A stochastic inventory model for some interconnected locations

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1. Introduction

The papers [5] and [7] were the starting point for the investigation of stochastic inventory models at the beginning of the 50th. Until the middle of the 60th were considered above all models with a single location (warehouse) and a single product. For practice however it is important to design models by which we can study the inventory of a product in a number of spatially distant warehouses. Different to former models we have to look for an inventory policy that minimises the cost of the whole system of warehouses. Typical for these models is the pooling of inventories with the aim to liquidate possible shortages by transhipments from one warehouse to another one. It is meaningful to consider following problems (see [12]):

1. How to organise the replenishment of inventories in all warehouses?
2. How to organise an optimal redistribution of available inventories?
3. How to define an optimal distribution of an arriving supply among all warehouses?

The easiest problems are problems 2 and 3. Therefore, the first papers, which investigate a system of warehouses, were dealing with these problems (see [2], [3], [4], [8], [9]). For the first time problem 1 is considered in [11]. However, there is investigated a special case of *n* warehouses with $h_i = h$, $p_i = p$, and $c_{ij} = c$, $i, j = 1, \ldots, n$ $(i \neq j)$, where h denotes the holding cost per product unit, p the shortage cost, and c the transhipment cost per unit transported from warehouse *i* to warehouse *j* (per unit of product).

Problem 1 is solved for two locations with different cost parameters in [1] and [10]. For more than two locations up to now could not be found a strong analytical solution. In the present paper we introduce conditions that allow proving some structural properties for the general case of *n* locations.

2. Description of the model

Let us assume a system of *n* connected warehouses, which store during a single period the same product. As a period we can understand different time intervals, a shift, a week, a month and so on. During that period each location has to meet a given demand for that product. The amount y_i of the demand is a random variable with distribution function $F_i(\cdot)$, $i=1, \ldots, n$. We assume that these functions are known and that the random variables y_1 to y_n are independent. If x_i denotes the starting inventory and y_i the demand at warehouse *i*, $i=1, \ldots, n$, then during the period following cost can incur:

1. If at the beginning of the period warehouse *i*, $i=1, \ldots, n$, gets z units of product, then we have to pay $K_i \cdot z$ ordering cost.
2. If at the end of the period for warehouse *i* holds $x_i > y_i$ then we have holding cost. We assume proportional cost, i.e., $h_i \cdot (x_i - y_i)$, $i=1, \ldots, n$.
3. If at the end of the period for warehouse *i* holds $x_i < y_i$ then we have to pay shortage cost, which are again assumed to be proportional, i.e., $p_i \cdot (y_i - x_i)$, $i=1, \ldots, n$.
4. If at the end of the period we have in location *i* that $x_i > y_i$ and in location *j* that $y_j > x_j$ then location *i* can "help" location *j* and tranship a given amount of product from *i* to *j*. Thereby arise transhipment cost c_{ij} per transferred unit.

Based on the economical background of the inventory process in such a system of warehouses we can introduce four conditions, which are of great significance for the solution.

1. *Condition of transhipment efficiency*

(1) $$c_{ij} < h_i + p_j \quad (i, j = 1, \ldots, n, i \neq j),$$

i.e., the transshipment cost c_{ij} for a unit must be smaller that the cost in case we do not tranship, $h_i + p_j$.

2. Condition of relative independence of the warehouses

(2.1) $K_i + c_{ij} > K_j \quad (i, j = 1, \ldots, n, i \neq j)$,

i.e., it is advantageous for each warehouse *j* to order directly than to use an intermediary warehouse *i*.

(2.2) $p_i + c_{ij} > p_j; \quad h_i + c_{ji} > h_j \qquad (i, j = 1, \ldots, n, i \neq j)$,

i.e., each location gets its own cost and does not shift the cost on to another location.

3. Condition of the shortest way

(3) $c_{ik} + c_{kj} \geq c_{ij} \quad (i, j, k = 1, \ldots, n, i \neq j \neq k)$.

With these conditions the dynamics of the system can be described as follows.

a) At the beginning of the period each warehouse orders x_i units of product without of regard for surplus inventories or shortages in other locations ($i = 1, \ldots, n$).
b) After the system has reached state $X = (x_1, \ldots, x_n)$ demand $Y = (y_1, \ldots, y_n)$ is observed.
c) If transshipments decrease cost then we transship.
d) Calculate the total cost, and a new period starts.

Our goal is to find such an inventory policy, which minimises the total expected cost of the system. In other words, we have to find such numbers x_i^*, $i = 1, \ldots, n$, up to which the inventory levels have to be raised by replenishments such that the expectation of the total cost for the period will be minimal.

3. Cost function for the single-period expected cost

Let $\overline{x_i}$ denote the initial inventory in location *i*, *i* = 1, …, *n*. Let further $L_{\overline{x_1}, \ldots, \overline{x_n}}(x_1, \ldots, x_n)$ denote the expected value of the period cost if the initial inventory was $\overline{X} = (\overline{x_1}, \ldots, \overline{x_n})$ and the inventory was raised up to the level $X = (x_1, \ldots, x_n)$. Obviously must be $x_i \geq \overline{x_i}$, *i* = 1, …, *n*. Then

$$L_{\overline{X}}(X) = \sum_{i=1}^{n} K_i(x_i - \overline{x_i}) + G(x_1, \ldots, x_n), \tag{4}$$

where $G(x_1, \ldots, x_n)$ denotes the expected period cost under inventory vector $X = (x_1, \ldots, x_n)$.
We prove following theorem.

Theorem 1.
In case of linear holding, shortage and transhipment cost function $L_{\overline{X}}(X)$ is convex in its arguments.
Proof:
Let $G(X, Y)$ denote the cost under inventory vector X and demand vector Y. It holds

$$G(X,Y) = \min_{\{z_{ij}\}} \left\{ \sum_{i=1}^{n} h_i \max\Big(0, x_i - y_i - \sum_{j=1}^{n} z_{ij} + \sum_{j=1}^{n} z_{ji}\Big) + \sum_{i,j=1}^{n} c_{ij} z_{ij} + \sum_{i=1}^{n} p_i \max\Big(0, y_i - x_i + \sum_{j=1}^{n} z_{ij} - \sum_{j=1}^{n} z_{ji}\Big) \right\}, \tag{5}$$

whereby $\{z_{ij}\}$ denotes the transhipment plan between the locations. We assume that $c_{ii} = 0$ for *i* = 1, …, *n*. Obviously holds

$$G(x_1, \ldots, x_n) = \int_0^\infty \ldots \int_0^\infty G(x_1, \ldots, x_n; y_1, \ldots, y_n)\, f_1(y_1) \ldots f_n(y_n)\, dy_1 \ldots dy_n. \tag{6}$$

Plan $\{z_{ij}\}$ must fulfil following constraints:

a) $\sum_{j=1}^{n} z_{ij} = x_i$, *i* = 1, …, *n*;

b) $z_{ij} \geq 0$, $i, j = 1, \ldots, n$.

From (5) we get

$$G(X,Y) = \min_{\substack{z_{ij} \geq 0 \\ \sum_{j=1}^{n} z_{ij} = x_i, i=1,..,n}} \left\{ \sum_{i=1}^{n} \left[h_i \, max(0, \sum_{j=1}^{n} z_{ji} - y_i) + p_i \, max(0, y_i - \sum_{j=1}^{n} z_{ji}) + \sum_{j=1}^{n} c_{ij} z_{ij} \right] \right\}.$$

If we set $u_i = max(0, \sum_{j=1}^{n} z_{ji} - y_i)$ and $v_i = max(0, y_i - \sum_{j=1}^{n} z_{ji})$ then we get

$$G(X,Y) = \min_{\substack{1.) z_{ij}, u_i, v_i \geq 0 \\ 2.) \sum_{j=1}^{n} z_{ij} = x_i \\ 3.) u_i \geq \sum_{j=1}^{n} z_{ji} - y_i \\ 4.) v_i \geq y_i - \sum_{j=1}^{n} z_{ji}}} \left\{ \sum_{i=1}^{n} \left[h_i \cdot u_i + p_i \cdot v_i + \sum_{j=1}^{n} c_{ij} \cdot z_{ij} \right] \right\}.$$

With auxiliary variables α_i and β_i this LP problem is equivalent to problem

$$G(X,Y) = \min_{\substack{1.) z_{ij}, u_i, v_i, \alpha_i, \beta_i \geq 0 \\ 2.) \sum_{j=1}^{n} z_{ij} = x_i \\ 3.) \sum_{j=1}^{n} z_{ji} - u_i + \alpha_i = y_i \\ 4.) \sum_{j=1}^{n} z_{ji} + v_i - \beta_i = y_i}} \left\{ \sum_{i=1}^{n} \left[h_i \cdot u_i + p_i \cdot v_i + \sum_{j=1}^{n} c_{ij} \cdot z_{ij} \right] \right\}. \tag{5*}$$

Let $T = (X, Y)$ and R be the vector consisting of all variables z_{ij}, u_i, v_i, α_i, and β_i. Then function $G(X,Y)$ can be represented as function $f(T) = \min_{\substack{A \cdot R = B \cdot T \\ R \geq 0}} (c, R)$.

Lemma: *f(T)* is a convex function of *T*.

Proof:

We have to show $f(\lambda T_1 + (1-\lambda) T_2) \leq \lambda f(T_1) + (1 - \lambda) f(T_2)$ for $0 \leq \lambda \leq 1$.

Let us assume now that $f(T_1) = \min_{\substack{A \cdot R = B \cdot T_1 \\ R \geq 0}} (c, R) = \min_{\substack{\lambda AR = \lambda BT \\ R \geq 0}} (c, R) = (c, R_1)$

and

$$f(T_2) = \min_{\substack{A \cdot R = B \cdot T_2 \\ R \geq 0}} (c, R) = \min_{\substack{(1-\lambda) AR = (1-\lambda) BT \\ R \geq 0}} (c, R) = (c, R_2),$$

i.e., R_1 and R_2 are the corresponding solutions of problems $f(T_1)$ and $f(T_2)$. Then it holds also $\lambda f(T_1) + (1 - \lambda) f(T_2) = (c, \lambda R_1 + (1 - \lambda) R_2)$.

On the other hand we have

$$f(\lambda\cdot T_1+(1-\lambda)\cdot T_2)=\min_{\substack{A\cdot R=B\cdot(\lambda T_1+(1-\lambda)T_2)\\ R\geq 0}}(c,R)=\min_{\substack{\lambda A\cdot R+(1-\lambda)A\cdot R=\lambda B\cdot T_1+(1-\lambda)B\cdot T_2\\ R\geq 0}}(c,R),$$

such that vector $\lambda\cdot R_1+(1-\lambda)\cdot R_2$ is an admissible solution for problem $f(\lambda T_1+(1-\lambda)T_2)$. This means that

$$f(\lambda T_1+(1-\lambda)T_2)\leq(c,\lambda\cdot R_1+(1-\lambda)\cdot R_2)=\lambda\cdot f(T_1)+(1-\lambda)\cdot f(T_2),$$

which we had to prove.

Thus we have that $G(X,Y)$ is a convex function of X and Y, especially $G(X,Y)$ is a convex function of x_1 to x_n for any Y. Since $f_i(y_i)\geq 0$ for all y_i, $i=1,\ldots,n$, function $G(X)$ is also convex in its arguments. From (4) follows now the proof.□

From Theorem 1 follow some statements on existence and form of the optimal policy.

1. From convexity of $L_{\overline{X}}(X)$ and the property $L_{\overline{X}}(X)\xrightarrow[\|X\|\to\infty]{}\infty$ follows that problem $\min\limits_{X\geq\overline{X}}L_{\overline{X}}(X)$ has a finite solution X*($\overline{X}$). The optimal policy has following structure: The inventory levels are raised up from $\overline{X}$ to X*($\overline{X}$), i.e., we order X*($\overline{X}$) - $\overline{X}$.

Let further X* = X*(0) denote the solution of problem $\min\limits_{X\geq 0}L_0(X)$. Then:

   a) for $\overline{X}\leq$ X* we have X*($\overline{X}$) = X* independent of $\overline{X}$;
   b) for $\overline{X}>$ X* we have X*($\overline{X}$) = $\overline{X}$;
   c) if $\overline{x_i}>x_i^*$ for some i and $\overline{x_j}<x_j^*$ for some j then X*($\overline{X}$) depends on $\overline{X}$.

2. If we allow randomised policies like

$$X=\begin{cases}\text{with probability } \lambda \text{ we increase inventory up to } X_1\\ \text{with probability } (1-\lambda) \text{ we increase inventory up to } X_2\end{cases}$$

then from convexity of function $L_{\overline{X}}(X)$ follows that the optimal policy is a pure policy, i.e., X*($\overline{X}$) still defines the optimal policy.

3. With additional conditions we get a statement on the solution of the multi-period model.

Let at the beginning of period one the initial inventory be $\overline{X}\leq X^*$*, where X* is the solution of the single-period problem* $\min\limits_{X\geq 0}L_0(X)$*. Assume further* $K_i=0$*, i=1, ..., n. Then the solution X* of the single-period model is also solution of the multi-period model.*

Proof:

From $K_i=0$, i=1, …,n, follows $L_0(X)=G(X)$. Since $G(X)$ is convex it holds

$$t\cdot G\left(\frac{X_1+\ldots+X_t}{t}\right)\leq G(X_1)+\ldots+G(X_t).$$

Let X_1, …, X_t denote the optimal policies for the first, …, and t-th period in the multi-period model. The corresponding costs are $\sum\limits_{s=1}^{t}G(X_s)$. These costs are greater or equal to the cost over t periods if in each period we apply policy $\left\{\frac{X_1+\ldots+X_t}{t}\right\}$.

Since $t\cdot G\left(\frac{X_1+\ldots+X_t}{t}\right)\geq t\cdot G(X^*)$ we proved the assertion.

We remark that this result corresponds with the case of a single location and K = 0.

4. Optimal redistribution of available inventory

In Chapter 2 we introduced conditions (1) – (3). In the present chapter we show how these conditions can help to solve problem (5) or (5*). For this we define three sets: $I^+=\{i: x_i>y_i\}$, $I^-=\{i: x_i < y_i\}$, and $I^0=\{i: x_i = y_i\}$. These sets characterise the state of the system before the stock redistribution, i.e., before the minimisation over $\{z_{ij}\}$. As assumed above we have $c_{ii} = 0$, $i = 1, \ldots, n$.

Theorem 2

Under the conditions (1), (2.2), and (3) holds:

a) if one of the sets I^+ and I^- is empty then there are no transshipments in the optimal plan, excluding $z_{ii} = x_i$, $i = 1, \ldots, n$;

b) if both sets I^+ and I^- are not empty then there exists at least one $z_{i_o j_0} > 0$ with $i_0 \in I^+$ and $j_o \in I^-$.

Proof:

a) 1) If both sets are empty then must be $y_i = x_i$, $i = 1, \ldots, n$. Then for the optimal transhipment plan holds $z_{ii} = y_i = x_i$, $i = 1, \ldots, n$. In this case $G(X, Y) = 0$. Any other plan generates cost.

2) Assume I^- to be empty. Then $y_i \le x_i$, $i = 1, \ldots, n$, and the cost before the redistribution are equal to

$$F(X,Y)=\sum_{i=1}^{n} h_i \cdot (x_i - y_i).$$

Assume now that after optimal transhipments $\{\bar{z}_{ij}\}$ we have following cost

$$G(X,Y)=\sum_{i\in U=\{i:u_i>0\}} h_i \cdot \left(\sum_{j=1}^{n} \bar{z}_{ji} - y_i \right) + \sum_{j\in V=\{i:v_j>0\}} p_j \cdot \left(y_j - \sum_{i=1}^{n} \bar{z}_{ij} \right) + \sum_{i,j=1}^{n} c_{ij} \bar{z}_{ij}.$$

Assume V to be not empty. Then in set V must exist a j_0 with $\bar{z}_{j_0 i_0} > 0$, whereby $i_0 \in U$. We have following picture:

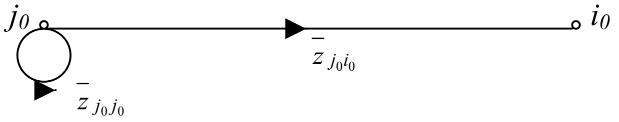

Figure 1

Now we design a new plan $Z = \{z_{ij}\}$ with $z_{j_0 i_0} = \bar{z}_{j_0 i_0} - \varepsilon$, $z_{j_0 j_0} = \bar{z}_{j_0 j_0} + \varepsilon$, and $z_{ij} = \bar{z}_{ij}$ for all others. We take $\varepsilon > 0$ sufficiently small. Thus plan Z is admissible because of $z_{ij} \ge 0$ and $\sum_{i=1}^{n} z_{j_0 i} = \sum_{i=1}^{n} \bar{z}_{ji} = x_{j_0}$. But, the costs for plan Z decrease by $\varepsilon \cdot (h_{i_0} + p_{j_0} + c_{j_0 i_0}) > 0$.

If set V is empty then exists a $\bar{z}_{j_0 i_0} > 0$, whereby $i_0, j_0 \in U$. Then plan Z decreases cost by $\varepsilon \cdot (h_{i_0} + c_{j_0 i_0} - h_{j_0}) > 0$ by (2.2). This is a contradiction to the optimality assumption of $\bar{Z}$. Consequently, there are no redistributions.

3) The proof is analogous in case I^+ is empty.

Remark: In the same way we can show that for $i \in I^0$ always holds $\bar{z}_{ij} = x_i$, i.e., set I^0 does not take part in the redistribution.

b) The proof of part b) goes by analogy and will be omitted here.□

Up to now we have shown that under the conditions (1), (2.2), and (3) there exist transhipments from I^+ to I^-; if one of the sets I^+ and I^- is empty then we have no transhipments excluding $z_{ii} = x_i$ ($i=1, \dots, n$), and for $i \in I^0$ we have always $z_{ii} = y_i = x_i$. To fully characterise plan $\{z_{ij}\}$ we prove Theorem 3.

Theorem 3

Under the conditions (1), (2.2), and (3) holds:

1. There are no transshipments from I^- to I^+.
2. For $i \in I^-$ we have $z_{ii} = x_i$; for $i \in I^+$ we have $y_i \le z_{ii} \le x_i$.
3. Within the sets I^+ and I^- we have no other transhipments than those from 2.

Proof:

As in the proof of Theorem 2 we start from the opposite. We assume an optimal transshipment plan $\overline{Z}$ and show that we can improve that plan. Since the proof is analogue to the proof of Theorem 2 we omit them here.□

Lets us summarise:

1. If I^+ and I^- are not empty then condition (1) means that transshipments are profitable. Thereby transshipments go, because of (2.2) and (3), only from I^+ into I^-; within the sets I^+ and I^- are no transshipments excluding $z_{ii} = x_i$ for $i \in I^-$ and $y_i \le z_{ii} \le x_i$ for $i \in I^+$. The set I^0 does not take part in the redistribution; here we have $z_{ii} = y_i = x_i$, $i \in I^0$. By (1) it is obvious that the total transhipped amount of product with positive coefficients in function (5) is equal to
$$\sum_{i\in I^+}\sum_{j\in I^-} z_{ij} = \min\left\{\sum_{i\in I^+}(x_i - y_i); \sum_{j\in I^-}(y_j - x_j)\right\}.$$
2. By (2.2) we have $y_i \le z_{ii} \le x_i$ for $i \in I^+$ and $z_{ii} = x_i$ for $i \in I^-$. Thus each location satisfies at first its own demand. Only afterwards lateral transshipments are allowed.

In other words, under conditions (1), (2.2), and (3) we get for (5) that

$$(7) \quad G(X,Y) = \min_{\{z_{ij}\}}\left\{\sum_{i\in I^+} h_i \cdot (x_i - y_i - \sum_{j\in I^-} z_{ij}) + \sum_{i\in I^+}\sum_{j\in I^-} c_{ij} z_{ij} + \sum_{j\in I^-} p_j \cdot (y_j - x_j - \sum_{i\in I^+} z_{ij})\right\},$$

whereby $Z = \{z_{ij}\}$ must fulfil following constraints:

1. $z_{ij} \ge 0, i \in I^+, j \in I^-,$
2. $\sum_{j\in I^-} z_{ij} \le x_i - y_i, i \in I^+,$
3. $\sum_{i\in I^+} z_{ij} \le y_j - x_j, j \in I^-,$
4. $\sum_{i\in I^+}\sum_{j\in I^-} z_{ij} = min\left\{\sum_{i\in I^+}(x_i - y_i); \sum_{j\in I^-}(y_j - x_j)\right\}.$

If $\sum_{i=1}^{n} x_i \ne \sum_{i=1}^{n} y_i$ then (7) represents an open linear transportation problem. In case $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ we have a classical transportation problem with delivery locations I^+ and receiver locations I^-.

Assume $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$, $|I^+| = k$, and $|I^-| = l$. Then the constraints 2 to 4 of problem (7) are equivalent to

2'. $\sum_{j=1}^{l} z_{ij} \le a_i, i = 1,\dots,k,$

3'. $\sum_{i=1}^{k} z_{ij} \le b_j, j = 1,\dots,l,$

4'. $\sum_{i=1}^{k}\sum_{j=1}^{l} z_{ij} = \sum_{j=1}^{l} b_j,$

whereby $a_i = x_i - y_i$ and $b_j = y_j - x_j$. As in Dantzig [6] we introduce auxiliary variables $z_{i0} = a_i - \sum_{j=1}^{l} z_{ij}, i = 1, \dots, k$. Obviously $z_{i0} \ge 0$. For the transportation tableau we have:

	Surplus offer		delivery locations
	z_{10}	$z_{11}\ z_{12} \dots z_{1l}$	$= a_1$
	z_{20}	$z_{21}\ z_{22} \dots z_{2l}$	$= a_2$
	:	: : :	:
	:	: : :	:
	z_{k0}	$z_{k1}\ z_{k2} \dots z_{kl}$	$= a_k$
receiver locations	$=$ $a_i - b_j$	$= \ = \dots =$ $b_1\ b_2 \dots b_l$	

Thereby we have to minimise the following goal function:

$$v = \sum_{i=1}^{k} h_i \cdot z_{i0} + \sum_{i=1}^{k}\sum_{j=1}^{l} c_{ij} \cdot z_{ij} \ .$$

This form of the goal function comes from the fact that in case $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$ by (1) the whole demand can be satisfied and therefore we will have only hooding and transhipment cost. The introduction of the variables z_{i0} means that there exists an additional (dummy) location "0" with demand $\sum_{i=1}^{n}(x_i - y_i)$. To that location all undemanded stock will be transferred by cost $c_{i0} = h_i$, $i=1, \dots, k$, per unit.

The case $\sum_{i=1}^{n} x_i < \sum_{i=1}^{n} y_i$ can be handled by analogy, but now we have a location "0" with stock $\sum_{i=1}^{n}(y_i - x_i)$ from which transhipments go into all locations with unsatisfied demand by cost $c_{0j} = p_j, j = 1, \dots, l$, per unit.

The corresponding problems can be solved for fixed X and Y by arbitrary methods of transportation optimisation. Since Y takes values in the positive orthant of the n-dimensional space we clearly can see from (6) and (7) the reason for the difficulties to get an analytical solution of the initial problem (to define optimal $x_1^*, \dots, x_n^*$) for the case $n > 2$. However, problem (7) can be used to solve the following problem: Assume that in the n warehouses we have the inventory levels $x_1, x_2, \dots, x_n$. If the demand is now equal to $Y = (y_1, \dots, y_n)$ then the solution of problem (7) tells us how to organise the transhipments such that the total cost will be minimal.

In the following we consider case $n = 2$.

5. Case N=2

In case $n = 2$ the solution of problem (7) is easy for general $Y = (y_1, y_2)$. For this we consider Figure 2.

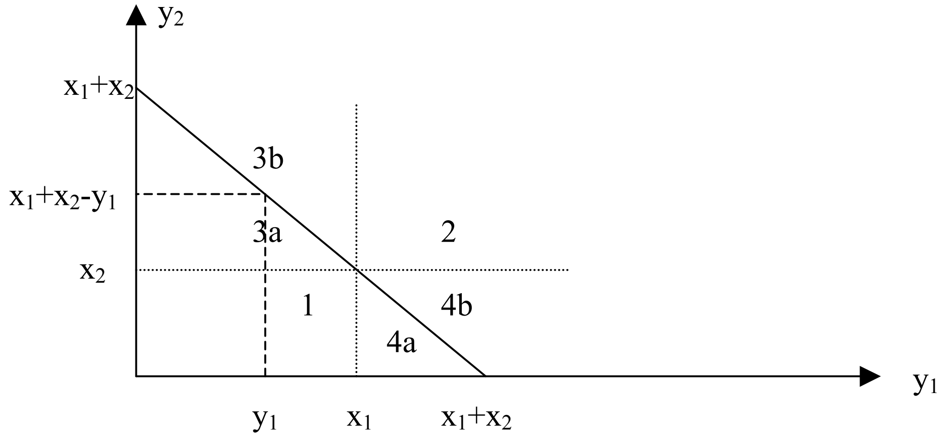

Figure 1

1. $Y \le X$. Set I^- is empty. Thus we have by Theorem 2 that $G(X, Y) = h_1 \cdot (x_1 - y_1) + h_2 \cdot (x_2 - y_2)$. Thus with region 1 the following average costs are connected:

$$\int_0^{x_1}\int_0^{x_2}[h_1 \cdot (x_1 - y_1) + h_2 \cdot (x_2 - y_2)]f_1(y_1)f_2(y_2)dy_1dy_2 =$$

$$= F_2(x_2)\cdot\int_0^{x_1} h_1 \cdot (x_1 - y_1)f_1(y_1)dy_1 + F_1(x_1)\cdot\int_0^{x_2} h_2 \cdot (x_2 - y_2)f_2(y_2)dy_2 .$$

2. $Y \ge X$. By analogy we get $G(X, Y) = p_1 \cdot (y_1 - x_1) + p_2 \cdot (y_2 - x_2)$ and the cost

$$\overline{F}_2(x_2)\cdot\int_{x_1}^{\infty} p_1 \cdot (y_1 - x_1)f_1(y_1)dy_1 + \overline{F}_1(x_1)\cdot\int_{x_2}^{\infty} p_2 \cdot (y_2 - x_2)f_2(y_2)dy_2 ,$$

whereby $\overline{F}(\cdot) = 1 - F(\cdot)$.

3.a) $x_1 \ge y_1$; $x_2 < y_2$; $x_1+x_2 \ge y_1+y_2$.

In this case, since $\sum_{i\in I^+}\sum_{j\in I^-} z_{ij} = z_{12} = min\{x_1 - y_1; y_2 - x_2\} = y_2 - x_2$, we get from (7) that

$G(X, Y) = h_1 \cdot (x_1 - y_1 - z_{12}) + c_{12} z_{12} + p_2 \cdot (y_2 - x_2 - z_{12}) = h_1 \cdot (x_1 + x_2 - y_1 - y_2) + c_{12} \cdot (y_2 - x_2)$, i.e., there exists a unique solution that fulfils all the constraints 1 to 4 of problem (7). $G(X,Y)$ has this form in the region $\{Y: x_1 \ge y_1; x_2 < y_2 ; x_1+x_2 \ge y_1+y_2\}$, i.e., where $0 \le y_1 \le x_1$ and $x_2 < y_2 < x_1 + x_2 - y_1$. The corresponding costs are

$$\int_0^{x_1}\int_{x_2}^{x_1+x_2-y_1}[h_1 \cdot (x_1 + x_2 - y_1 - y_2) + c_{12} \cdot (y_2 - x_2)]f_2(y_2)f_1(y_1)dy_2dy_1 .$$

3.b) $x_1 \ge y_1$; $x_2 < y_2$; $x_1+x_2 < y_1+y_2$.

In this case $z_{12} = x_1 - y_1$ and the costs are

$$\int_0^{x_1}\int_{x_1+x_2-y}^{\infty}[p_2 \cdot (y_1 + y_2 - x_1 - x_2) + c_{12} \cdot (x_1 - y_1)]f_2(y_2)f_1(y_1)dy_2dy_1 .$$

It remains region 4. In analogy to region 3 we get

$$\int_0^{x_2}\int_{x_1}^{x_1+x_2-y_2}[h_2 \cdot (x_1 + x_2 - y_1 - y_2) + c_{21} \cdot (y_1 - x_1)]f_1(y_1)f_2(y_2)dy_1dy_2 +$$

$$+ \int_{0}^{x_2} \int_{x_1+x_2-y_2}^{\infty} \left[p_1 \cdot (y_1 + y_2 - x_1 - x_2) + c_{21} \cdot (x_2 - y_2 \right] f_1(y_1) f_2(y_2) dy_1 dy_2 .$$

G(X) = G(x$_1$, x$_2$) is then equal to the sum of all these parts for the regions 1 to 4. Since the cost functions are linear it is easy to show that

$$G(x_1,x_2) = h_1 \int_0^{x_1} F_1(y_1) dy_1 + p_1 \int_{x_1}^{\infty} \overline{F}_1(y_1) dy_1 - (h_1 + p_2 - c_{12}) \int_0^{x_1} F_1(y_1) \overline{F}_2(x_1 + x_2 - y_1) dy_1 +$$

$$+ \; h_2 \int_0^{x_2} F_2(y_2) dy_2 + p_2 \int_{x_2}^{\infty} \overline{F}_2(y_2) dy_2 - (h_2 + p_1 - c_{21}) \int_0^{x_2} F_2(y_2) \overline{F}_1(x_1 + x_2 - y_2) dy_2 .$$

If we put this expression into (4) then it is seen that function $L_{\overline{X}}(X)$ is a function of all initially given parameters and functions. An algorithm, which yields after a finite number of steps the solution X* = (x$_1$*, x$_2$*) is described in [10].

6. Discussion of the model

To investigate a system of warehouses, which are connected by transporting routes, comes from the demand for the most efficient use of economic resources. This requirement will increase in the future because of the ongoing concentration in production and the implementation of automations in planning and control. However we must say that in the considered model we have assumed very simple cost functions. Future investigations have to remove this drawback.

As advantageous we consider the introduction of the conditions (1) to (3). These conditions have a simple economic interpretation and enable us to transform the problem of optimal redistribution for fixed inventory levels into a linear transporting problem.

A similar formula for $L_{\bar{x}_1,\bar{x}_2}(x_1,x_2)$ is developed in [1]. However, there were not considered conditions (1)-(3), i.e., problem (7) was not solved. There was simply assumed that transhipments will be realised independent on their effectiveness, and that there are no reallocations inside the sets I$^+$ and I$^-$. But we are of opinion that the efficiency of transshipments, i.e. condition (1), is important. The goal is not to organise cooperation at any price but to derive advantages for the whole system.

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Summary

This paper deals with an inventory problem in multilocation supply systems. We minimise the one-period expected costs. It is shown, that for more than two locations the problem is not solvable in the framework of our model. Constraints for the cost-parameters are also obtained. This allows us to solve the problem of an optimal redistribution of the present stock among all locations as a transportation problem of linear programming. Some assertions about the optimal inventory policy are made. The two-location supply system is considered in detail.