Retrospective Optimization of a Two-location Inventory Model with Lateral Transshipments

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Abstract

At present various methods for the simulation optimization of stochastic systems are developed. One of them is the so-called retrospective approach. The simple basic idea is to simulate (or to observe) some outcomes of the stochastic system under investigation and then to deterministically optimize over these outcomes.
In our paper we apply this idea to a two-location inventory model with lateral transshipments. Such a model can serve as a tool to allocate a divisible resource to a given number of locations with random demand and to redistribute not yet used resources between the locations. These are important control problems in organizing effective transportation systems. We propose an algorithm to find the optimal solution. An example shows the practicability of the developed algorithm.

Zusammenfassung

Gegenwärtig sind verschiedene Methoden der simulationsbasierten Optimierung stochastischer Systeme entwickelt. Eine davon ist die sogenannte retrospektive Methode. Deren einfache Grundidee ist es, einige Realisierungen des zu untersuchenden stochstischen Systems zu simulieren (oder zu beobachten) und danach bezüglich dieser Realisierungen eine deterministische Optimierung durchzuführen.
In der vorliegenden Arbeit wird diese Idee auf ein Zwei-Lager-Modell mit Transport-beziehungen angewandt. Ein derartiges Modell kann als Werkzeug dienen, um eine teilbare Ressource auf eine gegebene Anzahl Standorte mit zufälligem Bedarf aufzuteilen und nicht benötigte Ressourcen umzuverteilen. Das sind wichtige Steuerprobleme im Zusammenhang mit der Organisation effektiver Transportsysteme. Es wird ein Algorithmus vorgeschlagen, um eine optimale Lösung zu finden. Ein Beispiel zeigt die Anwendbarkeit des entwickelten Algorithmus.

1. Introduction

The allocation of a divisible resource to a given number of locations with random demand and the redistribution of not yet used resources between the locations are important control problems in organizing effective transportation systems. Such a resource for instance may be a fleet of vehicles, a set of reusable containers and others. In [7] such problems are modelled as a multi-location inventory model. An analytical solution could be derived in the case of full information about the random nature of the demand, for example distribution function or distribution parameters. In the present paper we apply the so-called retrospective approach to a two-location model. This approach allows to compute at least an approximate solution without any information about the random nature of the demand.

The paper is organized as follows. In the next section we describe the retrospective approach and its application to the classical single-location inventory problem. In Section 3 we present the multi-location inventory model with redistribution. We summarize important properties and equations necessary for the retrospective approach. The application of the retrospective approach to the 2-location model together with an example is given in Section 4. The concluding section contains some remarks on the applicability of the proposed approach and on directions for future research.

2. The retrospective approach

Let us consider a stochastic system that is to be controlled during a given planning horizon. The planning horizon is divided into T time periods. For each period t = 1(1)T we introduce the following notations [1]:

- $\mathbf{X}(t)$ - the state vector of the stochastic system;
- $\underline{\mathbf{D}}(t)$ - the random vector which describes the uncertainties of the considered stochastic system;
- g_t - a real valued cost function.

Further, let Θ denote a given set of control parameter vectors $\theta \in \Theta$. Under some less restrictive assumptions (see e.g. [8]) the expected average cost function

$$G(\theta) := 1/T \cdot E\left[\sum_{t=1}^{T} g_t(\underline{\mathbf{X}}(t), \underline{\mathbf{D}}(t), \theta) \right] \tag{2.1}$$

is well defined for each $\theta \in \Theta$, and it exists an optimal parameter vector

$$\theta^* := \arg \min_{\theta \in \Theta} G(\theta). \tag{2.2}$$

Often the cost function in (2.1) is not given in an analytical form or can not be computed with a reasonable expense. Then some approximation methods should be used. One of them is the retrospective approach that usually is known as a technique for optimization of stochastic systems via simulation (cp. [1], [2]). The underlying idea is very simple and can be described by the following three steps:

(i) Generate a sample $\mathcal{D} = \{ \mathbf{D}(1), \mathbf{D}(2), ..., \mathbf{D}(T) \}$ of T realizations of the uncertainties of the system (else by simulation or by observation of the stochastic system).

(ii) For given $\theta \in \Theta$ compute the corresponding average cost

T

[1] Bold characters denote sets or vectors, and random vectors are underlined.

$$G_T(\theta, \mathcal{D}) \;:=\; 1 / T \cdot \sum_{t=1} g_t(\, \mathbf{X}(t), \mathbf{D}(t), \theta\,) \tag{2.3}$$

and the minimizer

$$\theta^T \;:=\; \arg \min_{\theta \in \Theta} G_T(\theta, \mathcal{D}). \tag{2.4}$$

(iii) Take θ^T as an approximation for the optimal solution θ^*, and $G_T(\theta^T, \mathcal{D})$ as an approximation for the minimum of the expected average costs $G(\theta^*)$.

We demonstrate now the application of the retrospective approach to the classical single location single product inventory problem (see e.g. [9]). The decision problem is to find such order decisions which minimize the expected average costs over the planning horizon of length T. Thereby we assume:

1. Ordering is possible only at the beginning of a period with zero lead time.
2. During period t a random demand $\underline{D}(t)$ occurs, where $\{\underline{D}(t), t=1(1)T\}$ is a sequence of independent and identically distributed random variables with distribution function F and expectation $m = E[\underline{D}(t)]$.
3. At the end of a period cost incur, holding cost $h > 0$ per unit of not required product or shortage cost $p > 0$ per unsatisfied demand unit. Unsatisfied demand is backlogged.

For the above assumed cost structure the optimal inventory policy is an „order-up-to θ" policy, i.e., order up to θ for given constant $\theta > 0$ (see [9]). Furthermore, the optimal order-up-to level θ^* can be computed from the equation

$$F(\theta^*) = p / (h + p). \tag{2.5}$$

To solve equ. (2.5) we need the distribution function F of the demand. In the case that we do not have that information we can apply the retrospective approach.

Step (i):
Let us assume that we have observed T realizations D(1) to D(T) of the demand.

Step (ii):
For given order-up-to level $\theta > 0$ the cost incurred in period t are

$$g_t(X(t), D(t), \theta) = g_t(\, \theta - D(t)\,) = h \cdot \max[0; \theta - D(t)] + p \cdot \max[0; D(t) - \theta]$$

or

$$g_t(\, \theta - D(t)\,) = (h + p) \cdot \max[D(t), \theta] - h \cdot D(t) - p \cdot \theta. \tag{2.6}$$

Consequently, we have

$$G_T(\theta, \mathcal{D}) = \; 1 / T \;\cdot \{ \,(h + p)\, \sum_{t=1}^{T} \max[\, D(t), \theta\,] - h \sum_{t=1}^{T} D(t) - p \cdot \theta \cdot T \;\}. \tag{2.7}$$

To define the minimizer θ^T we compute the derivative $dG_T/d\theta$. From equ. (2.7) we get $dG_T/d\theta = 1 / T \cdot [\,(h + p) \cdot T(\theta) - p \cdot T\,]$, where $T(\theta)$ denotes the number of demand realizations $D(t) \le \theta$, i.e., the number of periods with no excess demand. However, the necessary optimality condition $dG_T/d\theta = 0$ or equivalently $T(\theta^T) / T = p / (h + p)$ does not have a solution in general. The reason is that $T(\theta)$ is an integer. To overcome this problem we use an important property of function G_T defined in (2.7). It is evident to see from equ. (2.6) that for given D(t) function g_t is a picewise linear and convex function of θ. The same property holds for G_T. Therefore, we have a simple procedure to minimize G_T (see Fig.1):

a) For the demand realizations D(t); t=1(1)T, we define the ordered sample

$$0 < d(1) \le d(2) \le ... \le d(T). \qquad (2.8)$$

b) The optimal order-up-to level is

$$\theta^T = d(t^T) \qquad \text{with} \qquad t^T = \min\{\, t=1(1)T:\; t / T > p / (h + p) \,\}. \qquad (2.9)$$

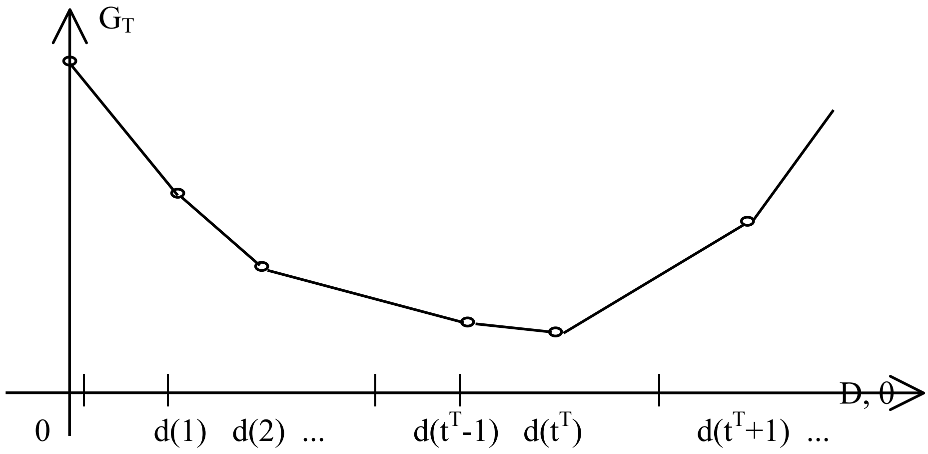


Figure 1. A typical graph for function $G_T(\,\cdot\, , \mathcal{D}\,)$.

We remark that the optimal rank number t^T is independent of the demand realizations. The optimal order-up-to level θ^T of cause depends on that realizations. Obviously it holds that $dG_T/d\theta = - p < 0$ for $\theta < d(1)$, and $dG_T/d\theta = h > 0$ for $\theta > d(T)$.

Step (iii):
To form an idea of the approximation quality we consider an example.

Example 2.1.

We assume exponentially distributed demand with an average value of 200 units, and we assume cost factors h = 1 and p = 10. For comparison we compute at first the exact solution. From equ. (2.5) we get $1 - \exp(- \theta^*/200) = p / (h+p)$ or $\theta^* = - 200 \ln(1/11) = 479,58 \approx 480$. Since $G(\theta^*) = h \cdot \theta^*$ we have $G(\theta^*) = 479,58 \approx 480$.
Now we apply the retrospective approach. We have simulated 100 realizations of the demand. The corresponding values for the first 10 periods (T = 10) and the ordered sample as well are given in Table 1.

D(1)	D(2)	D(3)	D(4)	D(5)	D(6)	D(7)	D(8)	D(9)	D(10)
279	79	297	173	157	76	137	342	149	11
d(1)	d(2)	d(3)	d(4)	d(5)	d(6)	d(7)	d(8)	d(9)	d(10)
11	76	79	137	149	157	173	279	297	342

Table 1. Data for the first 10 demand realizations of Example 2.1.

For these data it follows from formula (2.9) that

$$t^{10} = \min\{\, t = 1(1)10:\; t > 100 / 11 \,\} = 10 \text{ and } \theta^{10} = 342.$$

Furthermore, from equ. (2.7) we compute

$$G_{10}(\,342) = 1 / 10 \cdot \{\, (1 + 10) \cdot 10 \cdot 342 - 1 \cdot 1700 - 10 \cdot 342 \cdot 10 \,\} = 172.$$

The underestimation of the optimal order-up-to level as well as the minimal average costs is the consequence of a very small sample size and of the fact that the sample mean of the demand, which equals 170 in our example, has a great deviation from the true average demand. Since we can expect an improving approximation with increasing sample size we consider the next 20 demand realizations given in Table 2.

D(11)	D(12)	D(13)	D(14)	D(15)	D(16)	D(17)	D(18)	D(19)	D(20)
68	644	285	1104	311	23	214	13	513	411
D(21)	**D(22)**	**D(23)**	**D(24)**	**D(25)**	**D(26)**	**D(27)**	**D(28)**	**D(29)**	**D(30)**
198	106	273	105	374	61	64	26	429	244

Table 2. Data for the second and third 10 demand realizations of Example 2.1.

The corresponding results are:

$t^{20} = 19,\quad \theta^{20} = 644,\quad G_{20}(644) = 632,7$, and

$t^{30} = 28,\quad \theta^{30} = 513,\quad G_{30}(513) = 538,9$.

We summarize the results for all 100 demand realizations in Table 3 (μ^T denotes the sample average).

T	10	20	30	40	50	60	70	80	90	100
μ^T	170,0	358,6	188,0	200,0	206,0	192,0	198,2	196,9	199,3	193,7
t^T	10	19	28	37	46	55	64	73	82	91
θ^T	342	644	513	429	513	5 13	513	493	493	471
$G_T(\theta^T)$	172,0	632,7	538,9	496,8	522,4	512,8	498,2	456,4	464,2	468,1

Table 3. Results for Example 2.1.

From Table 3 we can see that the retrospective approach yields a reasonable accuracy for the simple problem of Example 2.1. Analogous results are observed for the more complex multi-location system (cp. Section 4).

3. The two-location model: definition and basic results

In the present section we define a two-location inventory model with transshipments. Also we summarize some important results from [4] and [6] which are necessary for our further investigations.

The two-location model with transshipments is a generalization of the single location model of Section 2. With respect to the latter one we have to change the following things:

a) The location parameters get an index, e.g., h_1, p_2 or m_1.
b) The demand in period t is now a random vector $\underline{\mathbf{D}}(t) = (\underline{D}_1(t), \underline{D}_2(t))$.
c) After realization of the demand it is possible to redistribute the on-hand inventory by a transshipment decision (TD) at the end of a period.

d) Transshipments occur immediately by cost $c_{ij} \geq 0$ for one unit transported from location i to location j. Demand still unsatisfied after the TD will be backlogged.

e) The problem is to find such a sequence of order decisions (OD´s) and TD´s that minimizes the expected average costs.

To solve this optimization problem we have to formalize the TD. Obviously, a TD is a transportation plan $\mathbf{b} = (b_{ij})_{i,j=1,2}$, where b_{ij} denotes the amount of product transferred from location i to location j. Let $\mathbf{Y} = \theta - \mathbf{D}$ denote the vector of inventory positions before the TD in the case if the OD was chosen in accordance with the order-up-to level θ and if demand realization **D** was observed. Then the set **B**(**Y**) of all admissible TD´s is given as

$$\mathbf{B}(\mathbf{Y}) = \{ \mathbf{b} = (b_{ij})_{i,j=1,2} : \ b_{ij} \geq 0 \ \text{ and } \ \sum_{i=1}^{2} b_{ij} = (Y_i)^+ , \ i, j = 1, 2 \},$$

where $(x)^+ = \max(0; x)$ for a real x. For each period t, OD θ, demand realization **D**, and TD $\mathbf{b} \in \mathbf{B}(\theta - \mathbf{D})$ the corresponding costs are given by function

$$G_t(\theta, \mathbf{D}, \mathbf{b}) = \sum_{i=1}^{2} [\, h_i \ (\theta_i - D_i - b_{i3-i})^+ + p_i \ (D_i - \theta_i - b_{3-ii})^+ + c_{i3-i}\, b_{i3-i} \,] \tag{3.1}$$

The function

$$G_t(\theta, \mathbf{D}) \quad = \quad \min_{\mathbf{b} \in \mathbf{B}(\theta - \mathbf{D})} G_t(\theta, \mathbf{D}, \mathbf{b}) \tag{3.2}$$

represents the costs for θ and **D** under optimal transshipments.

Now we want to characterize the optimal TD. To this end we introduce two additional conditions on the cost parameters of the model. The first condition follows from the natural economic demand that lateral transshipments from a location with not required product to a location with excess demand should be efficient. This leads to the condition „**E**fficiency of **T**ransshipments“

$$\text{(ET)} \qquad\qquad h_i + p_j > c_{ij} \quad , \ i, j = 1, 2.$$

On the other hand, transshipments between locations without excess demand should be inefficient. Thus we assume the condition „**R**elative **I**ndependence“ of the locations, i.e.,

$$\text{(RI)} \qquad\qquad c_{ij} + h_j > h_i \quad , \ i, j = 1, 2.$$

Finally, to describe the optimal TD we define for given $\theta=(\theta_1, \theta_2)$ the following sets (cp. Fig.2):

$$S_1(\theta) = \{ \mathbf{D} = (D_1, D_2): \ D_1 \leq \theta_1 ; D_2 \leq \theta_2 \},$$

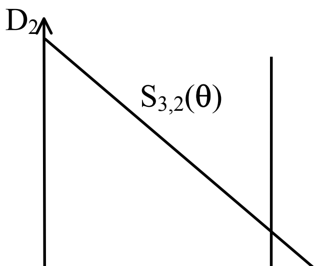
$$S_2(\theta) = \{ \mathbf{D} = (D_1, D_2): \ D_1 \geq \theta_1 ; D_2 \geq \theta_2 \} \setminus \{ \mathbf{D} = \theta \},$$

$$S_{3,1}(\theta) = \{ \mathbf{D} = (D_1, D_2): \ D_1 < \theta_1 ; D_2 > \theta_2 ; D_1 + D_2 \leq \theta_1 + \theta_2\},$$

$$S_{3,2}(\theta) = \{ \mathbf{D} = (D_1, D_2): \ D_1 < \theta_1 ; D_2 > \theta_2 ; D_1 + D_2 > \theta_1 + \theta_2\},$$

$$S_{4,1}(\theta) = \{ \mathbf{D} = (D_1, D_2): \ D_1 > \theta_1 ; D_2 < \theta_2 ; D_1 + D_2 \leq \theta_1 + \theta_2\}, \text{ and}$$

$$S_{4,2}(\theta) = \{ \mathbf{D} = (D_1, D_2): \ D_1 > \theta_1 ; D_2 < \theta_2 ; D_1 + D_2 > \theta_1 + \theta_2\}.$$



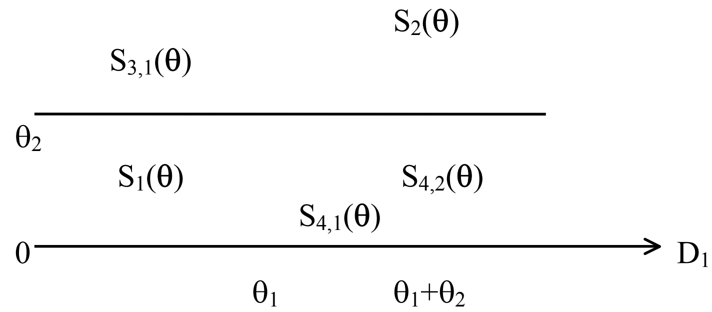


Figure 2. Draft for the arrangement of sets with different TD.

Using now Satz 4.1 in [4] we get the following

Lemma 3.1.
If for the considered two-location model with transshipments are fulfilled the conditions (ET) and (RI) then it holds

$$b^*_{12} = \begin{cases} D_2 - \theta_2 & for \quad D \in S_{3,1}(\theta), \\ \theta_1 - D_1 & for \quad D \in S_{3,2}(\theta), \\ 0 & else, \end{cases}$$

and

$$b^*_{21} = \begin{cases} D_1 - \theta_1 & for \quad D \in S_{4,1}(\theta), \\ \theta_2 - D_2 & for \quad D \in S_{4,2}(\theta), \\ 0 & else. \end{cases}$$

In the light of Lemma 3.1 we see that the above defined sets represent sets of demand realizations that lead to TD´s of identical type. For instance, we have no transshipments for $\mathbf{D} \in S_1(\theta) \cup S_2(\theta)$. In the case $\mathbf{D} \in S_{3,2}(\theta)$ we transfer all inventory from location 1 to location 2. From Lemma 3.1 we get for function $G_t(\theta, \mathbf{D})$, given in equ. (3.2), the following explicit form:

$$G_t(\theta, D) = \begin{cases} h_1 (\theta_1 - D_1) + h_2 (\theta_2 - D_2) & for \quad D \in S_1(\theta), \\ p_1 (D_1 - \theta_1) + p_2 (D_2 - \theta_2) & for \quad D \in S_2(\theta), \\ h_1(\theta_1 + \theta_2 - D_1 - D_2) + c_{12} (D_2 - \theta_2) & for \quad D \in S_{3,1}(\theta), \\ p_2(D_1 + D_2 - \theta_1 - \theta_2) + c_{12} (\theta_1 - D_1) & for \quad D \in S_{3,2}(\theta), \\ h_2(\theta_1 + \theta_2 - D_1 - D_2) + c_{21} (D_1 - \theta_1) & for \quad D \in S_{4,1}(\theta), \\ p_1(D_1 + D_2 - \theta_1 - \theta_2) + c_{21} (\theta_2 - D_2) & for \quad D \in S_{4,2}(\theta). \end{cases} \tag{3.3}$$

This explicit form for $G_t(\theta, \mathbf{D})$ allows us to compute for given OD $\theta \geq (0, 0)$ the expected cost in period t as

$$G_t(\theta) = \int_{\{\mathbf{D} \geq (0,0)\}} G_t(\theta, \mathbf{D}) \; dF(\mathbf{D}) = \sum_{i=1}^{2} L_i(\theta_i) - C(\theta) \tag{3.4}$$

with

$$L_i(\theta) = (h_i + p_i) \int_0^{\theta} F_i(x) \, dx + p_i (m_i - \theta), \qquad \theta \geq 0, \; i = 1, 2, \tag{3.5}$$

where $F_i(\cdot)$ denotes the marginal distribution function of $\underline{D}_i(t)$, i=1, 2. Function $L_i(\theta)$ represents the expected costs for location i without transshipments and order-up-to level θ, i=1, 2. For function $C(\cdot)$, which describes the maximal expected gain from transshipments,we have

$$C(\theta) = (h_1+p_2-c_{12}) \int_0^{\theta_1} F_1(x)\,[1-F_2(\theta_1+\theta_2-x)]\,dx + (h_2+p_1-c_{21}) \int_0^{\theta_2} F_2(x)\,[1-F_1(\theta_1+\theta_2-x)]\,dx. \quad (3.6)$$

Thus equ. (3.4) has a simple interpretation: The expected costs $G_t(\theta_1,\theta_2)$ are equal to the expected costs for independent locations minus the expected gain from transshipments.

In [5] it is shown that $G_t(\cdot, \cdot)$ is a convex function of $\boldsymbol{\theta}$ and $\mathbf{D}$ and consequently $G_t(\cdot)$ is a convex function of $\boldsymbol{\theta}$. Thus it holds (cp. [6])

Theorem 3.1

Let for the two-location model with transshipments hold the conditions (ET) and (RI). Then:

(1) The optimal OD is of the „order-up-to“ type.
(2) The optimal order-up-to point $\boldsymbol{\theta}^* = (\theta_1, \theta_2) > (0, 0)$ is the unique minimum of function $G_t(\cdot)$.
(3) For any vector $\mathbf{Y}(0) = (Y_1(0), Y_2(0))$ of starting inventory positions with $\mathbf{Y}(0) \le \boldsymbol{\theta}^*$ the minimal expected average costs $G(\boldsymbol{\theta}^*)$ are equal to $G_t(\boldsymbol{\theta}^*)$.

Let us finish the present section with an example.

Example 3.1

Let the demands in the two locations be independent and exponentially distributed. For location 1 we assume the data from Example 2.1. All the other necessary data are given in Table 4.

i	h_i	p_i	c_{i1}	c_{i2}	m_i
1	1	10	0	5	200
2	2	9	4	0	300

Table 4. Data for Example 3.1.

For independent locations without transshipments we compute from (2.5) and (3.5) the following solution:

$\boldsymbol{\theta}$’ = (479,6; 511,4) with $G(\boldsymbol{\theta}’)$ = 1 502, 46.

In the case of transshipments we get

$\boldsymbol{\theta}^*$ = (476 ; 448) with $G(\boldsymbol{\theta}^*)$ = 1 244, 03.

We see that the cooperation of the two locations has a twofold consequence - with 92,2 % of the recourses of the independent locations the cooperating locations produce only 82, 8 % of the costs.

4. The two-location model: retrospective approach

In the present section we will apply the retrospective approach to the two-location inventory model with transshipments. The analytic approach of Section 3 is not applicable if, for instance, we have a complicated distribution function for the demand or if we have no full information on the demand. Throughout this section we assume that we have a possibility to get a number of demand realizations, else by observing the system or by simulation. That is the necessary information for the retrospective approach. From Theorem 3.1 we know that a stationary order policy of the „order-up-to“ type is optimal for the considered situation. We denote such a policy by the order-up-to level $\theta = (\theta_1, \theta_2)$. Finally, we assume that the starting inventory position does not exceed θ. The consequence of this assumption is that in each period the order policy θ is applicable. Integer T denotes the sample size as well as the number of periods.

We rewrite equ.(2.3) now as

$$G_T(\theta, \mathscr{D}) = 1 / T \cdot \sum_{t=1}^{T} g_t(\theta, \mathbf{D}(t), \mathbf{b}^*), \tag{4.1}$$

where $\mathbf{b}^*$ denotes the optimal TD (see Lemma 3.1) and function g_t replaces function G_t from (3.1), i.e.,

$$g_t(\theta, \mathbf{D}(t), \mathbf{b}^*) = \sum_{i=1}^{2} [\, h_i\ (\theta_i - D_i - b^*_{i3-i})^+ + p_i\ (D_i - \theta_i - b_{3-ii})^+ + c_{i3-i}\, b^*_{i3-i}\ ]. \tag{4.2}$$

If we replace b^*_{i3-i} in (4.2) by the expression given in Lemma 3.1, we derive for the function $g_t(\theta, \mathbf{D}(t)) := g_t(\theta, \mathbf{D}(t), \mathbf{b}^*)$ an analogous form as for G_t in equ.(3.3). Furthermore, from equ.(4.2) it is evident that g_t and consequently G_T as well is a picewise linear and convex function of θ_1 and θ_2. To minimize such functions in the multi-dimensional case we have to overcome analogous problems as in the single-location case of Section 2. The mathematical background for the minimization problem of picewise linear, convex functions can be found for instance in [8]. We mention only the main problem that the partial derivatives do not exist for all points. Thus we omit these things and concentrate on the description of a simple algorithm. In addition to it we point out to some facts we have to take into account. Nevertheless, we need for function G_T the partial derivatives with respect to θ_1 and θ_2. For that purpose we introduce for given θ the following notations:

$I\{\mathbf{S}\}$ - indicator function for a set $\mathbf{S}$,

$I_t(k) := I\{\mathbf{D}(t) \in \mathbf{S}_k(\theta)\}$, $k = 1, 2$, $I_t(k, j) := I\{\mathbf{D}(t) \in \mathbf{S}_{k,j}(\theta)\}$, $k = 3, 4$, $j = 1, 2$, $t = 1(1)T$,

$N_T(k) := \sum_{t=1}^{T} I_t(k)$, $k = 1, 2$, and $N_T(k, j) := \sum_{t=1}^{T} I_t(k, j)$, $k = 3, 4$, $j = 1, 2$.

We remark that the integer variables N_T counts the number of realizations (or periods) for which the demand belongs to the corresponding set in Fig.2.

With these notations we can develop a formula for the derivatives $\delta g_t/\delta\theta_i$, i=1, 2. From (3.3) follows immediately that

$$\delta g_t/\delta\theta_1 = h_1 \cdot [I_t(1) + I_t(3, 1)] - p_1 \cdot [I_t(2) + I_t(4, 2)] + (h_2 - c_{21}) \cdot I_t(4, 1) - (p_2 - c_{12}) \cdot I_t(3,2) \tag{4.3}$$

and

$$\delta g_t/\delta\theta_2 = h_2 \cdot [I_t(1) + I_t(4, 1)] - p_2 \cdot [I_t(2) + I_t(3, 2)] + (h_1 - c_{12}) \cdot I_t(3, 1) - (p_1 - c_{21}) \cdot I_t(4,2). \tag{4.4}$$

With (4.1), (4.3) and (4.4) we derive the final result

$\delta G_T/\delta\theta_1=1/T\{h_1\cdot[N_T(1)+N_T(3,1)]-p_1\cdot[N_T(2)+N_T(4,2)]+(h_2-c_{21})\cdot N_T(4,1)-(p_2-c_{12})\cdot N_T(3,2)\}$ (4.5)
respectively
$\delta G_T/\delta\theta_2=1/T\{h_2\cdot[N_T(1)+N_T(4,1)]-p_2\cdot[N_T(2)+N_T(3,2)]+(h_1-c_{12})\cdot N_T(4,1)-(p_1-c_{21})\cdot N_T(4,2)\}$.(4.6)

The basic requirement to apply the retrospective approach is the existence of a given number T of demand realizations $\mathcal{D}_T = \{\mathbf{D}(1), \mathbf{D}(2), ..., \mathbf{D}(T)\}$. From this starting information we define for each location the ordered sample $\{d_i(t), t=1(1)T\}$ with $0 < d_i(1) \le d_i(2) \le ... \le d_i(T)$, $i=1, 2$. The search for the optimal OD θ^T we start with a coordinate-wise passing of points $\theta = (d_1(t'), d_2(t''))$, $t', t''=1(1)T$. In these points the corresponding partial derivatives $\delta G_T/\delta\theta_i$, may change their values. Obviously, there is no need to consider OD´s $\theta< (d_1(1), d_2(1))$ as well as $\theta \ge (d_1(T), d_2(T))$. In the first case it holds $N_T(2) = T$ and with equ. (4.5) and (4.6) we have $\delta G_T/\delta\theta_i = - p_i < 0$ for $i = 1, 2$. For the second case we see that $N_T(1) = T$. Hence it follows that $\delta G_T/\delta\theta_i = h_i > 0$ for $i = 1, 2$. Consequently, the optimal OD can not be in these

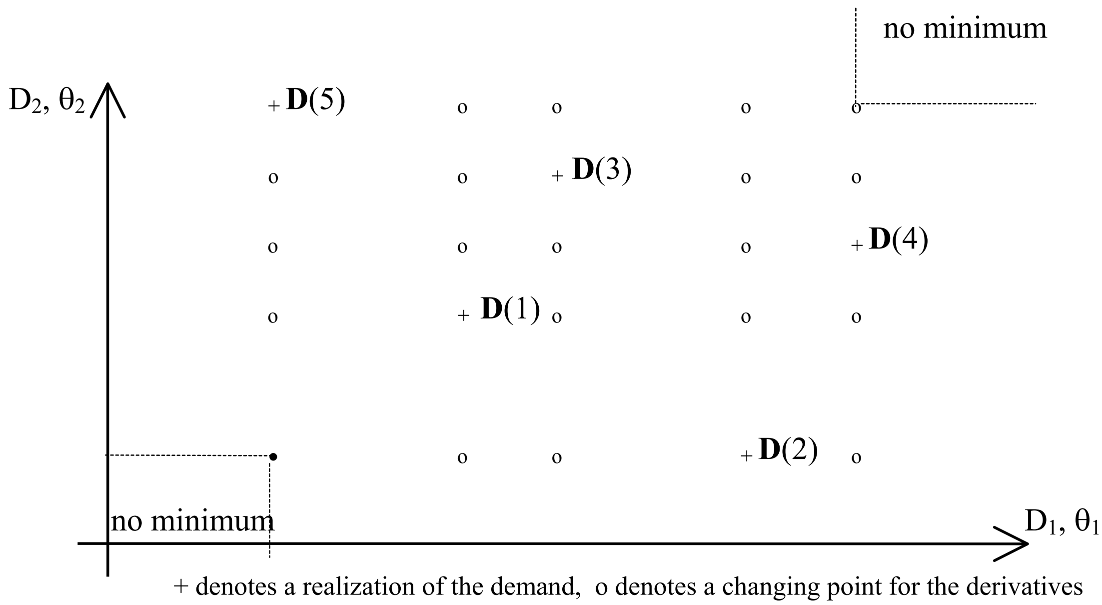


Figure 3. Possible demand realizations and up-to-order levels.

regions (see Fig.3). An other question is when to stop a given coordinate-wise passing. To give at least a partial answer we refer to Fig.4. If the sign of the partial derivative changes in a

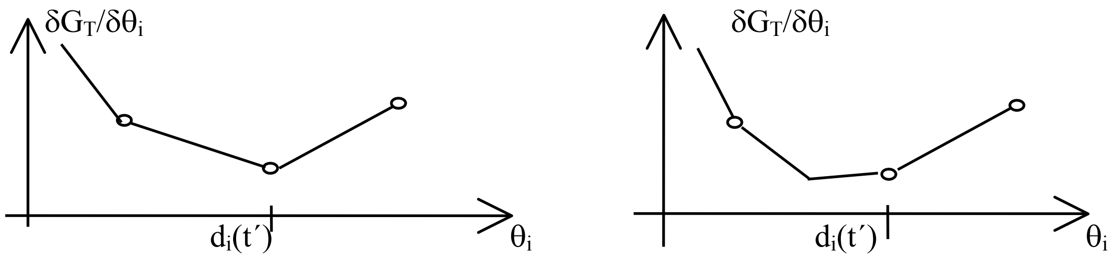


Figure 4. Two variants for the behaviour of the partial derivatives.

point $\theta_i' = d_i(t')$ then we consider the inventory level $\theta_i' - \varepsilon$, where $\varepsilon > 0$ is a small real number. In dependence on the sign of the partial derivative here either point $d_i(t')$ is minimizing or the minimizing point is located between $d_i(t')$ and $d_i(t'-1)$.

With these remarks in mind we formulate the following algorithm.

Algorithm „Retrospective optimization of the two-location inventory model with transshipments":

Input: Sample of T demand realizations $\mathcal{D}_T=\{\mathbf{D}(1), \mathbf{D}(2), ..., \mathbf{D}(T)\}$.

1. Initialization:

Formation of the ordered samples $\{d_i(t), t=1(1)T\}$, i=1, 2.
Selection of a starting solution θ^0.

2. Minimization:

$\theta' := \theta^0_1$;
Repeat
find $\theta'_2 = \arg\min G_T((\theta'_1, \theta_2); \mathcal{D})$;
find $\theta'_1 = \arg\min G_T((\theta_1, \theta'_2); \mathcal{D})$
Until
G_T can not decreased ;
$\theta^T := (\theta'_1, \theta'_2)$ and $G^T := G_T(\theta^T, \mathcal{D})$.

3. Return:

θ^T, G^T. ***STOP***.

To get some insights into the work of the proposed algorithm let us solve an example.
Example 4.1
We use the data of Example 3.1 with the only exception that we do not know the distribution function of the demand. Instead of this we assume that some realizations of the demand are given (of cause in accordance with the true distribution). For a better comparison we rewrite the data and the solution for known distribution function, that is,

i	h_i	p_i	c_{i1}	c_{i2}	m_i
1	1	10	0	5	200
2	2	9	4	0	300

respectively $\theta^* = (476 ; 448)$ and $G(\theta^*) = 1\,244, 03$.
To see how the retrospective solution changes with increasing sample size we consider the three cases T = 10, T = 20, and T = 30.
<u>I. T = 10:</u>
We have simulated 10 demand realizations. Their values are given below:

$D_1(t)$	279	79	297	173	157	76	137	342	149	11
$D_2(t)$	31	57	301	5	402	43	444	413	700	218 .

The sample averages are $m_1(10) = 170$ and $m_2(10) = 261, 4$. Now we apply our algorithm.

1. Initialization:
From the above given realizations we get the following ordered samples:

$d_1(t)$	11	76	79	137	149	157	173	279	297	342
$d_2(t)$	5	31	43	57	218	301	402	413	444	700 .

Let the starting solution be equal to $\theta^0 = (d_1(1), d_2(1)) = (11; 5)$. We remark here that we could take as well $\theta^0 = (d_1(\lceil T/2 \rceil), d_2(\lceil T/2 \rceil)) = (149; 218)$.

2. Minimization: We fix θ'_1 as $\theta'_1 := 11$, and we minimize with respect to θ_2. The computational results are given in the following table.

θ_2	$N_T(4,2)$	$N_T(3,1)$	$N_T(1)+N_T(4,1)$	$N_T(2)+N_T(3,2)$	$T\cdot \delta G_T/\delta\theta_2$
5	0	0	0	10	- 90
31	1	0	0	9	- 87
43	2	0	0	8	- 84
57	3	0	0	7	- 81
218	1	0	4	5	- 43
301	1	0	4	5	- 43
402	1	0	5	4	- 32
413	2	0	5	3	- 29
444	3	0	5	2	- 26
700	1	0	8	1	+ 1
700-ε	1	0	8	1	+ 1
587	1	0	8	1	+ 1
587-ε	2	0	7	1	- 7

We have here the situation sketched on the right of Fig.4. A more sophisticated search between 444 and 700 leads to $\theta_2 = 587$. We fix now $\theta_2 = 587$ and search in the θ_1 direction.

θ_1	$N_T(3,2)$	$N_T(4,1)$	$N_T(1)+N_T(3,1)$	$N_T(2)+N_T(4,2)$	$T\cdot \delta G_T/\delta\theta_1$
11	0	7	1	2	- 33
76	0	6	2	2	- 30
79	0	5	3	2	- 27
137	0	4	4	2	- 24
149	0	4	4	2	- 24
157	1	3	5	1	- 15
173	1	3	6	0	- 4
279	0	3	7	0	+ 1
279-ε	0	4	6	0	- 2

The situation here is that on the left of Fig.4. The result is $\theta_1 = 279$. For that θ_1-value we start the searching process again in the θ_2 - direction.

θ_2	$N_T(4,2)$	$N_T(3,1)$	$N_T(1)+N_T(4,1)$	$N_T(2)+N_T(3,2)$	$T\cdot \delta G_T/\delta\theta_2$

~~587~~	~~0~~	~~1~~	~~9~~	~~0~~	~~+ 14~~
444	1	0	8	1	+ 1
413	0	1	7	2	- 8
444-ε	1	1	7	1	- 5

We get $\theta_2 = 444$.

The following tables show how the searching process continues.

θ_1	$N_T(3,2)$	$N_T(4,1)$	$N_T(1)+N_T(3,1)$	$N_T(2)+N_T(4,2)$	$T\cdot \delta G_T/\delta\theta_1$
279	1	1	7	1	- 9
297	1	0	8	1	- 6
342	1	0	9	0	- 5
405	0	0	10	0	+ 10
405-ε	1	0	9	0	- 5

$\theta_1 = 405$:

θ_2	$N_T(4,2)$	$N_T(3,1)$	$N_T(1)+N_T(4,1)$	$N_T(2)+N_T(3,2)$	$T\cdot \delta G_T/\delta\theta_2$
444	0	1	9	0	+ 14
413	0	1	8	1	+ 3
402	0	2	7	1	- 3
413-ε	0	1	8	1	+ 3
402+ε	0	2	7	1	- 3

$\theta_2 = 402$:

θ_1	$N_T(3,2)$	$N_T(4,1)$	$N_T(1)+N_T(3,1)$	$N_T(2)+N_T(4,2)$	$T\cdot \delta G_T/\delta\theta_1$
405	1	0	9	0	+ 5
342	1	0	8	1	- 6
342-ε	1	0	9	0	+ 5

$\theta_1 = 342$:

θ_2	$N_T(4,2)$	$N_T(3,1)$	$N_T(1)+N_T(4,1)$	$N_T(2)+N_T(3,2)$	$T\cdot \delta G_T/\delta\theta_2$
402	0	1	7	2	- 8
413	0	1	8	1	+ 3
413-ε	0	1	7	2	- 8

$\theta_2 = 413$:

In this case $\theta_1 = 342$ remains optimal. Thus we have the solution $\boldsymbol{\theta}^{\mathbf{10}} = (342; 413)$.

From equ. (4.1) and (3.3) we get $G_{10}((342; 413), \mathcal{D}) = 1/10$ { h_1 (8 · 342 - 1 414) + h_2 (8 · 342 - 1 414) + [h_1 (755 - 581) + c_{12} (444 - 413)] + [p_2 (849 - 755) + c_{12} (342 - 149)]} = = 713.

3. Return:

The final result is $\theta^{10} = (342; 413)$ and $G^{10} = 713$.

<u>II. T = 20:</u>

We consider now 10 additional demand realizations. Their values are given below:

$D_1(t)$	68	644	285	1 104	311	23	214	13	513	411
$D_2(t)$	157	16	274	588	90	617	8	631	685	48 .

The overall sample averages are $m_1(20) = 264, 3$ and $m_2(20) = 286, 4$.
To start our algorithm we have to form the ordered samples for all 20 realizations together. We omit this and give only the result:

$$\theta^{20} = (513; 516) \text{ and } G^{20} = 1\,303, 65.$$

<u>III. T = 30:</u>

Finally, we will give the solution for T = 30 realizations. The last 10 demand realizations are the following ones:

$D_1(t)$	198	106	273	105	374	61	64	26	429	244
$D_2(t)$	61	164	575	120	101	575	207	148	603	160 .

The overall sample averages are now $m_1(30) = 238, 87$ and $m_2(30) = 281, 4$.
The final result is

$$\theta^{30} = (443; 575) \text{ and } G^{30} = 1\,177, 1.$$

5. Conclusion

We have considered the retrospective method as an approach to optimize a two-location inventory model with transshipments.The main advantage of that approach is that we do not need any information about the type and/or the parameters of the demand distribution. In the present paper we have used structural properties of the optimal decisions and the criterion function to simplify the resulting deterministic optimization problem. Therefore the proposed algorithm uses only simple numerical operations. This is shown by an example.

Nevertheless the algorithm works there exists the need for further research. Above all we will concentrate on two directions - to prove convergence properties for $T \to \infty$, and to apply the retrospective approach to a model with more than two locations.

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