > D$, we infer with $|a_1|, \ldots, |a_{d-1}| \leq |a_d|$ that $|a_d| > D/\sqrt{d}$. Summing over all possible choices for the grid-points P and R in the $T \times \cdots \times T$-grid, we obtain for constants $c, c', c_4 > 0$ on the number of unordered pairs $\{E, E'\}$ of distinct edges $E, E' \in \mathcal{E}_3$ with $|E \cap E'| = 2$ in the hypergraph $\mathcal{G} = \mathcal{G}(A, D) = (V, \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_3^0)$ the following upper bound

$$\begin{aligned}
& c \cdot T^d \cdot \sum_{|a_d| = \lceil D/\sqrt{d} \rceil}^{T} \; \sum_{a_{d-1} = -|a_d|}^{|a_d|} \cdots \sum_{a_1 = -|a_d|}^{|a_d|} \left(T \cdot \left(\frac{6 \cdot A}{|a_d|} \right)^{d-1} \right)^2 \\
& \leq c \cdot 6^{2d-2} \cdot A^{2d-2} \cdot T^{d+2} \cdot \sum_{|a_d| = \lceil D/\sqrt{d} \rceil}^{T} \frac{(2 \cdot |a_d| + 1)^{d-1}}{a_d^{2d-2}} \\
& \leq c' \cdot A^{2d-2} \cdot T^{d+2} \cdot \sum_{a_d = \lceil D/\sqrt{d} \rceil}^{T} \frac{1}{a_d^{d-1}} \qquad (23) \\
& \leq c_4 \cdot A^{2d-2} \cdot T^{d+2} / D^{d-2} \qquad \text{for } d \geq 3.
\end{aligned}$$

For dimension $d = 2$ we obtain from (23) the upper bound $c_4 \cdot A^2 \cdot T^4 \cdot \log T$, thus $s_2(\mathcal{G}; \mathcal{E}_3) \leq c_4 \cdot A^{2d-2} \cdot T^{d+2} \cdot \log T / D^{d-2}$ for each fixed dimension $d \geq 2$. $\square$

By (19) the average degree t^2 of the hypergraph $\mathcal{G} = (V, \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_3^0)$ for the 3-element edges from $\mathcal{E}_3$ satisfies

$$t^2 = \frac{3 \cdot |\mathcal{E}_3|}{|V|} \leq \frac{3 \cdot c_3 \cdot A^{d-1} \cdot T^{d+2}}{T^d} = 3 \cdot c_3 \cdot A^{d-1} \cdot T^2 =: t_0^2 . \qquad (24)$$

Set $p := T^{\varepsilon}/t_0 \leq 1$ for some small constant $\varepsilon > 0$, i.e., $p = \Theta(T^{\varepsilon}/(A^{(d-1)/2} \cdot T))$. Moreover, for some suitable constant $c > 0$, which will be fixed later, let

$$A := c \cdot \frac{T^2}{n^{\frac{2}{d-1}}} \cdot (\log n)^{\frac{1}{d-1}} \geq \frac{1}{2} , \qquad (25)$$

hence $p \leq 1$ provided that $\varepsilon \leq d - 1/\beta$.

To simplify the presentation we use a probabilistic argument, which will be derandomized shortly. With probability p we select uniformly at random and independently of each other vertices from the vertex-set V. Let V^* be the set of the chosen vertices. Let $\mathcal{G}^* = (V^*, \mathcal{E}_2^* \cup \mathcal{E}_3^* \cup \mathcal{E}_3^{0*})$ be the resulting induced subhypergraph of $\mathcal{G}$ with $\mathcal{E}_2^* := \mathcal{E}_2 \cap [V^*]^2$, $\mathcal{E}_3^* := \mathcal{E}_3 \cap [V^*]^3$ and $\mathcal{E}_3^{0*} := \mathcal{E}_3^0 \cap [V^*]^3$. Let $E(|V^*|), E(|\mathcal{E}_2^*|), E(|\mathcal{E}_3^*|), E(|\mathcal{E}_3^{0*}|), E(s_2(\mathcal{G}^*; \mathcal{E}_3^*))$ be the expected numbers of vertices, 2-element edges, 3-element edges (nondegenerate triangles with area at most A), collinear triples of grid-points and unordered pairs $\{E, E'\}$ of edges $E, E' \in \mathcal{E}_3^*$ with $|E \cap E'| = 2$ in $\mathcal{G}^* = (V^*, \mathcal{E}_2^* \cup \mathcal{E}_3^* \cup \mathcal{E}_3^{0*})$, respectively. With (17), (19), (21), (22), (24) we infer for some constants $c_1', c_2', c_3', c_{3,0}', c_4' > 0$:

$$
\begin{aligned}
E(|V^*|) &= p \cdot T^d = c_1' \cdot T^{d-1+\varepsilon}/A^{(d-1)/2} \\
E(|\mathcal{E}_2^*|) &= p^2 \cdot |\mathcal{E}_2| \le p^2 \cdot c_2 \cdot T^d \cdot D^d \le c_2' \cdot T^{d-2+2\varepsilon} \cdot D^d/A^{d-1} \\
E(|\mathcal{E}_3^*|) &= p^3 \cdot |\mathcal{E}_3| \le p^3 \cdot c_3 \cdot A^{d-1} \cdot T^{d+2} \le c_3' \cdot T^{d-1+3\varepsilon}/A^{(d-1)/2} \\
E(|\mathcal{E}_3^{0*}|) &= p^3 \cdot |\mathcal{E}_3^0| \le p^3 \cdot c_{3,0} \cdot T^{2d} \cdot \log T \le \\
&\le c_{3,0}' \cdot T^{2d-3+3\varepsilon} \cdot \log T/A^{3(d-1)/2} \\
E(s_2(\mathcal{G}^*; \mathcal{E}_3^*)) &= p^4 \cdot s_2(\mathcal{G}; \mathcal{E}_3) \le p^4 \cdot c_4 \cdot A^{2d-2} \cdot T^{d+2} \cdot \log T/D^{d-2} \le \\
&\le c_4' \cdot T^{d-2+4\varepsilon} \cdot \log T/D^{d-2}.
\end{aligned}
$$

By Chernoff's and Markov's inequalities, there exists a subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_2^* \cup \mathcal{E}_3^* \cup \mathcal{E}_3^{0*})$ of $\mathcal{G}$ such that

$$
\begin{aligned}
|V^*| &\ge c_1'/2 \cdot T^{d-1+\varepsilon}/A^{(d-1)/2} && (26) \\
|\mathcal{E}_2^*| &\le 5 \cdot c_2' \cdot T^{d-2+2\varepsilon} \cdot D^d/A^{d-1} && (27) \\
|\mathcal{E}_3^*| &\le 5 \cdot c_3' \cdot T^{d-1+3\varepsilon}/A^{(d-1)/2} && (28) \\
|\mathcal{E}_3^{0*}| &\le 5 \cdot c_{3,0}' \cdot T^{2d-3+3\varepsilon} \cdot \log T/A^{3(d-1)/2} && (29) \\
s_2(\mathcal{G}^*; \mathcal{E}_3^*) &\le 5 \cdot c_4' \cdot T^{d-2+4\varepsilon} \cdot \log T/D^{d-2}. && (30)
\end{aligned}
$$

This probabilistic argument can be turned into a deterministic polynomial time algorithm as follows. We use the method of conditional probabilities. Namely, let $\mathcal{C}$ be the set of all 4-element subsets $E \cup E'$ of V such that $E, E' \in \mathcal{E}_3$ and $|E \cap E'| = 2$. We enumerate the vertices of the $T \times \cdots \times T$-grid by $P_1, \ldots, P_{T^d}$. To each vertex P_i we associate a parameter $p_i \in [0, 1]$, $i = 1, \ldots, T^d$, and we define a potential function $F(p_1, \ldots, p_{T^d})$ by

$$
\begin{aligned}
F(p_1, \ldots, p_{T^d}) &:= 2^{p \cdot T^d/2} \cdot \prod_{i=1}^{T^d} \left(1 - \frac{p_i}{2}\right) + \frac{\sum_{\{i,j\} \in \mathcal{E}_2} p_i \cdot p_j}{5 \cdot c_2' \cdot T^{d-2+2\varepsilon} \cdot D^d/A^{d-1}} + \\
&+ \frac{\sum_{\{i,j,k\} \in \mathcal{E}_3} p_i \cdot p_j \cdot p_k}{5 \cdot c_3' \cdot T^{d-1+3\varepsilon}/A^{(d-1)/2}} + \frac{\sum_{\{i,j,k\} \in \mathcal{E}_3^0} p_i \cdot p_j \cdot p_k}{5 \cdot c_{3,0}' \cdot T^{2d-3+3\varepsilon} \cdot \log T/A^{3(d-1)/2}} + \\
&+ \frac{\sum_{\{i,j,k,l\} \in \mathcal{C}} p_i \cdot p_j \cdot p_k \cdot p_l}{5 \cdot c_4' \cdot T^{d-2+4\varepsilon} \cdot \log T/D^{d-2}}.
\end{aligned}
$$

With the initialisation $p_1 := \cdots := p_{T^d} := p := T^\varepsilon/t_0$ and using $1 + x \le e^x$ for $x \in \mathbb{R}$, we infer $F(p, \ldots, p) < (2/e)^{pT^d/2} + 4/5$, which is less than 1 for $p \cdot T^d \ge 12$. Hence, in the beginning we have $F(p_1, \ldots, p_{T^d}) < 1$. Using the linearity of $F(p_1, \ldots, p_{T^d})$ in each p_i, we minimize $F(p_1, \ldots, p_{T^d})$ successively with respect to each p_i by choosing one after the other $p_i := 0$ or $p_i := 1$ for $i = 1, \ldots, T^d$, and finally we obtain $p_1, \ldots, p_{T^d} \in \{0,1\}$ such that $F(p_1, \ldots, p_{T^d}) < 1$. The vertex-set $V^* = \{P_i \in V \mid p_i = 1\}$ yields an induced subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_2^* \cup \mathcal{E}_3^* \cup \mathcal{E}_3^{0*})$ of $\mathcal{G}$ with $\mathcal{E}_2^* := \mathcal{E}_2 \cap [V^*]^2$ and $\mathcal{E}_3^* := \mathcal{E}_3 \cap [V^*]^3$ and $\mathcal{E}_3^{0*} := \mathcal{E}_3^0 \cap [V^*]^3$, which fulfills (26)–(30). Namely, if $|V^*| < p \cdot T^d/2$, then $F(p_1, \ldots, p_{T^d}) \geq 2^{p \cdot T^d/2} \cdot \prod_{i=1}^{T^d} (1 - p_i/2) > 1$, but we have $F(p_1, \ldots, p_{T^d}) < 1$. Similarly, if say $|\mathcal{E}_2^*| > 5 \cdot c_2' \cdot T^{d-2+2\varepsilon} \cdot D^d/A^{d-1}$, then we have $F(p_1, \ldots, p_{T^d}) \geq \frac{\sum_{\{i,j\} \in \mathcal{E}_2} p_i \cdot p_j}{5 \cdot c_2' \cdot T^{d-2+2\varepsilon} \cdot D^d / A^{d-1}} > 1$, which again contradicts $F(p_1, \ldots, p_{T^d}) < 1$. By (17), (19), (21), (22), (25), and using $1 \leq D \leq T$ as well as $T \geq n^{2/d}$, the time for this derandomization is given by

$$\begin{aligned}
&O((|V| + |\mathcal{E}_2| + |\mathcal{E}_3| + |\mathcal{E}_3^0| + |\mathcal{C}|) = \\
=&O\left(T^d \cdot D^d + A^{d-1} \cdot T^{d+2} + T^{2d} \cdot \log T + A^{2d-2} \cdot T^{d+2} \cdot \log T / D^{d-2}\right) \\
=&O\left(A^{d-1} \cdot T^{d+2} + A^{2d-2} \cdot T^{d+2} \cdot \log T / D^{d-2}\right) \\
=&O\left(A^{2d-2} \cdot T^{d+2} \cdot \log T / D^{d-2}\right). \qquad (31)
\end{aligned}$$

Lemma 12 *For fixed $\beta, \varepsilon > 0$ with $\varepsilon < d/2 - 1/\beta$, it is*

$$|\mathcal{E}_3^{0*}| = o(|V^*|).$$

PROOF. By (25), (26), (29) and using $T = n^\beta$ for a constant $\beta > 0$, we obtain

$$\begin{aligned}
&|\mathcal{E}_3^{0*}| = o(|V^*|) \\
\Longleftarrow\, &T^{2d-3+3\varepsilon} \cdot \log T / A^{3(d-1)/2} = o(T^{d-1+\varepsilon}/A^{(d-1)/2}) \\
\Longleftrightarrow\, &\frac{T^{d-2+2\varepsilon} \cdot \log T}{A^{d-1}} = o(1) \\
\Longleftrightarrow\, &\frac{n^2}{T^{d-2\varepsilon}} \cdot \frac{\log T}{\log n} = o(1) \\
\Longleftrightarrow\, &n^{2-\beta(d-2\varepsilon)} = o(1) \\
\Longleftrightarrow\, &2 - \beta \cdot (d - 2 \cdot \varepsilon) < 0,
\end{aligned}$$

which holds for $\varepsilon < d/2 - 1/\beta$. □

Lemma 13 *For fixed $\beta, \gamma, \varepsilon > 0$ with $\varepsilon < 1/(3 \cdot \beta) - (1 - \gamma) \cdot (d-2)/3$, it is*

$$s_2(\mathcal{G}^*; \mathcal{E}_3^*) = o(|V^*|).$$

PROOF. By (25), (26), (30) and $T = n^\beta$ and $D = T^\gamma$ for constants $\beta, \gamma > 0$ with $\gamma < 1$, we infer that

$$
\begin{aligned}
& s_2(\mathcal{G}^*; \mathcal{E}_3^*) = o(|V^*|) \\
\Longleftarrow\; & T^{d-2+4\varepsilon} \cdot \log T / D^{d-2} = o(T^{d-1+\varepsilon}/A^{(d-1)/2}) \\
\Longleftrightarrow\; & \frac{A^{(d-1)/2}}{T^{1-3\varepsilon} \cdot D^{d-2}} \cdot \log T = o(1) \\
\Longleftrightarrow\; & \frac{T^{(1-\gamma)(d-2)+3\varepsilon}}{n} \cdot \log T \cdot \log^{1/2} n = o(1) \\
\Longleftrightarrow\; & n^{\beta((1-\gamma)(d-2)+3\varepsilon)-1} \cdot \log^{3/2} n = o(1) \\
\Longleftarrow\; & \beta \cdot ((1-\gamma) \cdot (d-2) + 3 \cdot \varepsilon) < 1\,,
\end{aligned}
$$

which holds for $0 < \varepsilon < 1/(3 \cdot \beta) - (1-\gamma) \cdot (d-2)/3$. □

Lemma 14 *For fixed $\beta, \gamma, \varepsilon > 0$ with $\varepsilon \leq (1-\gamma) \cdot d - 1/\beta$, it is*

$$|\mathcal{E}_2^*| = o(|V^*|).$$

PROOF. Using (25)–(27) with $T = n^\beta$ and $D = T^\gamma$ for constants $\beta, \gamma > 0$ with $\gamma < 1$, we infer

$$
\begin{aligned}
& |\mathcal{E}_2^*| = o(|V^*|) \\
\Longleftarrow\; & T^{d-2+2\varepsilon} \cdot D^d / A^{d-1} = o(T^{d-1+\varepsilon}/A^{(d-1)/2}) \\
\Longleftrightarrow\; & \frac{T^{-1+\varepsilon} \cdot D^d}{A^{(d-1)/2}} = o(1) \\
\Longleftrightarrow\; & \frac{T^{\varepsilon-(1-\gamma)d} \cdot n}{\log^{1/2} n} = o(1) \\
\Longleftrightarrow\; & \frac{n^{1+\beta(\varepsilon-(1-\gamma)d)}}{\log^{1/2} n} = o(1) \\
\Longleftarrow\; & \beta \cdot (\varepsilon - (1-\gamma) \cdot d) \leq -1\,,
\end{aligned}
$$

which holds for $0 < \varepsilon \leq (1-\gamma) \cdot d - 1/\beta$. □

Now we set $\gamma := 1/2$ and $\beta := 2/(d-1)$. For $\varepsilon := 1/7$ all assumptions in Lemmas 12, 13 and 14 and $p = T^\varepsilon/t_0 \leq 1$ are fulfilled. From each 2-element edge $E \in \mathcal{E}_2^*$, each 3-element edge $E \in \mathcal{E}_3^{0*}$, and each unordered pair $\{E, E'\}$ of distinct edges $E, E' \in \mathcal{E}_3^*$ with $|E \cap E'| = 2$ in $\mathcal{G}^*$ we delete one vertex. By Lemmas 12, 13 and 14 the resulting induced subhypergraph $\mathcal{G}^{**} = (V^{**}, \mathcal{E}_3^{**})$ of $\mathcal{G}^*$ with $|V^{**}| = (1-o(1)) \cdot |V^*|$ contains only 3-element edges from $\mathcal{E}_3^*$ but no other edges from $\mathcal{E}_2^*$ or $\mathcal{E}_3^{0*}$, and is linear. By (26) and (28) we have

$$
\begin{aligned}
|V^{**}| &\geq (c_1'/2 - o(1)) \cdot T^{d-1+\varepsilon}/A^{(d-1)/2} \\
|\mathcal{E}_3^{**}| &\leq 5 \cdot c_3' \cdot T^{d-1+3\varepsilon}/A^{(d-1)/2},
\end{aligned}
$$

and the average degree t^2 of the 3-uniform hypergraph $\mathcal{G}^{**} = (V^{**}, \mathcal{E}_3^{**})$ satisfies

$$t^2 = \frac{3 \cdot |\mathcal{E}_3^{**}|}{|V^{**}|} \le \frac{15 \cdot c_3' \cdot T^{d-1+3\varepsilon}/A^{(d-1)/2}}{(c_1'/2 - o(1)) \cdot T^{d-1+\varepsilon}/A^{(d-1)/2}} = \frac{30 \cdot c_3'}{c_1' - o(1)} \cdot T^{2\varepsilon} =: t_1^2 \,.$$

Since $\mathcal{G}^{**}$ is linear we can apply Theorem 7 and, using (28), we find for $t \ge 2$ and for any δ with $0 < \delta < 3$ in time

$$O\left(|\mathcal{E}_3^{**}| + \frac{|V^{**}|^3}{t^{3-\delta}}\right) = O\left(\frac{n^3 \cdot T^{3\varepsilon}}{\log^{3/2} n}\right) = O\left(\frac{n^{3+6/(7(d-1))}}{\log^{3/2} n}\right) \tag{32}$$

an independent set I of size

$$\begin{aligned}|I| &= \Omega\left(\frac{|V^{**}|}{t} \cdot \log^{1/2} t\right) = \Omega\left(\frac{|V^{**}|}{t_1} \cdot \log^{1/2} t_1\right) = \\ &= \Omega\left(\frac{T^{d-1+\varepsilon}/A^{(d-1)/2}}{T^{\varepsilon}} \cdot \log^{1/2} T^{\varepsilon}\right) \\ &= \Omega\left(\frac{n}{\log^{1/2} n} \cdot \log^{1/2} n^{\beta\varepsilon}\right) = \Omega(n) \,,\end{aligned}$$

since $T = n^{\beta}$ and $\beta, \varepsilon > 0$ are constants. By choosing the constant $c > 0$ in (25) sufficiently small, we obtain an independent set of size at least n. For $t < 2$ we have $|\mathcal{E}_3^{**}| \le 4 \cdot |V^{**}|/3$, hence

$$\alpha(\mathcal{G}) \ge \alpha(\mathcal{G}^*) \ge 2 \cdot |V^{**}|/9 = \Omega(T^{\varepsilon} \cdot n/\log^{1/2} n) = \Omega(n) \,.$$

since $T = n^{2/(d-1)}$ and $\varepsilon > 0$. Such an independent set can be found easily in time $O(|V^{**}| + |\mathcal{E}_3^{**}|) = O(n^{1+6/(7(d-1))}/\log^{1/2} n)$.

After rescaling, an independent set $I \subseteq V^{**}$ with $|V| = n$ yields a desired set of n points in $[0,1]^d$ such that the area of each triangle arising from these n points is $\Omega((\log n)^{1/(d-1)}/n^{2/(d-1)})$.

For $\beta = 2/(d-1)$ and $\gamma = 1/2$ and $\varepsilon = 1/7$, the running times in (31) and (32) yield the time bound $O(n^{5+7/(d-1)} \cdot \log^3 n + n^{3+6/(7(d-1))}/\log^{3/2} n) = O(n^{5+7/(d-1)} \cdot \log^3 n)$. Indeed, by choosing $\gamma = 1/2$ and $\beta = 2/(d-\delta)$ for constants $0 < \varepsilon < \delta/2 \le 1/10$, we obtain the time bound $O(n^{5-2/d+\gamma^*})$ for any fixed $\gamma^* > 0$ with small enough $\delta, \epsilon > 0$, which finishes the proof of Theorem 2. $\square$

3 The On-Line Case

In this section we consider the on-line situation and we show the lower bound in (2) from Theorem 1, namely that for fixed integers $d \geq 2$ and for some constant $c_2 = c_2(d) > 0$:

$$\Delta_d^{\text{on-line}}(n) \geq \frac{c_2}{n^{2/(d-1)}} \,. \tag{33}$$

PROOF. Successively we construct an arbitrary long sequence $P_1, P_2, \ldots$ of points in the d-dimensional unit cube $[0,1]^d$ for fixed integers $d \geq 2$ such that for suitable constants $b, c, \beta, \gamma > 0$, which will be fixed later, for every n the set $S_n = \{P_1, \ldots, P_n\}$ has the following properties

(i) dist $(P_i, P_j) > b/n^\beta$ for all $1 \leq i < j \leq n$, and
(ii) area $(P_i, P_j, P_k) > c/n^\gamma$ for all $1 \leq i < j < k \leq n$.

Assume that already a set $S_{n-1} = \{P_1, P_2, \ldots, P_{n-1}\} \subset [0,1]^d$ of $n-1$ points with (i') dist $(P_i, P_j) > b/(n-1)^\beta$ for all $1 \leq i < j \leq n-1$, and (ii') area $(P_i, P_j, P_k) > c/(n-1)^\gamma$ for all $1 \leq i < j < k \leq n-1$ has been constructed.

To have available some space in $[0,1]^d$ for choosing a new point $P_n \in [0,1]^d$ such that (i) is fulfilled, this new point P_n must not lie within any of the d-dimensional balls $B_r(P_i)$ of radius $r := b/n^\beta$ with center P_i, $i = 1, \ldots, n-1$. Adding the volumes of these balls yields

$$\sum_{i=1}^{n-1} \text{vol } (B_r(P_i)) < n \cdot C_d \cdot r^d = b^d \cdot C_d \cdot n^{1-\beta d}.$$

For $\beta := 1/d$ and $b^d \cdot C_d < 1/2$ we have $\sum_{i=1}^{n-1} \text{vol } (B_r(P_i)) < 1/2$.

We prove next that those regions within $[0,1]^d$, where condition (ii) is violated, altogether have volume less than $1/2$. The regions, where condition (ii) is violated by points $P \in [0,1]^d$, are given by sets $C_{i,j} \cap [0,1]^d$, $1 \leq i < j \leq n-1$, where $C_{i,j}$ is a d-dimensional cylinder centered at the line P_iP_j. These sets $C_{i,j} \cap [0,1]^d$ are contained in cylinders of height $\sqrt{d}$ and radius $2 \cdot c/(n^\gamma \cdot$ dist $(P_i, P_j))$. Summing up their volumes yields

$$
\begin{aligned}
&\sum_{1 \le i < j \le n-1} \text{vol } (C_{i,j} \cap [0,1]^d) \\
&\le \sum_{1 \le i < j \le n-1} \sqrt{d} \cdot C_{d-1} \cdot \left(\frac{2 \cdot c}{n^{\gamma} \cdot \text{dist } (P_i, P_j)} \right)^{d-1} \\
&= \frac{(2 \cdot c)^{d-1} \cdot \sqrt{d} \cdot C_{d-1}}{2 \cdot n^{\gamma(d-1)}} \cdot \sum_{i=1}^{n-1} \sum_{j=1; j \ne i}^{n-1} \left(\frac{1}{\text{dist } (P_i, P_j)} \right)^{d-1} . \qquad (34)
\end{aligned}
$$

We fix some point P_i, $i = 1, \ldots, n-1$. To give an upper bound on the last sum in (34), we use a packing argument, similar to an argument of Barequet [2]. Consider the balls $B_{r_t}(P_i)$ with center P_i and radius $r_t := b \cdot t / n^{\beta}$, $t = 0, 1, \ldots$ with $t \le \sqrt{d} \cdot n^{\beta}/b$. Clearly vol $(B_{r_0}(P_i)) = 0$, and for some constant $c_1 > 0$ and $t = 1, 2, \ldots$ we have

$$
\text{vol } (B_{r_t}(P_i) \setminus B_{r_{t-1}}(P_i)) \le c_1 \cdot \frac{t^{d-1}}{n^{\beta d}} . \qquad (35)
$$

Notice that for every ball $B_r(P_j)$ with radius $r = \Theta(n^{-\beta})$ and center $P_j \in B_{r_t}(P_i) \setminus B_{r_{t-1}}(P_i)$ with $i \ne j$ we have vol $(B_r(P_j) \cap (B_{r_t}(P_i) \setminus B_{r_{t-1}}(P_i)) = \Theta(n^{-\beta d})$. Set $n_t := |S_{n-1} \cap (B_{r_t}(P_i) \setminus B_{r_{t-1}}(P_i))|$. By inequalities (i') we have $n_1 = 1$ and by (35) each shell $B_{r_t}(P_i) \setminus B_{r_{t-1}}(P_i)$, $t = 2, 3, \ldots$, contains $n_t \le c_2 \cdot t^{d-1}$ points from the set S_{n-1}, where $c_2 > 0$ is a constant. We obtain for some constant $c_3 > 0$:

$$
\begin{aligned}
&\sum_{j=1; j \ne i}^{n-1} \left(\frac{1}{\text{dist } (P_i, P_j)} \right)^{d-1} \le \sum_{t=2}^{\lceil \sqrt{d} \cdot n^{\beta}/b \rceil} n_t \cdot \left(\frac{1}{b \cdot (t-1)/n^{\beta}} \right)^{d-1} \le \\
&\le \sum_{t=2}^{\lceil \sqrt{d} \cdot n^{\beta}/b \rceil} c_2 \cdot t^{d-1} \cdot \left(\frac{1}{b \cdot (t-1)/n^{\beta}} \right)^{d-1} \le \sum_{t=2}^{\lceil \sqrt{d} \cdot n^{\beta}/b \rceil} \frac{c_2}{b^{d-1}} \cdot 2^{d-1} \cdot n^{\beta(d-1)} \le \\
&\le c_3 \cdot n^{\beta d}. \qquad (36)
\end{aligned}
$$

We set $\gamma := 2/(d-1)$ and, using $\beta = 1/d$ and (36), inequality (34) becomes for a sufficiently small constant $c > 0$:

$$
\begin{aligned}
&\sum_{1 \le i < j \le n-1} \text{vol } (C_{i,j} \cap [0,1]^d) \\
&\le \frac{(2 \cdot c)^{d-1} \cdot \sqrt{d} \cdot C_{d-1}}{2 \cdot n^{\gamma(d-1)}} \cdot \sum_{i=1}^{n-1} \sum_{j=1; j \ne i}^{n-1} \left(\frac{1}{\text{dist } (P_i, P_j)} \right)^{d-1} \\
&\le \frac{(2 \cdot c)^{d-1} \cdot \sqrt{d} \cdot C_{d-1}}{2 \cdot n^{\gamma(d-1)}} \cdot \sum_{i=1}^{n-1} c_3 \cdot n^{\beta d} \\
&\le 1/2 \cdot (2 \cdot c)^{d-1} \cdot \sqrt{d} \cdot C_{d-1} \cdot c_3 \cdot n^{1 + \beta d - \gamma(d-1)} < 1/2 .
\end{aligned}
$$

Together all forbidden regions have volume less than 1, hence there exists a point $P_n \in [0,1]^d$ such that (i) and (ii) are satisfied. Thus we have $\Delta_d^{\text{on-line}}(n) = \Omega(1/n^{2/(d-1)})$, which proves (33). $\Box$

4 An Upper Bound

Here we show with a simple argument the upper bounds from Theorem 1 on the smallest area of a triangle arising from any n points in the d-dimensional unit cube $[0,1]^d$, namely that for fixed $d \geq 2$ and for some constant $c_1' > 0$:

$$\Delta_d^{\text{on-line}}(n) \leq \Delta_d^{\text{off-line}}(n) \leq \frac{c_1'}{n^{2/d}} \,. \tag{37}$$

PROOF. It is obvious that $\Delta_d^{\text{on-line}}(n) \leq \Delta_d^{\text{off-line}}(n)$, hence it suffices to prove $\Delta_d^{\text{off-line}}(n) \leq c_1'/n^{2/d}$. Given any n points $P_1, \ldots, P_n \in [0,1]^d$, for $D := c/n^{1/d}$, where $c > (2^{d+1}/C_d)^{1/d}$ is a constant, we consider the balls $B_D(P_j)$ with center P_j and radius D, $j = 1, \ldots, n$.

If there exist distinct i, j, k such that $B_D(P_i) \cap B_D(P_j) \neq \emptyset$ and $B_D(P_i) \cap B_D(P_k) \neq \emptyset$, then dist $(P_i, P_j) \leq 2 \cdot D$ and dist $(P_i, P_k) \leq 2 \cdot D$. Thus the Euclidean distance of the point P_k from the line $P_i P_j$ is at most $2 \cdot D$, hence area $(P_i, P_j, P_k) \leq 2 \cdot D^2 = O(1/n^{2/d})$.

Otherwise, each ball $B_D(P_i)$ has with at most one other ball $B_D(P_j)$, $j \neq i$, a nonempty intersection. Each ball $B_D(P_i)$ with center $P_i \in [0,1]^d$ and radius $D \leq 1$ satisfies vol $(B_D(P_i) \cap [0,1]^d) \geq$ vol $(B_D(P_i))/2^d$. Thus,

$$n/2 \cdot C_d \cdot D^d/2^d \leq 1$$

and we infer $D \leq (2^{d+1}/C_d)^{1/d}/n^{1/d}$, which contradicts our choice of D. Hence $\Delta_d^{\text{off-line}}(n) = O(1/n^{2/d})$, which proves (37) and hence Theorem 1. $\Box$

5 Concluding Remarks

Certainly it is of interest to narrow the gap between the lower and upper bounds given in this paper, in particular improving the existing upper bounds. It might be also of interest to investigate the minimum areas or volumes of more complex geometrical structures than triangles for distributions of n points in $[0,1]^d$.

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