ON AN ANTI-RAMSEY TYPE RESULT

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Abstract. We consider anti-Ramsey type results. For a given coloring Δ of the k-element subsets of an n-element set X, where two k-element sets with nonempty intersection are colored differently, let $inj_\Delta(k,n)$ be the largest size of a subset $Y \subseteq X$, such that the k-element subsets of Y are colored pairwise differently. Taking the minimum over all colorings, i.e. $inj(k,n) = \min_\Delta \{inj_\Delta(k,n)\}$, it is shown that for every positive integer k there exist positive constants $c_k, c_k^* > 0$ such that for all integers n, n large, the following inequality holds

$$c_k \cdot (ln\ n)^{\frac{1}{2k-1}} \cdot n^{\frac{k-1}{2k-1}} \leq inj(k,n) \leq c_k^* \cdot (ln\ n)^{\frac{1}{2k-1}} \cdot n^{\frac{k-1}{2k-1}} .$$

1. Introduction

In recent years some interest was drawn towards the study of anti-Ramsey type results, cf. for example [ESS 75], [Ra 75], [Al 83], [SS 84], [GRS 89]. In particular, they include topics like Canonical Ramsey Theory and spectra of colorings. Some of these results show a close connection to the theory of Sidon-sequences, cf. [BS 85], [So 90], and recently they turned out to be also fruitful in determining bounds for canonical Ramsey numbers, cf. [LR 90]. The general problem is: given a graph or hypergraph, where the edges are colored with certain restrictions on the coloring like, for example, edges of the same color cannot intersect nontrivially, one is interested in the largest size of some totally multicolored subgraph of a given type. In this paper we study this question for colorings of complete uniform hypergraphs.

Let $\Delta\colon [X]^k \longrightarrow \omega$, where $\omega = \{0, 1, \ldots\}$, be a coloring of the k-element subsets of X. A subset $Y \subseteq X$ is called totally multicolored if the restriction of Δ to $[Y]^k$ is a one-to-one coloring.

For coloring edges in complete graphs, where the coloring is such that edges of the same color are not incident, Babai gave bounds for the largest totally multicolored complete subgraph K_k:

Theorem 1.1. [Ba 85] *Let n be a positive integer. Let X be an n-element set and let $\Delta\colon E(K_n) \longrightarrow \omega$, where $\omega = \{0, 1, 2, \ldots\}$, be a coloring of the edges of the complete graph K_n on n vertices, such that incident edges get different colors. Then there exists a totally* *multicolored complete subgraph K_k, which satisfies*

$$k \geq (2 \cdot n)^{\frac{1}{3}} . \tag{1}$$

Moreover, for n large, there exists a coloring $\Delta\colon E(K_n) \longrightarrow \omega$, where incident edges are colored differently, such that the largest totally multicolored complete subgraph K_k satisfies

$$k \leq 8 \cdot (n \cdot ln\ n)^{\frac{1}{3}} . \tag{2}$$

Here we improve the lower bound in Theorem (1.1) and show that the upper bound is tight, up to a constant factor. Moreover, we generalize Babai's result to colorings of k-uniform complete hypergraphs. Specifically, we prove

Theorem 1.2. *Let k be a positive integer with $k \geq 2$. Then there exist positive constants $c_k, c_k^* > 0$ such that for all positive integers n with $n \geq n_0(k)$ the following holds.*

Let X be a set with $|X| = n$ and let $\Delta\colon [X]^k \longrightarrow \omega$ be a coloring, where $\Delta(S) \neq \Delta(T)$ for all sets $S, T \in [X]^k$ with $S \neq T$ and $|S \cap T| \geq 1$. Then there exists a totally multicolored set $Y \subset X$ of size

$$|Y| \geq c_k \cdot (ln\ n)^{\frac{1}{2k-1}} \cdot n^{\frac{k-1}{2k-1}} . \tag{3}$$

Moreover, there exists a coloring $\Delta\colon [X]^k \longrightarrow \omega$, where $\Delta(S) \neq \Delta(T)$ for all different $S, T \in [X]^k$ with $|S \cap T| \geq 1$, such that every totally multicolored subset $Y \subset X$ satisfies

$$|Y| \leq c_k^* \cdot (ln\ n)^{\frac{1}{2k-1}} \cdot n^{\frac{k-1}{2k-1}} . \tag{4}$$

The proof relies heavily on probabilistic arguments. The lower bound is proved is Section 2 and the upper bound is established in Section 3.

2. The proof of the lower bound

Let $\mathcal{F} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. For a vertex $x \in \mathcal{V}$ let $deg_{\mathcal{F}}(x)$ denote the degree of x in $\mathcal{F}$, i.e. the number of edges $e \in \mathcal{E}$ containing x, and let $\Delta(\mathcal{F}) = max\ \{deg_{\mathcal{F}}(x) \mid x \in \mathcal{V}\}$. Our lower bound is based on the following result of Ajtai, Komlós, Pintz, Spencer and Szemerédi:

Theorem 2.1. **[AKPSS 82]** *Let $\mathcal{F}$ be an $(r+1)$-uniform hypergraph on N vertices. Assume that*

(i) *$\mathcal{F}$ is uncrowded (i.e. contains no cycles of length 2, 3 or 4) and*

(ii) *$\Delta(\mathcal{F}) \leq t^r$, where $t \geq t_0(r)$, and*

(iii) *$N \geq N_0(r, t)$,*

then the maximum independent set of $\mathcal{F}$ has size

$$\alpha(\mathcal{F}) \geq \frac{0.98}{e} \cdot 10^{-\frac{5}{r}} \cdot \frac{N}{t} \cdot (ln\ t)^{\frac{1}{r}} \ . \tag{5}$$

We want to apply this Theorem with the parameters $r+1 = 2 \cdot k$ and $t = N^{\delta}$, where $\delta \approx \frac{1}{r \cdot (2r-1)}$. The proof given in [AKPSS 82] suggests that $N_0(r,t)$ should be at least $t^{4r+O(1)}$. Indeed, these calculations can be avoided, as the statement of Theorem (2.1) is strong enough to imply the same statement, where condition (iii) is dropped.

Theorem 2.1' . *Let $\mathcal{G}$ be an $(r+1)$-uniform hypergraph on n vertices. Assume that*

(i) *$\mathcal{G}$ is uncrowded and*

(ii) *$\Delta(\mathcal{G}) \leq t^r$, where $t \geq t_0(r)$,*

then

$$\alpha(G) \geq \frac{0.98}{e} \cdot 10^{-\frac{5}{r}} \cdot \frac{n}{t} \cdot (ln\ t)^{\frac{1}{r}} \ . \tag{6}$$

To see that Theorem (2.1) implies (the seemingly stronger) Theorem (2.1') consider a hypergraph $\mathcal{G}$ on n vertices, which satisfies the assumptions of Theorem (2.1'). Let m be a positive integer with

$$m > \frac{N_0(r,t)}{n}$$

and consider the hypergraph $\mathcal{F}_m$ consisting of m vertex disjoint copies of $\mathcal{G}$. Then $\mathcal{F}_m$ has $N = m \cdot n$ vertices and satisfies the assumptions (i), (ii) and (iii) of Theorem (2.1). We infer that

$$m \cdot \alpha(\mathcal{G}) = \alpha(\mathcal{F}_m) \geq \frac{0.98}{e} \cdot 10^{-\frac{5}{r}} \cdot \frac{N}{t} \cdot (ln\ t)^{\frac{1}{r}}$$

and thus

$$\alpha(\mathcal{G}) \geq \frac{0.98}{e} \cdot 10^{-\frac{5}{r}} \cdot \frac{n}{t} \cdot (ln\ t)^{\frac{1}{r}} \ .$$

We can now prove the lower bound in Theorem (1.2). The basic idea is to define a hypergraph on the set of vertices X whose edges are all unions of pairs of members of $[X]^k$ which have the same color. Then we choose an appropriate random subset of the set of vertices and show that with positive probability the induced hypergraph on this set does not contain too many edges, and contains only a small number of cycles of length 2,3 or 4. We can then delete these cycles and apply Theorem (2.1') to obtain the desired result. The actual calculations are somewhat complicated and are described below. A similar application of Theorem (2.1') is given in [RS 91].

Proof. Let $\Delta\colon [X]^k \longrightarrow \omega$ be a coloring, which satisfies the assumptions in Theorem (1.2). We may assume that $|X| = n$ is divisible by k. This will not change the calculations asymptotically. Put $r + 1 = 2 \cdot k$ and consider an $(r + 1)$-uniform hypergraph $\mathcal{H}$ with vertex set X and with $U \in E(\mathcal{H})$, where $E \subseteq [X]^{r+1}$, being an edge of $\mathcal{H}$ if and only if there exist different k-element subsets $S, T \subset U$ with $\Delta(S) = \Delta(T)$. Indeed, by assumption on the coloring Δ the sets S and T are disjoint. Note that if $Z \subseteq X$ is an independent set of $\mathcal{H}$, then Z is totally multicolored. Our aim is to find a large independent set in $\mathcal{H}$.

As there are at most $\frac{n}{k}$ k-element subsets of the same color, we infer that the number of edges of $\mathcal{H}$ satisfies

$$|E(\mathcal{H})| \leq \frac{\binom{n}{k}}{\frac{n}{k}} \cdot \binom{\frac{n}{k}}{2} \leq \frac{1}{2 \cdot k^2 \cdot (k-1)!} \cdot n^{k+1} \ . \tag{7}$$

For $i = 2, 3, 4$ we will further bound the number μ_i of i-cycles in $\mathcal{H}$.

For distinct vertices x, y of $\mathcal{H}$ let $deg_{\mathcal{H}}(x, y)$ be the number of edges of $\mathcal{H}$ containing both vertices x and y. Let

$$\{x, y\} \subset U \in E(\mathcal{H}) \tag{8}$$

for some set U. Let $S, T \in [X]^k$ be such that $S \cup T = U$ and $\Delta(S) = \Delta(T)$. As S and T are by assumption disjoint sets, we may assume without loss of generality that one of the following possibilities happens:

(i) either $x \in S$ and $y \in T$

(ii) or $\{x, y\} \subset S$.

The number of edges $U \in E(\mathcal{H})$ satisfying (8) and (i) ((8) and (ii)) is bounded from above by $\binom{n-2}{k-1}$ ($(\frac{n}{k} - 1) \cdot \binom{n-2}{k-2}$ respectively).

Summing, we obtain that

$$deg_{\mathcal{H}}(x, y) \leq \binom{n-2}{k-1} + \left(\frac{n}{k} - 1\right) \cdot \binom{n-2}{k-2} < \frac{2 \cdot k - 1}{k!} \cdot n^{k-1} \ . \tag{9}$$

Using (9) one easily sees that

$$\begin{aligned} \mu_3 &\leq c_3 \cdot n^{3k} \\ \mu_4 &\leq c_4 \cdot n^{4k} \ . \end{aligned}$$

We will now discuss the 2-cycles: for $j = 2, 3, \ldots, 2k-1$ let ν_j be the number of (unordered) pairs of edges of $\mathcal{H}$ intersecting in a j-element set. We clearly have $\mu_2 = \sum_{j=2}^{2k-1} \nu_j$.

Set

$$p = n^{-\frac{1}{2} - \frac{1}{4k-1}} \ .$$

Let Y be a random subset of X with vertices chosen independently, each with probability p. Then

$$Prob\ (|Y| \approx p \cdot n) > 0.9\ . \tag{10}$$

For $i = 3, 4$ let $\mu_i(Y)$ be the random variable counting the number of i-cycles no two of whose edges form a two cycle of the subgraph of $\mathcal{H}$ induced on a set Y. Similarily, for $j = 2, 3, \ldots, 2k-1$ let $\nu_j(Y)$ be the random variable counting the number of (unordered) pairs of edges of $\mathcal{H}$ induced on Y, which intersect in j vertices. Let $E(\mu_i(Y))$ and $E(\nu_j(Y))$ be the corresponding expected values.

It is easy to see that

$$E(\mu_i(Y)) \leq p^{(2k-1)\cdot i} \cdot c_i \cdot n^{ki} = o(p \cdot n) \tag{11}$$

for $i = 3, 4$ (and $k \geq 2$).

In order to give an upper bound on $E(\nu_j(Y))$ we will first estimate ν_j.

Fix an edge $S \in E(\mathcal{H}) \subseteq [X]^{2k}$ and fix nonnegative integers j_0 and j_1 with $j_0, j_1 \leq k$. We will count the number of unordered pairs $\{T_0, T_1\}$, which satisfy

$$\Delta(T_0) = \Delta(T_1) \tag{12}$$

and

$$j_0 = |S \cap T_0|\ , \quad j_1 = |S \cap T_1|\ . \tag{13}$$

Assume first that $j_0, j_1 \geq 1$. Fixing the set T_0, there are at most $\left\lfloor \frac{2k-j_0}{j_1} \right\rfloor$ sets T_1 satisfying (12) and (13). If we apply the same argument also when T_1 is fixed we infer that the number of pairs $\{T_0, T_1\}$, which satisfy (12) and (13), is bounded from above by

$$\min\ \left\{ \binom{2k}{j_0} \cdot \left\lfloor \frac{2k-j_0}{j_1} \right\rfloor \cdot \binom{n}{k-j_0}, \binom{2k}{j_1} \cdot \left\lfloor \frac{2k-j_1}{j_0} \right\rfloor \cdot \binom{n}{k-j_1} \right\}\ .$$

Assume now that $j_0 = 0$ and $j_1 \geq 1$. Having fixed the set T_1 there are at most $(\frac{n}{k} - 2)$ sets T_0, which satisfy (12) and (13). Therefore, the number of such pairs $\{T_0, T_1\}$ is bounded from above by

$$\binom{2k}{j_1} \cdot \binom{n}{k-j_1} \cdot \frac{n}{k}\ .$$

Setting $j = j_0 + j_1 \geq 2$ and summing this means, that for every edge $S \in E(\mathcal{H})$ there are at most

$$\begin{aligned} & c_{j,k} \cdot n^{k - \lceil \frac{j}{2} \rceil} + c'_{j,k} \cdot n^{k-j+1} \\ \leq\ & \bar{c}_{j,k} \cdot n^{k - \lceil \frac{j}{2} \rceil} \end{aligned}$$

edges $T \in E(\mathcal{H})$ such that $|S \cap T| = j$, and hence with (7) we have

$$\begin{aligned} \nu_j &\leq \bar{c}_{j,k} \cdot n^{k - \lceil \frac{j}{2} \rceil} \cdot |E(\mathcal{H})| \\ &\leq c^*_{j,k} \cdot n^{2k+1-\lceil \frac{j}{2} \rceil} , \end{aligned}$$

where $c^*_{j,k}$ is a constant depending on j and k only. Thus, for $j = 2, 3, \ldots, 2k-1$ it follows that

$$\begin{aligned} E(\mu_2(Y)) &= \sum_{j=2}^{2k-1} E(\nu_j(Y)) \\ &\leq \sum_{j=2}^{2k-1} p^{4k-j} \cdot c^*_{j,k} \cdot n^{2k+1-\lceil \frac{j}{2} \rceil} \\ &= o(p \cdot n) . \end{aligned} \tag{14}$$

Summarizing (10), (11) and (14) and the fact that

$$E(|E(\mathcal{H}) \cap [Y]^{2k}|) = p^{2k} \cdot |E(\mathcal{H})| , \tag{15}$$

we infer that there exists a subset $Y_0 \subset X$ with $|Y_0| \approx p \cdot n$, $\mu_i(Y_0) = o(p \cdot n)$ and $\nu_j(Y_0) = o(p \cdot n)$ for $i = 3, 4$ and $j = 2, 3, \ldots, 2k-1$ and with $|E(\mathcal{H}) \cap [Y_0]^{2k}| \leq 2 \cdot p^{2k} \cdot |E(\mathcal{H})|$.

We delete from Y_0 all vertices, which are contained in i-cycles of length $i = 2, 3, 4$ to obtain a subset $Y_1 \subseteq Y_0$ with $|Y_1| \approx p \cdot n$, such that the subgraph induced on Y_1 is uncrowded and has at most $2 \cdot p^{2k} \cdot |E(\mathcal{H})|$ edges.

Finally, delete all vertices Y_1 with degree bigger than

$$\frac{8 \cdot k \cdot p^{2k} \cdot |E(\mathcal{H})|}{p \cdot n} .$$

We obtain a subset $Z \subseteq Y_1$ with at least $\frac{p \cdot n}{2} \cdot (1 - o(1))$ vertices such that the subgraph $\mathcal{G}$ of $\mathcal{H}$ induced on the set Z satisfies the assumptions of Theorem (2.1') with

$$\Delta(\mathcal{G}) \leq t^{2k-1} = \frac{8 \cdot k \cdot p^{2k} \cdot |E(\mathcal{H})|}{p \cdot n} \leq \frac{4}{k!} \cdot p^{2k-1} \cdot n^k ,$$

hence,

$$t \leq \left(\frac{4}{k!} \right)^{\frac{1}{2k-1}} \cdot p \cdot n^{\frac{k}{2k-1}} .$$

We apply Theorem (2.1') to the hypergraph $\mathcal{G}$ and obtain

$$\begin{aligned}\alpha(\mathcal{H}) &\geq \alpha(\mathcal{G})\\ &\geq \frac{0.98}{e}\cdot 10^{-\frac{5}{r}}\cdot\frac{|Z|}{t}\cdot(ln\ t)^{\frac{1}{r}}\\ &\geq (1-o(1))\cdot\frac{0.49}{e}\cdot\left(\frac{k!}{4\cdot 10^5}\right)^{\frac{1}{2k-1}}\cdot\left(\frac{1}{(4k-1)\cdot(4k-2)}\right)^{\frac{1}{2k-1}}\cdot n^{\frac{k-1}{2k-1}}\cdot(ln\ n)^{\frac{1}{2k-1}}\ .\end{aligned}$$

This completes the proof of the lower bound. □

3. The proof of the upper bound

Let k, n be positive integers, where n is divisible by k. Let X be a set of vertices with $|X| = n$. A perfect k-matching on X is a collection of $\frac{n}{k}$ pairwise disjoint k-element subsets of X.

The number of perfect k-matchings of an n-element set, n divisible by k, is given by

$$\begin{aligned}&\frac{\binom{n}{k}\cdot\binom{n-k}{k}\cdot\binom{n-2k}{k}\cdot\ldots\cdot\binom{k}{k}}{\frac{n}{k}!}\\ =\ &\frac{n!}{(k!)^{\frac{n}{k}}\cdot\frac{n}{k}!}\ .\end{aligned}$$

In the following we prove the upper bound given in Theorem (1.2). The basic idea is similar to the one used by Babai in [Ba 85], but there are several complications.

Proof. Let X be a set with $|X| = n$, and assume that n is divisible by k. As long as k is fixed, this will not change the calculations asymptotically. Let $m = \lceil cn^{k-1}\rceil$, where $c = \frac{1}{(k-1)!\cdot 8}$. Let $M_1, M_2, \ldots, M_m$ be random perfect k-matchings, chosen uniformly randomly and independently from the set of all perfect k-matchings. Put $H_j = \cup_{i<j} M_i$, hence H_j is the set of all k-element subsets occurring in one of the first $(j-1)$ perfect k-matchings M_i, $i < j$. Define a coloring $\Delta\colon [X]^k \longrightarrow \omega$ in the following way: for $j = 1, 2, \ldots, m$ color the sets in $M_j \setminus H_j$ with color j, and, in order to complete the coloring, color those sets in $[X]^k \setminus H_{m+1}$ with new colors in an arbitrary way, compatible with the assumptions. (E.g., using a different color for each edge).

Now let $Y \subseteq X$ be a subset of X with $|Y| = l$, where $l > n^{\frac{1}{k+1}}$ but $l = o(n^{\frac{k-1}{k}})$. Our objective is to estimate the probability that Y is totally multicolored. Very roughly, this is done as follows. We show that with a high probability Y does not contain too many edges of any single matching, and hence it does not contain too many edges of H_j for every j. If this occurs, then for each j, with a reasonably high probability Y does contain two edges of M_j

that do not lie in H_j (and thus have the same color). This implies that with extremely high probability such two edges exist for some j, and hence Y is not totally multicolored.

The actual proof is rather complicated, and is described in the following sequence of lemmas.

Lemma 3.1. *For all integers j, t, $1 \le j \le m$ and $t \ge 0$, the following inequality holds*

$$\text{Prob}\,(|M_j \cap [Y]^k| \ge t) \le \left(\frac{l^k}{k \cdot n^{k-1}}\right)^t . \tag{16}$$

Proof. The k-matchings M_j above have been chosen at random and the set Y was fixed. In order to give an upper bound for the probability Prob $(|M_j \cap [Y]^k| \ge t)$, view it the other way around: Let M_j be fixed and let Y be chosen at random. This does not change the corresponding probabilities. Now, Y can be chosen in $\binom{n}{l}$ ways. From M_j we can choose t k-element sets in $\binom{\frac{n}{k}}{t}$ ways and the remaining elements of Y in at most $\binom{n-kt}{l-kt}$ ways. This implies

$$\begin{aligned}
\text{Prob}\,(|M_j \cap [Y]^k| \ge t) &\le \frac{\binom{\frac{n}{k}}{t} \cdot \binom{n-kt}{l-kt}}{\binom{n}{l}} \\
&\le \left(\frac{n}{k}\right)^t \cdot \left(\frac{l}{n}\right)^{kt} \\
&= \left(\frac{l^k}{k \cdot n^{k-1}}\right)^t .
\end{aligned}$$

□

Let E_i denote the event

$$|H_i \cap [Y]^k| \le \frac{6c}{k} l^k = \frac{3}{4 \cdot (k!)} \cdot l^k . \tag{17}$$

Notice, that Prob $(E_i) \ge$ Prob (E_{i+1}) for every i, $1 \le i \le m$. The following Lemma gives a lower bound for the probability that E_{m+1} occurs:

Lemma 3.2. *For n large, Prob $(E_{m+1}) \ge 1 - 2^{-\frac{6c}{k} l^k}$.*

Proof. For $i = 1, 2, \ldots, m$ define random variables x_i by $x_i = |M_i \cap [Y]^k|$. Thus $|E_{m+1} \cap [Y]^k| \le \sum_{i=1}^m x_i$. As the k-matchings are chosen independently, the x_i's are independent random variables too. Therefore,

$$\begin{aligned}
\text{Prob}\,(|H_{m+1} \cap [Y]^k| \ge t) &\le \text{Prob}\,(\sum_{i=1}^m x_i \ge t) \\
&\le \sum_{(t_i)_{i=1}^m, t_i \ge 0, \sum t_i = t} \ \prod_{i=1}^m \text{Prob}\,(x_i \ge t_i) . \qquad (18)
\end{aligned}$$

The number of sequences $(t_i)_{i=1}^m$ with $t_i \geq 0$ and $\sum_{i=1}^m t_i = t$ is given by the binomial coefficient $\binom{t+m-1}{t}$. By Lemma 1 we know

$$\text{Prob } (x_i \geq t_i) \leq (\frac{l^k}{k \cdot n^{k-1}})^{t_i} ,$$

hence with (18) we have

$$\begin{aligned} \text{Prob } (|H_{m+1} \cap [Y]^k| \geq t) &\leq \binom{t+m-1}{t} \cdot (\frac{l^k}{k \cdot n^{k-1}})^t \\ &\leq (\frac{e(t+m)}{t})^t \cdot (\frac{l^k}{k \cdot n^{k-1}})^t \\ &\leq (\frac{e(t+m)l^k}{tkn^{k-1}})^t \end{aligned} \tag{19}$$

For $t = \frac{6c}{k} l^k$ and $l = o(n^{\frac{k-1}{k}})$ the quotient $\frac{e(t+m) \cdot l^k}{tkn^{k-1}}$ occurring in (19) is less than 1/2 for n large. Hence

$$\text{Prob } (E_{m+1}) \geq 1 - 2^{-\frac{6c}{k} l^k} .$$

□

Next, define another random variable y_j by

$$y_j = |[M_j]^2 \cap [[Y]^k \setminus H_j]^2| . \tag{20}$$

Clearly, y_j counts the number of those pairs of disjoint k-element sets i $[Y]^k \setminus H_j$ which have both elements in the k-matching M_j (and hence have the same color). Let $E(y_j \mid M)$ denote the conditional expected value of y_j given M.

Lemma 3.3. *For positive integers n, n large,*

$$E(y_j \mid E_j, M_1, M_2, \ldots, M_{j-1}) > \frac{1}{130 \cdot k^2} \cdot \frac{l^{2k}}{n^{2k-2}} . \tag{21}$$

Proof. As E_j holds, (17) implies for n large that

$$|[Y]^k \setminus H_j| \geq \binom{l}{k} - \frac{6c}{k} l^k \geq c' l^k , \tag{22}$$

where, say, $c' = \frac{1}{8 \cdot (k!)}$. For every set $S \in [Y]$ at most $k\binom{l-1}{k-1}$ k-element subsets of Y, which are not disjoint from S, hence the the number of sets $\{S, T\} \in [[Y]^k \setminus H_j]^2$, where S and T are disjoint, is for n large at least

$$\frac{1}{2} c' l^k \cdot (c' l^k - k\binom{l-1}{k-1}) > \frac{1}{130 \cdot (k!)^2} \cdot l^{2k} . \tag{23}$$

Now, for given disjoint k-element sets S, T, the probability that both are in M_j is given by

$$\begin{aligned}
\text{Prob}\,(S, T \in M_j) &= \frac{\frac{n}{k} \cdot (\frac{n}{k} - 1)}{\binom{n}{k} \cdot \binom{n-k}{k}} \\
&= \frac{1}{\binom{n-1}{k-1} \cdot \binom{n-k-1}{k-1}} \\
&\geq \frac{1}{\binom{n-1}{k-1}^2} \\
&\geq (\frac{(k-1)!}{n^{k-1}})^2 .
\end{aligned}$$

Therefore,

$$E(y_j \mid E_j, M_1, M_2, \ldots, M_{j-1}) \quad \geq \quad \frac{1}{130 \cdot k^2} \cdot \frac{l^{2k}}{n^{2k-2}} .$$

$\square$

Lemma 3.4. *For every positive integer j, $j \leq m$, and n large*

$$\textit{Prob}\,(y_j = 1 \mid E_j, M_1, M_2, \ldots, M_{j-1}) > \frac{1}{131 \cdot k^2} \cdot \frac{l^{2k}}{n^{2k-2}} . \tag{24}$$

Proof. First we claim that

$$\text{Prob}\,(y_j \geq t \mid E_j, M_1, M_2, \ldots, M_{j-1}) \leq (\frac{l^k}{k \cdot n^{k-1}})^{\lceil \sqrt{2t+1}\, \rceil} \tag{25}$$

for every positive integer t.

This follows from the fact that t pairwise different two-element sets imply that the underlying set has cardinality at least $\lceil \sqrt{2t+1}\, \rceil$. Hence, $y_j \geq t$ implies $|M_j \cap [Y]^k| \geq \lceil \sqrt{2t+1} \rceil$. By Lemma 1 this gives

$$\begin{aligned}
\text{Prob}\,(y_j \geq t \mid E_j, M_1, M_2, \ldots, M_{j-1}) &\leq \text{Prob}\,(|M_j \cap [Y]^k| \geq \lceil \sqrt{2t+1}\, \rceil) \\
&\leq (\frac{l^k}{k \cdot n^{k-1}})^{\lceil \sqrt{2t+1}\, \rceil} ,
\end{aligned}$$

proving (25).

For every positive integer i put $p_i = \text{Prob}\,(y_j = i \mid E_j, M_1, M_2, \ldots, M_{j-1})$. Clearly, we have $E(y_j \mid E_j, M_1, M_2, \ldots, M_{j-1}) = \sum_{i<\omega} i \cdot p_i$ and by (25) this implies

$$\begin{aligned}
E(y_j \mid E_j, M_1, M_2, \ldots, M_{j-1}) &\leq p_1 + \sum_{i \geq 2} i \cdot (\frac{l^k}{k \cdot n^{k-1}})^{\lceil \sqrt{2i+1} \rceil} \\
&\leq p_1 + O((\frac{l^k}{n^{k-1}})^3) \\
&\leq p_1 + o\left(\frac{l^{2k}}{n^{2k-2}}\right) .
\end{aligned} \tag{26}$$

The last inequality (26) follows from the fact that $l = o(n^{\frac{k-1}{k}})$. By Lemma 3 this implies that, say, $p_1 > \frac{1}{131 \cdot k^2} \cdot \frac{l^{2k}}{n^{2k-2}}$ for n large. $\square$

Let F_j denote the event ($y_j = 0$ and E_{j+1}). Then $F_1 \wedge F_2 \wedge \ldots \wedge F_{j-1}$ is the event that E_j and $y_i = 0$ for $i = 1, 2, \ldots, j-1$.

Let $\mathcal{M}$ be the set of all mutually exclusive events $(E_j, M_1, M_2, \ldots, M_{j-1})$ for which $y_i = 0$, $i = 1, 2, \ldots, j-1$, holds. Then clearly

$$F_1 \wedge F_2 \wedge \ldots \wedge F_{j-1} = \bigvee_{\mathcal{M}} (E_j, M_1, M_2, \ldots, M_{j-1})$$

and thus

$$\begin{aligned}
& \text{Prob } (y_j = 1 \mid F_1 \wedge F_2 \wedge \ldots \wedge F_{j-1}) \\
= \; & \frac{\sum_{\mathcal{M}} \text{ Prob } (y_j = 1 \; \wedge \; (E_j, M_1 \wedge M_2 \wedge \ldots \wedge M_{j-1}))}{\sum_{\mathcal{M}} \text{ Prob } (E_j, M_1, M_2, \ldots, M_{j-1})} \\
\geq \; & \min_{\mathcal{M}} \frac{\text{Prob } (y_j = 1 \; \wedge (E_j, M_1, M_2, \ldots, M_{j-1}))}{\text{Prob } (E_j, M_1, M_2, \ldots, M_{j-1})} \\
= \; & \min_{\mathcal{M}} \text{Prob}\, (y_j = 1 \mid E_j, M_1, M_2, \ldots, M_{j-1}) \; .
\end{aligned}$$

Hence, by Lemma 4 we infer that

$$\text{Prob } (y_j = 1 \mid F_1 \wedge F_2 \wedge \ldots \wedge F_{j-1}) > \frac{1}{131 \cdot k^2} \cdot \frac{l^{2k}}{n^{2k-2}} \; . \tag{27}$$

Now we are ready to prove

Lemma 3.5. *For every positive integer n, n large,*

$$\textit{Prob } (F_1 \wedge F_2 \wedge \ldots \wedge F_m) < \textit{exp}(-\frac{1}{1048 \cdot k \cdot (k!)} \cdot \frac{l^{2k}}{n^{k-1}}) \; . \tag{28}$$

Proof. As

$$\text{Prob } (F_1 \wedge F_2 \wedge \ldots \wedge F_m) = \text{Prob } (F_1) \cdot \prod_{j=2}^{m} \text{Prob } (F_j \mid F_1 \wedge F_2 \wedge \ldots \wedge F_{j-1}) \; , \tag{29}$$

we infer that

$$\begin{aligned}
\text{Prob } (F_j \,|F_1 \wedge F_2 \wedge \ldots \wedge F_{j-1})) \;\; & \leq \;\; \text{Prob } (y_j = 0 \mid F_1 \wedge F_2 \wedge \ldots \wedge F_{j-1}) \\
& \leq \;\; \text{Prob } (y_j \neq 1 \mid F_1 \wedge F_2 \wedge \ldots \wedge F_{j-1}) \\
& = \;\; 1 - \text{Prob } (y_j = 1 \mid F_1 \wedge F_2 \wedge \ldots \wedge F_{j-1}) \\
& < \;\; 1 - \frac{1}{131 \cdot k^2} \cdot \frac{l^{2k}}{n^{2k-2}} \qquad\qquad \text{by (27).}
\end{aligned}$$

With (29) it follows that

$$\begin{aligned}\text{Prob }(F_1 \wedge F_2 \wedge \ldots \wedge F_m) &< (1 - \frac{1}{131 \cdot k^2} \cdot \frac{l^{2k}}{n^{2k-2}})^m \\ &\leq exp(-\frac{1}{131 \cdot k^2} \cdot m \cdot \frac{l^{2k}}{n^{2k-2}}) \\ &\leq exp(-\frac{1}{1048 \cdot k \cdot (k!)} \cdot \frac{l^{2k}}{n^{k-1}}) \qquad \text{as } m = \left\lceil \frac{1}{8 \cdot ((k-1)!)} \cdot n^{k-1} \right\rceil .\end{aligned}$$

□

We finish the proof of Theorem 2 by bounding the probability that there exists a totally multicolored subset $Y \subseteq X$ with $|Y| = l$. If a fixed subset $Y \subseteq X$ is totally multicolored, then either $F_1 \wedge F_2 \wedge \ldots \wedge F_m$ is true or for some j, $1 \leq j \leq m$, the event E_{j+1} does not occur and therefore by (17) also not the event E_{m+1}. With Lemma 2 and Lemma 5 we infer that

$$\text{Prob }(Y \text{ is totally multicolored }) < 2^{-\frac{6c}{k} l^k} + exp(-\frac{1}{1048 \cdot k \cdot (k!)} \cdot \frac{l^{2k}}{n^{k-1}}) ,$$

and doing this for all $\binom{n}{l}$ l-element subsets $Y \subseteq X$ implies

$$\begin{aligned}&\text{Prob (there exists } Y \subseteq X \text{ with } |Y| = l \text{ and } Y \text{ is totally multicolored)} \\ &< \binom{n}{l} \cdot (exp(-\frac{1}{1048 \cdot k \cdot (k!)} \cdot \frac{l^{2k}}{n^{k-1}}) + 2^{-\frac{6c}{k} l^k}) . \qquad (30)\end{aligned}$$

For $l \geq (1049 \cdot k \cdot (k!))^{\frac{1}{2k-1}} \cdot n^{\frac{k-1}{2k-1}} \cdot (ln\ n)^{\frac{1}{2k-1}}$ (where the constant can be easily improved) expression (30) goes to 0 with n going to infinity. Thus for $|X| = n$ and n large there exists a coloring $\Delta \colon [X]^k \longrightarrow \omega$ such that every totally multicolored subset $Y \subseteq X$ has size $|Y| \leq (1049 \cdot k \cdot (k!))^{\frac{1}{2k-1}} \cdot n^{\frac{k-1}{2k-1}} \cdot (ln\ n)^{\frac{1}{2k-1}}$. □

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