Solution 4: Rate coded neurons

A. Theoretical evaluation of a winner-takes-all network

Question 1 (points: 2)**:** *For which p, q values do we see a response of some of the cells n due to an input?*

Answer 1: Let us first consider the steady state of a neuron i:

$$z_i + q \cdot \sum_{j=1}^{n} [f(z_j)]^+ \quad = \quad I_i^{ext} + p\,[f(z_i)]^+ \tag{1}$$

Since we want a response even for small inputs, we estimate $I_i^{ext} = 0$. We know that $z_i < f(z_i)$, thus in the limit we obtain:

$$q < p \tag{2}$$

Question 2 (points: 3)**:** *For which p, q values does this network guarantee that only a single cell is active (winner-takes-all behavior)?*

Answer 2: To ensure that only a single neuron fires, there has to be a winning neuron w with the rate z_w that inhibits all other neurons such that they will not fire anymore. Thus, we estimate that the inhibitory term is only driven by the winning neuron w and the neuron i, since more active neurons would make the inhibition even larger. With these assumptions the ODE of a neuron i writes:

$$\tau \frac{\delta z_i}{\delta t} \quad = \quad -z_i + I_i^{ext} + (p-q)\,[f(z_i)]^+ - q \cdot [f(z_w)]^+ \tag{3}$$

As long as the change in the firing rate of the neuron i is negative, sooner or later this neuron will be inactive:

$$0 \quad > \quad -z_i + I_i^{ext} + (p-q)\,[f(z_i)]^+ - q \cdot [f(z_w)]^+ \tag{4}$$

We can further simplify the expression by setting $z_i = 0$ and $I_i^{ext} = 1$:

$$0 \quad > \quad 1 + (p-q)\,[f(z_i)]^+ - q \cdot [f(z_w)]^+ \tag{5}$$

Since $(p - q) > 0$ as required to have activity in the network at all, we can estimate $[f(z_i)]^+$ with the largest possible value which is $[f(z_w)]^+$ which leads us to the answer:

$$q > \frac{1}{2}\left(p + \frac{1}{[f(z_w)]^+}\right) \tag{6}$$

When we assume that c is small (e.g. $c = 0.1$) we can furthermore estimate $[f(z_w)]^+$ with 1 which leads to the final answer:

$$q > \frac{1}{2}(p + 1) \tag{7}$$

B. Emperical evaluation of a winner-takes-all network

Answer 2: The last step of the solution to A.2 was to estimate $[f(z_w)]^+$ with 1. Depending on the choice of c, $[f(z_w)]^+$ might be much smaller than 1. When for example $[f(z_w)]^+ < 0.5$ it follows from 6:

$$q > \frac{p}{2} + 1 \tag{8}$$

Previously we estimated

$$q > \frac{1}{2}(p + 1) \tag{9}$$

If q is chosen such that it obeys both constraints and we estimate $z_i = 0$, $I_i^{ext} = 1$ and $z_i = z_w$, the ODE becomes

$$\begin{aligned}
1 + (p - q)\,[f(z_w)]^+ - q \cdot [f(z_w)]^+ &= \qquad (10)\\
1 + p\,[f(z_w)]^+ - 2q\,[f(z_w)]^+ &< \qquad (11)\\
1 + p\,[f(z_w)]^+ - p\,[f(z_w)]^+ - 2\,[f(z_w)]^+ &= 1 - 2\,[f(z_w)]^+ \qquad (12)
\end{aligned}$$

If $[f(z_w)]^+ < 0.5$ this is positive and thus z_i is raising.