Point Sets in the Unit Square and Large Areas of Convex Hulls of Subsets of Points

Hanno Lefmann

Fakultät für Informatik, TU Chemnitz, D-09107 Chemnitz, Germany
lefmann@informatik.tu-chemnitz.de

Abstract. In this paper generalizations of Heilbronn's triangle problem to convex hulls of j points in the unit square $[0,1]^2$ are considered. By using results on the independence number of linear hypergraphs, for fixed integers $k \geq 3$ and any integers $n \geq k$ a deterministic $o(n^{6k-4})$ time algorithm is given, which finds distributions of n points in $[0,1]^2$ such that, simultaneously for $j = 3, \ldots, k$, the areas of the convex hulls determined by any j of these n points are $\Omega((\log n)^{1/(j-2)}/n^{(j-1)/(j-2)})$.

1 Introduction

Distributions of n points in the unit square $[0,1]^2$, where the minimum area of a triangle determined by three of these n points is large, have been investigated by Heilbronn. Let $\Delta_3(n)$ denote the supremum – over all distributions of n points in $[0,1]^2$ – of the minimum area of a triangle among n points. Since no three of the points $(1/n) \cdot (i \bmod n, i^2 \bmod n)$, $i = 0, \ldots, n-1$, are collinear, we infer $\Delta_3(n) = \Omega(1/n^2)$, provided n is prime, as has been observed by Erdős. For quite a while this lower bound was believed to be also the upper bound. However, Komlós, Pintz and Szemerédi [13] proved that $\Delta_3(n) = \Omega(\log n/n^2)$. In [6] a deterministic polynomial in n time algorithm has been given, which achieves this lower bound. Upper bounds on $\Delta_3(n)$ were given by Roth [18]–[21] and Schmidt [23] and, improving these earlier results, the currently best upper bound $\Delta_3(n) = O(2^{c\sqrt{\log n}}/n^{8/7})$ for a constant $c > 0$, has been obtained by Komlós, Pintz and Szemerédi [12]. We remark that, if n points are uniformly at random and independently of each other distributed in $[0,1]^2$, then the expected value of the minimum area of a triangle formed by three of n points has been shown in [11] to be equal to $\Theta(1/n^3)$.
Variants of Heilbronn's triangle problem in higher dimensions have been investigated by Barequet [2, 3], who considered the minimum volumes of simplices among n points in the d-dimensional unit cube $[0,1]^d$, see also [14] and Brass [7]. Recently, Barequet and Shaikhet [4, 22] considered the on-line situation, where the points have to be positioned one after the other and suddenly this process stops. For this situation they showed by a packing argument the existence of configurations of n points in $[0,1]^d$, where the volume of any $(d+1)$-point simplex among these n points is $\Omega(1/n^{(d+1)\ln(d-2)-0.265d+2.269})$ for fixed $d \geq 5$.
In generalizing Heilbronn's triangle problem to k-gons, see Schmidt [23], asks, given an integer $k \geq 3$, to maximize the minimum area of the convex hull of any

k distinct points in a distribution of n points in $[0,1]^2$. In particular, let $\Delta_k(n)$ be the supremum – over all distributions of n points in $[0,1]^2$ – of the minimum area of the convex hull determined by some k of n points. For $k=4$, Schmidt [23] proved the lower bound $\Delta_4(n)=\Omega(1/n^{3/2})$. In [6] a deterministic algorithm has been given, which shows the lower bound $\Delta_k(n)=\Omega(1/n^{(k-1)/(k-2)})$ has been shown for fixed integers $k\geq 3$. Also in [6] a deterministic polynomial in n time algorithm was given which achieves this lower bound. This has been improved in [15] to $\Delta_k(n)=\Omega((\log n)^{1/(k-2)}/n^{(k-1)/(k-2)})$ for fixed $k\geq 3$.
We remark that for k a function of n, Chazelle proved in [8] in connection with range searching problems that $\Delta_k(n)=\Theta(k/n)$ for $\log n\leq k\leq n$.
In [16] a deterministic algorithm has been given, which finds for fixed integers $k\geq 2$ and any integers $n\geq k$ in time polynomial in n a distribution of n points in the unit square $[0,1]^2$ such that, simultaneously for $j=2,\ldots,k$, the areas of the convex hulls of any j among the n points are $\Omega(1/n^{(j-1)/(j-2)})$. In [17] these simultaneously achievable lower bounds on the minimum areas of the convex hull of any j among n points in $[0,1]^2$ have been improved by using non-discrete probabilistic existence arguments by a logarithmic factor to $\Omega((\log n)^{1/(j-2)}/n^{(j-1)/(j-2)})$ for $j=3,\ldots,k$. (Note that $\Delta_2(n)=\Theta(1/n^{1/2})$.) Here we give a constructive argument, which provides deterministically such configurations of points in $[0,1]^2$:

Theorem 1. *Let $k\geq 3$ be a fixed integer. For each integer $n\geq k$ one can find deterministically in time $o(n^{6k-4})$ some n points in the unit square $[0,1]^2$ such that, simultaneously for $j=3,\ldots,k$, the minimum area of the convex hull determined by some j of these n points is $\Omega((\log n)^{1/(j-2)}/n^{(j-1)/(j-2)})$.*

Concerning upper bounds, we remark that for fixed $j\geq 4$ only the simple bounds $\Delta_j(n)=O(1/n)$ are known, compare [23].

2 The Independence Number of a Linear Hypergraph

In our considerations we transform the geometric problem into a problem on hypergraphs. Before doing so, we take a closer look at hypergraphs and their independence numbers.

Definition 1. *A hypergraph is a pair $\mathcal{G}=(V,\mathcal{E})$ with vertex-set V and edge-set $\mathcal{E}$, where $E\subseteq V$ for each edge $E\in\mathcal{E}$. For a hypergraph $\mathcal{G}$ the notation $\mathcal{G}=(V,\mathcal{E}_2\cup\cdots\cup\mathcal{E}_k)$ means that $\mathcal{E}_i$ is the set of all i-element edges in $\mathcal{G}$, $i=2,\ldots,k$. A hypergraph $\mathcal{G}=(V,\mathcal{E})$ is called k-uniform if $|E|=k$ for each edge $E\in\mathcal{E}$. The independence number $\alpha(\mathcal{G})$ of $\mathcal{G}=(V,\mathcal{E})$ is the largest size of a subset $I\subseteq V$ which contains no edges from $\mathcal{E}$.*

For hypergraphs $\mathcal{G}$ a lower bound on the independence number $\alpha(\mathcal{G})$ is given by Turán's theorem for arbitrary hypergraphs, see [24]:

Theorem 2. *Let $\mathcal{G}=(V,\mathcal{E}_2\cup\cdots\cup\mathcal{E}_k)$ be a hypergraph on $|V|=N$ vertices with average degrees $t_i^{i-1}:=i\cdot|\mathcal{E}_i|/N$ for the i-element edges, $i=2,\ldots,k$. Let $t_{i_0}:=$ max $\{t_i\mid 2\leq i\leq k\}\geq 1/2$.*

Then, the independence nunber $\alpha(\mathcal{G})$ of $\mathcal{G}$ satisfies

$$\alpha(\mathcal{G}) \geq N/(4 \cdot t_{i_0}). \tag{1}$$

An independent set $I \subseteq V$ in $\mathcal{G}$ with $|I| \geq N/(4 \cdot t_{i_0})$ can be found deterministically in time $O(|V| + |\mathcal{E}_2| + \cdots + |\mathcal{E}_k|)$.

For convenience we include the short proof, as a related strategy is used in the proof of Theorem 1.

Proof. Choose uniformly at random and independently of each other vertices from the vertex-set V with probability $p := 1/(2 \cdot t_{i_0})$. Let $V^* \subseteq V$ be the random set of chosen vertices and let $\mathcal{E}_i^* := \mathcal{E}_i \cap [V^*]^i$, $i = 2, \ldots, k$, be the sets of induced i-element edges. Then, the difference of the expected numbers $E[|V^*|]$ and $E[|\mathcal{E}_2^*| + \cdots + |\mathcal{E}_k^*|]$ of chosen vertices and induced edges, respectively, satisfies

$$E\left[|V^*| - \sum_{i=2}^{k} |\mathcal{E}_i^*|\right] = E[|V^*|] - \sum_{i=2}^{k} E[|\mathcal{E}_i^*|] = p \cdot N - \sum_{i=2}^{k} p^i \cdot N \cdot t_i^{i-1}/i \geq$$

$$\geq p \cdot N - \sum_{i=2}^{k} p^i \cdot N \cdot t_{i_0}^{i-1}/i \geq \frac{N}{2 \cdot t_{i_0}} - \sum_{i=2}^{k} \frac{1}{i \cdot 2^i} \cdot \frac{N}{t_{i_0}} \geq \frac{N}{4 \cdot t_{i_0}}.$$

Thus there exists a subset $V^* \subseteq V$ such that $|V^*| - \sum_{i=2}^{k} |\mathcal{E}_i^*| \geq N/(4 \cdot t_{i_0})$. Delete one vertex from each edge $E \in \mathcal{E}_i^*$, $i = 2, \ldots, k$, hence all edges have been destroyed, and we obtain an independent set $V^{**} \subseteq V^*$ with $|V^{**}| \geq N/(4 \cdot t_{i_0})$. This probabilistic argument can be turned into a deterministic algorithm with running time $O(|V| + \sum_{i=2}^{k} |\mathcal{E}_i|)$ by using the method of conditional probabilities, compare [5] for example. □

For fixed integers $k \geq 2$, one can show by Theorem 2, Proposition 2 and Lemma 2 below, that one can find deterministically n points in the unit square $[0,1]^2$ such that the areas of the convex hulls of any j of these n points are $\Omega(1/n^{(j-1)/(j-2)})$, simultaneously for $j = 2, \ldots, k$. However, we are aiming for better lower bounds. To achieve these, we consider the independence number of hypergraphs, which do not contain cycles of small lengths.

Definition 2. *A j-cycle in a hypergraph $\mathcal{G} = (V, \mathcal{E})$ is given by a sequence $E_1, \ldots, E_j$ of distinct edges $E_1, \ldots, E_j \in \mathcal{E}$, such that $E_i \cap E_{i+1} \neq \emptyset$ for $i = 1, \ldots, j-1$, and $E_j \cap E_1 \neq \emptyset$, and a sequence $v_1, \ldots, v_j$ of distinct vertices with $v_{i+1} \in \mathcal{E}_i \cap \mathcal{E}_{i+1}$ for $i = 1, \ldots, j-1$, and $v_1 \in \mathcal{E}_1 \cap \mathcal{E}_j$. An unordered pair $\{E, E'\}$ of distinct edges $E, E' \in \mathcal{E}$ with $|E \cap E'| \geq 2$ is called a 2-cycle.*
A hypergraph $\mathcal{G} = (V, \mathcal{E})$ is called linear if it does not contain any 2-cycles, and it is called uncrowded if it does not contain any 2-, 3- or 4-cycles.

For uncrowded, uniform hypergraphs the next lower bound on the independence number, which has been proved by Ajtai, Komlós, Pintz, Spencer and Szemerédi [1], is better than the one in (1), compare [5] and [10] for a deterministic polynomial time algorithm.

Theorem 3. *Let $k \geq 3$ be a fixed integer. Let $\mathcal{G} = (V, \mathcal{E}_k)$ be an uncrowded, k-uniform hypergraph with $|V| = N$ vertices and average degree $t^{k-1} := k \cdot |\mathcal{E}_k|/N$. Then, for some constant $C_k > 0$, the independence number $\alpha(\mathcal{G})$ of $\mathcal{G}$ satisfies*

$$\alpha(\mathcal{G}) \geq C_k \cdot (N/t) \cdot (\log t)^{\frac{1}{k-1}}. \tag{2}$$

Hence, for fixed $k \geq 3$ and uncrowded, k-uniform hypergraphs with average degree t^{k-1} the lower bound (2) improves on (1) by a factor of $\Theta((\log t)^{1/(k-1)})$. In [9] it has been shown that it suffices in Theorem 3 to relax the assumption of having an uncrowded hypergraph to having a linear hypergraph.
We use the following extension of Theorem 3 to non-uniform hypergraphs – moreover, instead of an uncrowded hypergraph we require only a linear one –, see [17].

Theorem 4. *Let $k \geq 3$ be a fixed integer. Let $\mathcal{G} = (V, \mathcal{E}_3 \cup \cdots \cup \mathcal{E}_k)$ be a linear hypergraph on $|V| = N$ vertices, where the average degrees $t_i^{i-1} := i \cdot |\mathcal{E}_i|/N$ for the i-element edges satisfy $t_i^{i-1} \leq S^{i-1} \cdot (\log S)^{(k-i)/(k-1)}$ for some number S. Then, for some constant $C_k > 0$, the independence number $\alpha(\mathcal{G})$ of $\mathcal{G}$ satisfies*

$$\alpha(\mathcal{G}) \geq C_k \cdot \frac{N}{S} \cdot (\log S)^{\frac{1}{k-1}}. \tag{3}$$

An independent set of size $\Omega((N/S) \cdot (\log S)^{1/(k-1)})$ can be found deterministically in time $O(N \cdot S^{4k-2})$.

Both Theorems 3 and 4 are provable best possible for a certain range of the parameters $k < T < N$ as can be seen by a random hypergraph argument.
Here we use Theorem 4 in our arguments to prove Theorem 1.

3 A Deterministic Algorithm

To give an algorithm, which for fixed integers $k \geq 3$ and any integers $n \geq k$ finds deterministically n points in the unit square $[0, 1]^2$ such that, simultaneously for $j = 3, \ldots, k$, the areas of the convex hulls of any j of these n points are $\Omega((\log n)^{1/(j-2)}/n^{(j-1)/(j-2)})$, we discretize $[0, 1]^2$ by considering the standard $T \times T$-grid, i.e., the set $\{(i, j) \in \mathbb{Z}^2 \mid 0 \leq i, j \leq T - 1\}$, where $T = n^{1+\beta}$ for some constant $\beta > 0$, which will be specified later.
For distinct grid-points P, Q in the $T \times T$-grid let PQ denote the *line* through P and Q, and let $[P, Q]$ be the *segment* between P and Q. Let dist $(P, Q) := ((p_x - q_x)^2 + (p_y - q_y)^2)^{1/2}$ denote the *Euclidean distance* between the grid-points $P = (p_x, p_y)$ and $Q = (q_x, q_y)$. For grid-points $P_1, \ldots, P_l$ in the $T \times T$-grid let area $(P_1, \ldots, P_l)$ be the area of the convex hull of the points $P_1, \ldots, P_l$. A *strip* centered at the line PQ of width w is the set of all points in $\mathbb{R}^2$, which are at Euclidean distance at most $w/2$ from the line PQ. Let $\leq_l$ be a total order on the $T \times T$-grid, which is defined as follows: for grid-points $P = (p_x, p_y)$ and $Q = (q_x, q_y)$ in the $T \times T$-grid let $P \leq_l Q :\Longleftrightarrow (p_x < q_x)$ or $(p_x = q_x$ and $p_y < q_y)$.
First notice the following simple observation, which is used in our arguments.

Proposition 1. *Let $P_1, \ldots, P_l$ be grid-points in the $T \times T$-grid, $l \geq 3$.*

(i) Then, it is area $(P_1, \ldots, P_l) \geq$ area $(P_1, \ldots, P_{l-1})$.
(ii) If area $(P_1, \ldots, P_l) \leq A$, then for any distinct grid-points P_i, P_j every grid-point P_k, $k = 1, \ldots, l$, is contained in a strip centered at the line P_iP_j of width $(4 \cdot A)/dist\ (P_i, P_j)$.

Next we prove Theorem 1.

Proof. For suitable constants $c_j^* > 0$, $j = 3, \ldots, k$, which are fixed later in connection with inequality (29), we set

$$A_j := \frac{c_j^* \cdot T^2 \cdot (\log n)^{1/(j-2)}}{n^{(j-1)/(j-2)}} > 1. \tag{4}$$

Then, it is $0 < A_3 \leq \cdots \leq A_k$ for $n \geq n_0$.
We form a non-uniform hypergraph $\mathcal{G} = \mathcal{G}(A_3, \ldots, A_k) = (V, \mathcal{E}_3^0 \cup \mathcal{E}_3 \cup \mathcal{E}_4 \cup \cdots \cup \mathcal{E}_k)$, which contains two types of 3-element edges, and (one type of) j-element edges, $j = 4, \ldots, k$. The vertex-set V of $\mathcal{G}$ consists of all T^2 grid-points in the $T \times T$-grid. The edge-sets are defined as follows. For distinct grid-points $P, Q, R \in V$ in the $T \times T$-grid let $\{P, Q, R\} \in \mathcal{E}_3^0$ if and only if P, Q, R are collinear. Moreover, for $j = 3, \ldots, k$, and distinct grid-points $P_1, \ldots, P_j \in V$ in the $T \times T$-grid let $\{P_1, \ldots, P_j\} \in \mathcal{E}_j$ if and only if area $(P_1, \ldots, P_j) \leq A_j$ and no three of the grid-points $P_1, \ldots, P_j$ are collinear.
We want to find a large independent set in the hypergraph $\mathcal{G} = (V, \mathcal{E}_3^0 \cup \mathcal{E}_3 \cup \mathcal{E}_4 \cup \cdots \cup \mathcal{E}_k)$, as an independent set $I \subseteq V$ in $\mathcal{G}$ corresponds to $|I|$ many grid-points in the $T \times T$-grid, such that the areas of the convex hulls of any j of these $|I|$ grid-points are bigger than A_j, $j = 3, \ldots, k$. To find a suitable, induced subhypergraph of $\mathcal{G}$ to which Theorem 4 may be applied, in a first step we estimate the numbers $|\mathcal{E}_3^0|$ and $|\mathcal{E}_j|$, $j = 3, \ldots, k$, of 3- and j-element edges, respectively, and the numbers of 2-cycles in $\mathcal{G}$. Then, in a certain induced subhypergraph $\mathcal{G}^*$ of $\mathcal{G}$ we destroy all 3-element edges in $\mathcal{E}_3^0$ and all 2-cycles. The resulting induced subhypergraph $\mathcal{G}^{**}$ is linear, and then we may apply Theorem 4 to $\mathcal{G}^{**}$.

3.1 The Numbers of Edges in $\mathcal{G}$

The next estimate is quite crude but it suffices for our purposes.

Proposition 2. *The number $|\mathcal{E}_3^0|$ of unordered collinear triples of grid-points in the $T \times T$-grid satisfies*

$$|\mathcal{E}_3^0| \leq T^5. \tag{5}$$

Proof. Each line is determined by two distinct grid-points in the $T \times T$-grid, for which there are at most T^4 choices. Each line contains at most T grid-points from the $T \times T$-grid, and the upper bound $|\mathcal{E}_3^0| \leq T^5$ on the number of collinear triples follows. □

To estimate $|\mathcal{E}_j|$, $j = 3, \ldots, k$, we use the following observation from [6], compare Proposition 1.

Lemma 1. *For distinct grid-points $P = (p_x, p_y)$ and $R = (r_x, r_y)$ with $P \leq_l R$ from the $T \times T$-grid, where $s := r_x - p_x \geq 0$ and $h := r_y - p_y$, it holds:*

(a) *There are at most $4 \cdot A$ grid-points Q in the $T \times T$-grid such that*
  (i) *$P \leq_l Q \leq_l R$, and*
  (ii) *P, Q, R are not collinear, and area $(P, Q, R) \leq A$.*

(b) *The number of grid-points Q in the $T \times T$-grid which fulfill only (ii) from (a) is at most $(12 \cdot A \cdot T)/s$ for $s > 0$, and at most $(12 \cdot A \cdot T)/|h|$ for $|h| > s$.*

Lemma 2. *For $j = 3, \ldots, k$, the numbers $|\mathcal{E}_j|$ of unordered j-tuples $P_1, \ldots, P_j$ of pairwise distinct grid-points in the $T \times T$-grid with area $(P_1, \ldots, P_j) \leq A_j$, where no three of $P_1, \ldots, P_j$ are collinear, satisfy for some constants $c_j > 0$:*

$$|\mathcal{E}_j| \leq c_j \cdot A_j^{j-2} \cdot T^4. \tag{6}$$

Proof. Let $P_1, \ldots, P_j$ be pairwise distinct grid-points in the $T \times T$-grid, no three on a line and with area $(P_1, \ldots, P_j) \leq A_j$. We may assume that $P_1 \leq_l \cdots \leq_l P_j$. For $P_1 = (p_{1,x}, p_{1,y})$ and $P_j = (p_{j,x}, p_{j,y})$ let $s := p_{j,x} - p_{1,x} \geq 0$ and $h := p_{j,y} - p_{1,y}$. Then $s > 0$, as otherwise $P_1, \ldots, P_j$ are collinear.
There are T^2 choices for the grid-point P_1. Given P_1, any grid-point P_j with $P_1 \leq_l P_j$ is determined by a pair $(s, h) \neq (0, 0)$ of integers with $1 \leq s \leq T$ and $-T \leq h \leq T$. By Proposition 1(i) we have area $(P_1, P_i, P_j) \leq A_j$ for $i = 2, \ldots, j-1$. Given the grid-points P_1 and P_j, since $P_1 \leq_l P_i \leq_l P_j$ for $i = 2, \ldots, j-1$, by Lemma 1(a) there are at most $4 \cdot A_j$ choices for each grid-point P_i, hence for $j = 3, \ldots, k$ and constants $c_j > 0$ we obtain

$$|\mathcal{E}_j| \leq T^2 \cdot \sum_{s=1}^{T} \sum_{h=-T}^{T} (4 \cdot A_j)^{j-2} \leq c_j \cdot A_j^{j-2} \cdot T^4. \qquad \square$$

For later use, observe that by (6) the average degrees t_j^{j-1} for the j-element edges $E \in \mathcal{E}_j$, $j = 3, \ldots, k$, of $\mathcal{G}$ satisfy

$$t_j^{j-1} = j \cdot |\mathcal{E}_j|/|V| \leq j \cdot c_j \cdot A_j^{j-2} \cdot T^2 =: (t_j(0))^{j-1}. \tag{7}$$

3.2 The Numbers of 2-Cycles in the Hypergraph $\mathcal{G}$

Here we take care of the number of 2-cycles in the hypergraph $\mathcal{G}$. Let $s_{2;(g,i,j)}(\mathcal{G})$ denote the number of $(2; (g, i, j))$-cycles in $\mathcal{G}$, i.e., the number of unordered pairs $\{E, E'\}$ of edges with $E \in \mathcal{E}_i$ and $E' \in \mathcal{E}_j$ and $|E \cap E'| = g$, $2 \leq g \leq i \leq j \leq k$ and $g < j$. Note that we do not take into account the edges from $\mathcal{E}_3^0$, i.e., collinear triples of grid-points, as these are treated separately.

Lemma 3. *For $2 \leq g \leq i \leq j \leq k$ with $g < j$, the numbers $s_{2;(g,i,j)}(\mathcal{G})$ of $(2;(g,i,j))$-cycles in the hypergraph $\mathcal{G} = (V, \mathcal{E}_3 \cup \mathcal{E}_4 \cup \cdots \cup \mathcal{E}_k)$ fulfill for some constants $c_{2;(g,i,j)} > 0$:*

$$s_{2;(g,i,j)}(\mathcal{G}) \leq c_{2;(g,i,j)} \cdot A_i^{i-2} \cdot A_j^{j-g} \cdot T^4 \cdot \log^3 T. \tag{8}$$

Proof. For $2 \leq g \leq i \leq j \leq k$ with $g < j$, let $\{E, E'\}$ be a $(2;(g,i,j))$-cycle in $\mathcal{G}$, where $E \in \mathcal{E}_i$ and $E' \in \mathcal{E}_j$. Let the grid-points, which correspond to the vertices in E and E', respectively, be $P_1, \ldots, P_i$ and $P_1, \ldots, P_g, Q_{g+1}, \ldots, Q_j$ with $P_1 \leq_l \cdots \leq_l P_g$. By definition of the edge-set of $\mathcal{G}$ no three of the grid-points $P_1, \ldots, P_i$ and of $P_1, \ldots, P_g, Q_{g+1}, \ldots, Q_j$ are collinear, and area $(P_1, \ldots, P_i) \leq A_i$ as well as area $(P_1, \ldots, P_g, Q_{g+1}, \ldots, Q_j) \leq A_j$.
There are T^2 choices for the grid-point P_1. Given $P_1 := (p_{1,x}, p_{1,y})$, any pair $(s, h) \neq (0, 0)$ of integers determines at most one grid-point $P_g := (p_{1,x}+s, p_{1,y}+h)$ in the $T \times T$-grid. By symmetry we may assume that $s > 0$ and $0 \leq h \leq s \leq T$, which is taken into account by an additional constant factor of 2. Given the grid-points P_1 and P_g, since area $(P_1, P_f, P_g) \leq A_i$ for $f = 2, \ldots, g-1$ by Proposition 1(i), with $P_1 \leq_l P_f \leq_l P_g$ and by Lemma 1(a) there are at most $4 \cdot A_i$ choices for each grid-point P_f in the $T \times T$-grid, hence, given h, s, the number of choices for the grid-points $P_1, \ldots, P_g$ is at most

$$(4 \cdot A_i)^{g-2} \cdot T^2. \tag{9}$$

For the convex hulls of the grid-points $P_1, \ldots, P_i$ and $P_1, \ldots, P_g, Q_{g+1}, \ldots, Q_j$ let their (w.r.t. $\leq_l$) *extremal* points be $P', P'' \in \{P_1, \ldots, P_i\}$ and $Q', Q'' \in \{P_1, \ldots, P_g, Q_{g+1}, \ldots, Q_j\}$, respectively, i.e., with $P' \leq_l P''$ and $Q' \leq_l Q''$ we have $P' \leq_l P_1, \ldots, P_i \leq_l P''$ and $Q' \leq_l P_1, \ldots, P_g, Q_{g+1}, \ldots, Q_j \leq_l Q''$.
Given the grid-points $P_1 \leq_l \cdots \leq_l P_g$, there are three possibilities for the convex hulls of the grid-points $P_1, \ldots, P_i$ and $P_1, \ldots, P_j, Q_{j+1}, \ldots, Q_k$, respectively:
(*i*) P_1 and P_g are extremal, or (*ii*) exactly one grid-point, P_1 or P_g, is extremal, or (*iii*) neither P_1 nor P_g is extremal.
We only consider the convex hull of $P_1, \ldots, P_i$, as the considerations for the convex hull of $P_1, \ldots, P_g, Q_{g+1}, \ldots, Q_j$ are essentially the same.
In case (i) the grid-points P_1 and P_g are extremal for the convex hull of $P_1, \ldots, P_i$, hence $P_1 \leq_l P_{g+1}, \ldots, P_i \leq_l P_g$. By Lemma 1(a), since area $(P_1, P_f, P_g) \leq A_i$ by Proposition 1(i), $f = g+1, \ldots, i$, and no three of the grid-points $P_1, \ldots, P_i$ are collinear, there are at most $4 \cdot A_i$ choices for each grid-point P_f, hence the number of choices for the grid-points $P_{g+1}, \ldots, P_i$ is at most

$$\text{case (i):} \qquad (4 \cdot A_i)^{i-g}. \tag{10}$$

In case (ii) exactly one of the grid-points P_1 or P_g is extremal for the convex hull of $P_1, \ldots, P_i$. By Lemma 1(b) there are at most $(12 \cdot A_i \cdot T)/s$ choices for the second extremal grid-point P' or P''. Having chosen this second extremal grid-point, for each of the $(i-g-1)$ remaining grid-points $P_{g+1}, \ldots, P_i \neq P', P''$ there are by Lemma 1(a) at most $4 \cdot A_i$ choices, hence the number of choices for the grid-points $P_{g+1}, \ldots, P_i$ is at most

$$\text{case (ii):} \qquad (4 \cdot A_i)^{i-g-1} \cdot \frac{12 \cdot A_i \cdot T}{s} = \frac{3 \cdot (4 \cdot A_i)^{i-g} \cdot T}{s}. \tag{11}$$

In case (iii) none of the grid-points P_1, P_g is extremal for the convex hull of $P_1, \ldots, P_i$. By Proposition 1(ii) all grid-points $P_{g+1}, \ldots, P_i$ are contained in a strip S_i, which is centered at the line $P_1 P_g$, of width $(4 \cdot A_i)/\sqrt{h^2 + s^2}$. Consider the parallelogram $\mathcal{P}_0 = \{(p_x, p_y) \in S_i \mid p_{1,x} \leq p_x \leq p_{1,x} + s\}$ within the strip S_i. We partition the strip S_i within the $T \times T$-grid into pairwise congruent parallelograms $\mathcal{P}_l$, $-L \leq l \leq L$ with $L := \lceil T/s \rceil + 1$, where $\mathcal{P}_l := \{(p_x, p_y) \in S_i \mid p_{1,x} + l \cdot s \leq p_x \leq p_{1,x} + (l+1) \cdot s\}$. Each parallelogram has side-lengths $(4 \cdot A_i)/s$ and $\sqrt{h^2 + s^2}$ and its area is $4 \cdot A_i$.

Since by assumption neither $P_1 \in \mathcal{P}_0$ nor $P_g \in \mathcal{P}_0$ are extremal, each extremal grid-point, P' or P'', is contained in some parallelogram $\mathcal{P}_l$ for some $l \neq 0$, for which there are by Lemma 1(a) at most $4 \cdot A_i$ choices. Each grid-point $P = (p_x, p_y) \in \mathcal{P}_l$ satisfies $|p_x - p_{1,x}| \geq l \cdot s$ or $|p_x - p_{j,x}| \geq l \cdot s$. Thus, if one of the grid-points P' or P'' is contained in some parallelogram $\mathcal{P}_l$, $l \neq 0$, then by Lemma 1(b) there are at most $(12 \cdot A_i \cdot T)/(l \cdot s)$ choices for the second extremal grid-point. Having fixed both extremal grid-points P' and P'' in at most $(4 \cdot A_i) \cdot ((12 \cdot A_i \cdot T)/(l \cdot s)) = (48 \cdot A_i^2 \cdot T)/(l \cdot s)$ ways, for the remaining $(i - g - 2)$ grid-points $P_{g+1}, \ldots, P_i \neq P', P''$ there are by Lemma 1(a) at most $(4 \cdot A_i)^{i-g-2}$ choices. Hence, by summing over all possible choices of $l \neq 0$, the number of choices for the grid-points $P_{g+1}, \ldots, P_i$ is at most

$$\text{case (iii):} \qquad (4 \cdot A_i)^{i-g-2} \cdot 2 \cdot \sum_{l=1}^{\lceil T/s \rceil + 1} \frac{48 \cdot A_i^2 \cdot T}{l \cdot s} =$$

$$= (4 \cdot A_i)^{i-g} \cdot \frac{6 \cdot T}{s} \cdot \sum_{l=1}^{\lceil T/s \rceil + 1} \frac{1}{l} \leq (4 \cdot A_i)^{i-g} \cdot \frac{10 \cdot T \cdot \log T}{s}. \tag{12}$$

Thus, given the grid-points $P_1, \ldots, P_g$, by (10)–(12) and using $s \leq T$, altogether the number of choices for the grid-points $P_{g+1}, \ldots, P_i$ is at most

$$(4 \cdot A_i)^{i-g} \cdot \left(1 + \frac{3 \cdot T}{s} + \frac{10 \cdot T \cdot \log T}{s}\right) \leq \frac{14 \cdot (4 \cdot A_i)^{i-g} \cdot T \cdot \log T}{s}. \tag{13}$$

Similar to (13), for the number of choices of the grid-points $Q_{g+1}, \ldots, Q_j$ the following upper bound holds:

$$\frac{14 \cdot (4 \cdot A_j)^{j-g} \cdot T \cdot \log T}{s}. \tag{14}$$

Hence, with (9), (13), and (14) for $2 \leq g \leq i \leq j \leq k$ and $g < j$ we obtain for constants $c_{2;(g,i,j)} > 0$:

$$s_{2;(g,i,j)}(\mathcal{G}) \leq 2 \cdot (4 \cdot A_i)^{g-2} \cdot T^2 \cdot \sum_{s=1}^{T} \sum_{h=0}^{s} \left(\frac{14 \cdot (4 \cdot A_i)^{i-g} \cdot T \cdot \log T}{s} \right) \cdot \\ \cdot \left(\frac{14 \cdot (4 \cdot A_j)^{j-g} \cdot T \cdot \log T}{s} \right) \leq \\ = 392 \cdot 4^{i+j-g-2} \cdot A_i^{i-2} \cdot A_j^{j-g} \cdot T^4 \cdot \log^2 T \cdot \sum_{s=1}^{T} \sum_{h=0}^{s} \frac{1}{s^2} \\ \leq c_{2;(g,i,j)} \cdot A_i^{i-2} \cdot A_j^{j-g} \cdot T^4 \cdot \log^3 T. \qquad \square$$

3.3 Choosing a Subhypergraph in $\mathcal{G}$

With probability $p := T^\varepsilon / t_k(0) \leq 1$, hence $p = \Theta(T^\varepsilon/(A_k^{(k-2)/(k-1)} \cdot T^{2/(k-1)})$ by (7), where $\varepsilon > 0$ is a small constant, we select uniformly at random and independently of each other vertices from V. Let $V^* \subseteq V$ be the random set of chosen vertices. Let $\mathcal{G}^* = (V^*, \mathcal{E}_3^{0*} \cup \mathcal{E}_3^* \cup \mathcal{E}_4^* \cup \cdots \cup \mathcal{E}_k^*)$ with $\mathcal{E}_3^{0*} := \mathcal{E}_3^0 \cap [V^*]^3$ and $\mathcal{E}_j^* := \mathcal{E}_j \cap [V^*]^j$, $j = 3, \ldots, k$, be the random subhypergraph of $\mathcal{G}$, which is induced by V^*. Let $E[|V^*|]$, $E[|\mathcal{E}_3^{0*}|]$, $E[|\mathcal{E}_j^*|]$, $j = 3, \ldots, k$, and $E[s_{2;(g,i,j)}(\mathcal{G}^*)]$, $2 \leq g \leq i \leq j \leq k$ but $g < j$, be the expected numbers of vertices, induced collinear triples, j-element edges and $(2;(g,i,j))$-cycles, respectively, in $\mathcal{G}^*$. By (5), (6), and (8) we infer for constants $c_1', c_3^{0'} c_j', c_{2;(g,i,j)}' > 0$:

$$E[|V^*|] = p \cdot T^2 = (c_1' \cdot T^{\frac{2k-4}{k-1}+\varepsilon})/A_k^{\frac{k-2}{k-1}} \tag{15}$$

$$E[|\mathcal{E}_3^{0*}|] = p^3 \cdot |\mathcal{E}_3^0| \leq (c_3^{0'} \cdot T^{\frac{5k-11}{k-1}+3\varepsilon})/A_k^{\frac{3k-6}{k-1}} \tag{16}$$

$$E[|\mathcal{E}_j^*|] = p^j \cdot |\mathcal{E}_j| \leq (c_j' \cdot T^{\frac{4k-2j-4}{k-1}+j\varepsilon} \cdot A_j^{j-2})/A_k^{\frac{j(k-2)}{k-1}} \tag{17}$$

$$E[s_{2;(g,i,j)}(\mathcal{G}^*)] = p^{i+j-g} \cdot s_{2;(g,i,j)}(\mathcal{G}) \leq \\ \leq \frac{c_{2;(g,i,j)}' \cdot T^{4-\frac{2(i+j-g)}{k-1}+(i+j-g)\varepsilon} \cdot \log^3 T \cdot A_i^{i-2} \cdot A_j^{j-g}}{A_k^{\frac{(k-2)(i+j-g)}{k-1}}}. \tag{18}$$

By (15)–(18) and by Chernoff's and Markov's inequality there exists a subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_3^{0*} \cup \mathcal{E}_3^* \cup \mathcal{E}_4^* \cup \cdots \cup \mathcal{E}_k^*)$ of $\mathcal{G}$ such that

$$|V^*| \geq ((c_1'/2) \cdot T^{\frac{2k-4}{k-1}+\varepsilon})/A_k^{\frac{k-2}{k-1}} \tag{19}$$

$$|\mathcal{E}_3^{0*}| \leq (k^3 \cdot c_3^{0'} \cdot T^{\frac{5k-11}{k-1}+3\varepsilon})/A_k^{\frac{3k-6}{k-1}} \tag{20}$$

$$|\mathcal{E}_j^*| \leq (k^3 \cdot c_j' \cdot T^{\frac{4k-2j-4}{k-1}+j\varepsilon} \cdot A_j^{j-2})/A_k^{\frac{j(k-2)}{k-1}} \tag{21}$$

$$s_{2;(g,i,j)}(\mathcal{G}^*) \leq \frac{k^3 \cdot c_{2;(g,i,j)}' \cdot T^{4-\frac{2(i+j-g)}{k-1}+(i+j-g)\varepsilon} \cdot \log^3 T \cdot A_i^{i-2} \cdot A_j^{j-g}}{A_k^{\frac{(k-2)(i+j-g)}{k-1}}}. \tag{22}$$

This probabilistic argument can be turned into a deterministic polynomial time algorithm by using the method of conditional probabilities. For $2 \leq g \leq i \leq j \leq k$ but $g < j$, let $\mathcal{C}_{2;(g,i,j)}$ denote the multiset of all $(i+j-g)$-element subsets $E \cup E'$ of V with $E \in \mathcal{E}_i$ and $E' \in \mathcal{E}_j$ and $|E \cap E'| = g$. Let the grid-points in the $T \times T$-grid be $P_1, \ldots, P_{T^2}$. To each grid-point P_i associate a variable $p_i \in [0,1]$, $i = 1, \ldots, T^2$, and let $F\colon [0,1]^{T^2} \longrightarrow \mathbb{R}$ be a function, which is defined by

$$F(p_1, \ldots, p_{T^2}) := 2^{pT^2/2} \cdot \prod_{i=1}^{T^2} \left(1 - \frac{p_i}{2}\right) +$$

$$+ \frac{\sum_{\{P_i,P_j,P_k\} \in \mathcal{E}_3^0} p_i \cdot p_j \cdot p_k}{(k^3 \cdot c_3^{0'} \cdot T^{\frac{5k-11}{k-1}+3\varepsilon})/A_k^{\frac{3k-6}{k-1}}} + \sum_{j=3}^{k} \frac{\sum_{\{P_{i_1},\ldots,P_{i_j}\} \in \mathcal{E}_j} \prod_{l=1}^{j} p_{i_l}}{(k^3 \cdot c_j' \cdot T^{\frac{4k-2j-4}{k-1}+j\varepsilon} \cdot A_j^{j-2})/A_k^{\frac{j(k-2)}{k-1}}} +$$

$$+ \sum_{2 \leq g \leq i \leq j \leq k; g < j} \frac{A_k^{\frac{(k-2)(i+j-g)}{k-1}} \cdot \sum_{\{P_{i_1},\ldots,P_{i_{i+j-g}}\} \in \mathcal{C}_{2;(g,i,j)}} \prod_{l=1}^{i+j-g} p_{i_l}}{k^3 \cdot c'_{2;(g,i,j)} \cdot T^{4-\frac{2(i+j-g)}{k-1}+(i+j-g)\varepsilon} \cdot \log^3 T \cdot A_i^{i-2} \cdot A_j^{j-g}}.$$

For convenience we assume that $p \cdot T^2$ is an integer. In the beginning we set $p_1 := \cdots := p_{T^2} := p = T^\varepsilon / t_k(0)$. We infer by (15)–(18) and using $1 + x \leq e^x$ that $F(p, \ldots, p) < (2/e)^{pT^2/2} + 1/3$, hence $F(p, \ldots, p) < 1$ for $p \cdot T^2 \geq 3$. By using the linearity of the function $F(p_1, \ldots, p_{T^2})$ in each p_i, we minimize $F(p_1, \ldots, p_{T^2})$ step by step by choosing one after the other $p_i := 0$ or $p_i := 1$, $i = 1, \ldots, T^2$. Finally we obtain $p_1, \ldots, p_{T^2} \in \{0,1\}$ such that $F(p_1, \ldots, p_{T^2}) < 1$. The set $V^* = \{P_i \in V \mid p_i = 1\}$ yields an induced subhypergraph $\mathcal{G}^* = (V^*, \mathcal{E}_3^{0*} \cup \mathcal{E}_3^* \cup \cdots \cup \mathcal{E}_k^*)$ of $\mathcal{G}$ with $\mathcal{E}_j^* := \mathcal{E}_j \cap [V^*]^j$ for $j = 3, \ldots, k$, and $\mathcal{E}_3^{0*} := \mathcal{E}_3^0 \cap [V^*]^3$, which satisfies (19)–(22), as otherwise $F(p_1, \ldots, p_{T^2}) > 1$ gives a contradiction. Namely, for example if $|V^*| < p \cdot T^2/2$, hence $|V^*| \leq (p \cdot T^2 - 1)/2$ since we assumed that $p \cdot T^2$ is an integer, then $F(p_1, \ldots, p_{T^2}) \geq 2^{pT^2/2} \cdot \prod_{i=1}^{T^2} (1 - p_i/2) \geq 2^{1/2} > 1$, which contradicts the fact that the final value of the function F is less than 1.

By (4)–(6) and (8) and using $T = n^{1+\beta}$ for fixed $\beta > 0$, the running time of this derandomization is given by

$$O\left(|V| + |\mathcal{E}_3^0| + \sum_{j=3}^{k} |\mathcal{E}_j| + \sum_{2 \leq g \leq i \leq j \leq k; g < j} |\mathcal{C}_{2;(g,i,j)}|\right) = O(|\mathcal{C}_{2;(2,k,k)}|) =$$

$$= O(A_k^{2k-4} \cdot T^4 \cdot \log^3 T) = O((T^{4k-4} \cdot \log^5 n)/n^{2k-2}). \tag{23}$$

Lemma 4. *For fixed β, ε with $\beta > 1$ and $0 < \varepsilon \leq (\beta - 1)/(2 \cdot (1 + \beta))$ it is*

$$|\mathcal{E}_3^{0*}| = o(|V^*|). \tag{24}$$

Proof. By (4), (19), and (20), and with $T = n^{1+\beta}$ we have

$$\begin{aligned} & |\mathcal{E}_3^{0*}| = o(|V^*|) \\ \Longleftarrow\ & T^{\frac{5k-11}{k-1}+3\varepsilon}/A_k^{\frac{3k-6}{k-1}} = o(T^{\frac{2k-4}{k-1}+\varepsilon}/A_k^{\frac{k-2}{k-1}}) \\ \Longleftrightarrow\ & n^{2-(1+\beta)(1-2\varepsilon)}/(\log n)^{\frac{2}{k-1}} = o(1) \\ \Longleftrightarrow\ & (1+\beta)\cdot(1-2\cdot\varepsilon) \geq 2, \end{aligned}$$

which holds for $\varepsilon \leq (\beta-1)/(2\cdot(1+\beta))$. □

Lemma 5. *For fixed* $2 \leq g \leq i \leq j \leq k$ *with* $g < j$*, and for fixed* ε *with* $0 < \varepsilon < \frac{j-g}{(i+j-g-1)(j-2)(1+\beta)}$ *it is*

$$s_{2;(g,i,j)}(\mathcal{G}^*) = o(|V^*|). \tag{25}$$

Proof. With (4), (19), and (22) and by using $T = n^{1+\beta}$ we infer

$$\begin{aligned} & s_{2;(g,i,j)}(\mathcal{G}^*) = o(|V^*|) \\ \Longleftarrow\ & \frac{T^{4-\frac{2(i+j-g)}{k-1}+(i+j-g)\varepsilon}\cdot\log^3 T\cdot A_i^{i-2}\cdot A_j^{j-g}}{A_k^{\frac{(k-2)(i+j-g)}{k-1}}} = o\left(\frac{T^{\frac{2k-4}{k-1}+\varepsilon}}{A_k^{\frac{k-2}{k-1}}}\right) \\ \Longleftrightarrow\ & n^{\varepsilon(1+\beta)(i+j-g-1)-\frac{j-g}{j-2}}\cdot(\log n)^{4+\frac{j-g}{j-2}-\frac{i+j-g-1}{k-1}} = o(1) \\ \Longleftrightarrow\ & \varepsilon < \frac{j-g}{(j-2)(i+j-g-1)(1+\beta)}. \end{aligned}$$

□

Set $\varepsilon := 1/(2\cdot k^2\cdot(1+\beta))$ and $\beta := 1+(2/k^2)$. Then, all assumptions in Lemmas 4 and 5 and also $p = T^\varepsilon/t_k(0) \leq 1$ are fulfilled. We delete one vertex from each edge $E \in \mathcal{E}_3^{0*}$, and from each 2-cycle in $\mathcal{G}^*$. Let $V^{**} \subseteq V^*$ be the set of remaining vertices. By Lemmas 4 and 5 and (19) we infer

$$|V^{**}| = (1-o(1))\cdot|V^*| \geq |V^*|/2 \geq ((c_1/4)\cdot T^{\frac{2k-4}{k-1}+\varepsilon})/A_k^{\frac{k-2}{k-1}}, \tag{26}$$

and the induced subhypergraph $\mathcal{G}^{**} = (V^{**}, \mathcal{E}_3^{**}\cup\cdots\cup\mathcal{E}_k^{**})$ with $\mathcal{E}_j^{**} := \mathcal{E}_j^* \cap [V^{**}]^j$, $j = 3,\ldots,k$, does not contain any edges from $\mathcal{E}_3^0$ or 2-cycles anymore, i.e., $\mathcal{G}^{**}$ is a linear hypergraph. Since $|\mathcal{E}_j^{**}| \leq |\mathcal{E}_j^*|$ with (4), (21), and (26) and $T = n^{1+\beta}$ the average degrees t_j^{j-1} for the j-element edges of $\mathcal{G}^{**}$, $j = 3,\ldots,k$, fulfill

$$\begin{aligned} t_j^{j-1} = \frac{j\cdot|\mathcal{E}_j^{**}|}{|V^{**}|} &\leq \frac{(j\cdot k^3\cdot c'_j\cdot T^{\frac{4k-2j-4}{k-1}+j\varepsilon}\cdot A_j^{j-2})/A_k^{\frac{j(k-2)}{k-1}}}{((c'_1/4)\cdot T^{\frac{2k-4}{k-1}+\varepsilon})/A_k^{\frac{k-2}{k-1}}} \leq \\ &\leq \frac{4\cdot k^4\cdot c'_j\cdot(c_j^*)^{j-2}}{c'_1\cdot(c_k^*)^{\frac{(j-1)(k-2)}{k-1}}\cdot(1+\beta)^{\frac{k-j}{k-1}}}\cdot T^{(j-1)\varepsilon}\cdot(\log T)^{\frac{k-j}{k-1}} =: t_j^{j-1}(1). \end{aligned} \tag{27}$$

Set

$$c := \max\left\{1, \frac{(4\cdot k^4\cdot c'_k)^{\frac{1}{k-1}}}{(c'_1)^{\frac{1}{k-1}}}\right\}. \tag{28}$$

Dependent on the choice of the constant $c_k^* > 0$ below, choose constants $c_j^* > 0$ arising from (4), $j = 3, \ldots, k-1$, such that

$$6 \cdot k^2 \cdot \frac{(4 \cdot k^4 \cdot c_j')^{\frac{1}{j-1}} \cdot (c_j^*)^{\frac{j-2}{j-1}}}{(c_1')^{\frac{1}{j-1}} \cdot (c_k^*)^{\frac{k-2}{k-1}} \cdot (1+\beta)^{\frac{k-j}{(k-1)(j-1)}}} \leq c. \tag{29}$$

With $S := c \cdot T^{\varepsilon}$ we infer from (27)–(29) that $t_j^{j-1}(1) \leq S^{j-1} \cdot (\log S)^{(k-j)/(k-1)}$ for $j = 3, \ldots, k$, as can be easily seen with $(1/\varepsilon)^{(k-j)/(k-1)} < 6 \cdot k^2$. Hence, as the subhypergraph $\mathcal{G}^{**}$ is linear, the assumptions in Theorem 4 are fulfilled, and we apply it. By using (4) we find by choice of $\beta, \varepsilon > 0$ in time

$$O((T^{\frac{2k-4}{k-1}+\varepsilon}/A_k^{\frac{k-2}{k-1}}) \cdot S^{4k-2}) = O(n \cdot T^{(4k-1)\varepsilon}) = o(T^2) \tag{30}$$

with (26), (28), $c \geq 1$, and $T = n^{1+\beta}$, and $\varepsilon = 1/(2 \cdot k^2 \cdot (1+\beta))$ an independent set I of size

$$\begin{aligned}
|I| &\geq C_k \cdot \frac{|V^{**}|}{S} \cdot (\log S)^{\frac{1}{k-1}} \\
&\geq C_k \cdot \frac{((c_1/4) \cdot T^{\frac{2k-4}{k-1}+\varepsilon})/A_k^{\frac{k-2}{k-1}}}{c \cdot T^{\varepsilon}} \cdot (\log(c \cdot T^{\varepsilon}))^{\frac{1}{k-1}} \\
&= \frac{C_k \cdot (c_1/4)}{c} \cdot \frac{T^{\frac{2k-4}{k-1}}}{((c_k^*)^{\frac{k-2}{k-1}} \cdot T^{\frac{2k-4}{k-1}} \cdot (\log n)^{\frac{1}{k-1}})/n} \cdot (\log(c \cdot T^{\varepsilon}))^{\frac{1}{k-1}} \\
&\geq \frac{C_k \cdot (c_1/4)}{c} \cdot \frac{(1/(2 \cdot k^2 \cdot (1+\beta)))^{\frac{1}{k-1}}}{(c_k^*)^{\frac{k-2}{k-1}}} \cdot \frac{n}{(\log n)^{\frac{1}{k-1}}} \cdot (\log n)^{\frac{1}{k-1}} \\
&> \frac{C_k \cdot (c_1/4)}{c} \cdot \frac{1}{7 \cdot (c_k^*)^{\frac{k-2}{k-1}}} \cdot n,
\end{aligned}$$

as $(2 \cdot k^2 \cdot (1+\beta))^{1/(k-1)} < 7$. The constants c, c_1, C_k do not depend on the constant c_k^*. Therefore, by choosing the constant $c_k^* > 0$ in (4) sufficiently small, we obtain an independent set of size n. This yields, after rescaling the areas A_j by a factor of T^2, a desired set of n points in $[0,1]^2$ such that, simultaneously for $j = 3, \ldots, k$, the areas of the convex hulls of every j distinct of these n points are $\Omega((\log n)^{1/(j-2)}/n^{(j-1)/(j-2)})$. Adding the times in (23) and (30) we obtain with $\beta = 1 + (2/k^2)$ the time bound $O(T^{4k-4} \cdot \log^5 n / n^{2k-2} + T^2) = (n^{(2k-2)(1+2\beta)}) = o(n^{6k-4})$. □

We remark that the bound $o(n^{6k-4})$ on the running time might be improved a little, for example by using the better estimate $O(T^4 \cdot \log T)$ on the number of collinear triples of grid-points in the $T \times T$-grid or by a random preselection of grid-points. However, with this approach we cannot do better than $O(n^{ck})$ for some constant $c > 0$ due to the need of constructing the edges and 2-cycles in the hypergraph $\mathcal{G}$.

References

1. M. Ajtai, J. Komlós, J. Pintz, J. Spencer and E. Szemerédi, *Extremal Uncrowded Hypergraphs*, Journal of Combinatorial Theory Ser. A, 32, 1982, 321–335.
2. G. Barequet, *A Lower Bound for Heilbronn's Triangle Problem in d Dimensions,* SIAM Journal on Discrete Mathematics 14, 2001, 230–236.
3. G. Barequet, *The On-Line Heilbronn's Triangle Problem,* Discrete Mathematics 283, 2004, 7–14.
4. G. Barequet and A. Shaikhet, *The On-Line Heilbronn's Triangle Problem in d Dimensions,* Discrete & Computational Geometry 38, 2007, 51–60.
5. C. Bertram–Kretzberg and H. Lefmann, *The Algorithmic Aspects of Uncrowded Hypergraphs,* SIAM Journal on Computing 29, 1999, 201–230.
6. C. Bertram-Kretzberg, T. Hofmeister and H. Lefmann, *An Algorithm for Heilbronn's Problem,* SIAM Journal on Computing 30, 2000, 383–390.
7. P. Brass, *An Upper Bound for the d-Dimensional Analogue of Heilbronn's Triangle Problem,* SIAM Journal on Discrete Mathematics 19, 192–195, 2005.
8. B. Chazelle, *Lower Bounds on The Complexity of Polytope Range Searching,* Journal of the American Mathematical Society 2, 1989, 637–666.
9. R. A. Duke, H. Lefmann and V. Rödl, *On Uncrowded Hypergraphs,* Random Structures & Algorithms 6, 1995, 209–212.
10. A. Fundia, *Derandomizing Chebychev's Inequality to find Independent Sets in Uncrowded Hypergraphs*, Random Structures & Algorithms, 8, 1996, 131–147.
11. T. Jiang, M. Li and P. Vitany, *The Average Case Area of Heilbronn-type Triangles,* Random Structures & Algorithms 20, 2002, 206–219.
12. J. Komlós, J. Pintz and E. Szemerédi, *On Heilbronn's Triangle Problem,* Journal of the London Mathematical Society, 24, 1981, 385–396.
13. J. Komlós, J. Pintz and E. Szemerédi, *A Lower Bound for Heilbronn's Problem,* Journal of the London Mathematical Society, 25, 1982, 13–24.
14. H. Lefmann, *On Heilbronn's Problem in Higher Dimension*, Combinatorica 23, 2003, 669–680.
15. H. Lefmann, *Distributions of Points in the Unit-Square and Large k-Gons,* Proceedings '16th ACM-SIAM Symposium on Discrete Algorithms SODA'05', ACM and SIAM, 2005, 241–250; Long version in European Journal of Combinatorics, to appear.
16. H. Lefmann, *Distributions of Points and Large Convex Hulls of k Points,* Proceedings Second Int. Conference 'Algorithmic Aspects in Information and Management AAIM'06', LNCS 4041, Springer Verlag, 2006, 173–184.
17. H. Lefmann, *Convex Hulls of Point Sets and Non-Uniform Uncrowded Hypergraphs,* Proceedings Third Int. Conference 'Algorithmic Aspects in Information and Management AAIM'07', Springer Verlag, LNCS 4508, 2007, 285–295.
18. K. F. Roth, *On a Problem of Heilbronn,* Journal of the London Mathematical Society 26, 1951, 198–204.
19. K. F. Roth, *On a Problem of Heilbronn, II, and III,* Proceedings of the London Mathematical Society (3), 25, 1972, 193–212, and 543–549.
20. K. F. Roth, *Estimation of the Area of the Smallest Triangle Obtained by Selecting Three out of n Points in a Disc of Unit Area,* Proceedings of Symposia in Pure Mathematics, 24, 1973, AMS, Providence, 251–262.
21. K. F. Roth, *Developments in Heilbronn's Triangle Problem,* Advances in Mathematics, 22, 1976, 364–385.

22. A. Shaikhet, *The On-line Heilbronn's Triangle Problem in d Dimensions,* M.Sc. Thesis, Department of Computer Science, The Technion – Haifa, Israel, 2007.
23. W. M. Schmidt, *On a Problem of Heilbronn,* Journal of the London Mathematical Society (2), 4, 1972, 545–550.
24. J. Spencer, *Turán's Theorem for k-Graphs*, Discrete Mathematics, 2, 1972, 183–186.