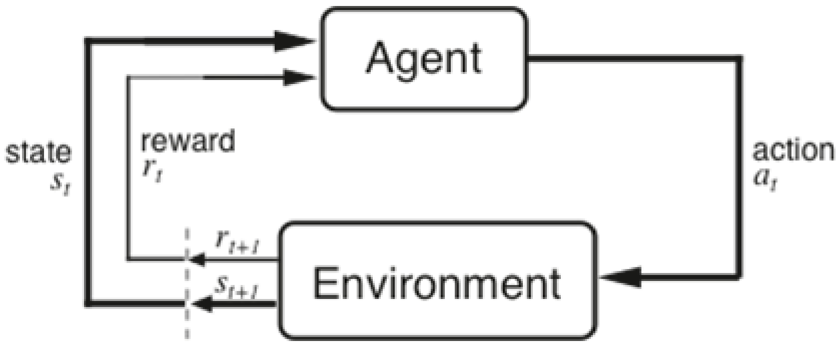
Dynamic Programming

Suggested reading:

Chapter 4 in R. S. Sutton, A. G. Barto: Reinforcement Learning: An Introduction
MIT Press, 1998.

Dynamic Programming



Contents:

- Policy evaluation
- Policy improvement
- Policy iteration
- Value interation
- Asynchronous dynamic programming
- Generalized policy iteration
- Efficiency of dynamic programming

Dynamic Programming

Objectives of this chapter:

- Overview of a collection of classical solution methods for MDPs known as dynamic programming (DP)
- Show how DP can be used to compute value functions, and hence, optimal policies
- Discuss efficiency and utility of DP

Key idea of Dynamic Programming

- Environment finite MDP (not necessary but typical)
- the use of value functions to organize and structure the search for good policies
- Once we have the value function we can easily obtain optimal policies
- DP algorithms are obtained by turning Bellman equations such as these

$$V^{\pi}(s) = \sum_{a} \pi(s,a) \sum_{s'} P_{ss'}^{a} \left[R_{ss'}^{a} + \gamma V^{\pi}(s') \right]$$

into assignments, that is, into update rules for improving approximations of the desired value functions.

Policy Evaluation

Policy Evaluation: for a given policy π, compute the state-value function V^π

Recall: **State - value function for policy π :**

$$V^\pi(s) = E_\pi\{R_t \mid s_t = s\} = E_\pi\left\{\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \mid s_t = s\right\}$$

Bellman equation for V^π :

$$V^\pi(s) = \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V^\pi(s')\right] \qquad s \in S$$

— a system of $|S|$ simultaneous linear equations

Iterative Methods

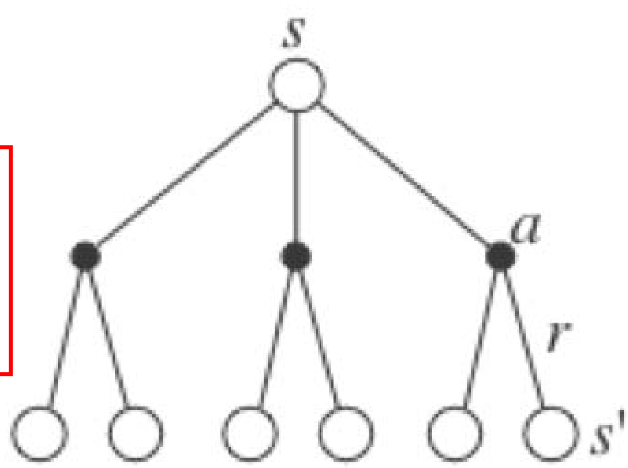
$$V_0 \rightarrow V_1 \rightarrow \ldots \rightarrow V_k \rightarrow V_{k+1} \rightarrow \ldots \rightarrow V^{\pi}$$

a “sweep”

A sweep consists of applying a full **backup operation** to each state.

A **full policy-evaluation backup**:

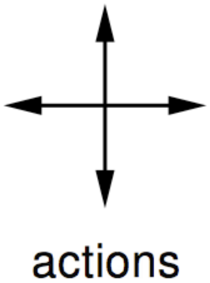
$$\boxed{V_{k+1}(s) \leftarrow \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V_k(s') \right]}$$

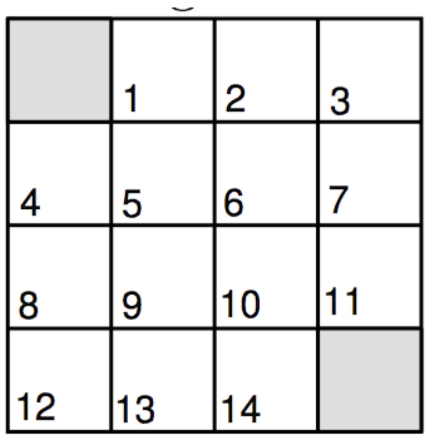


Iterative Policy Evaluation

Input π, the policy to be evaluated
Initialize $V(s) = 0$, for all $s \in \mathcal{S}^+$
Repeat
  $\Delta \leftarrow 0$
  For each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} [\mathcal{R}^a_{ss'} + \gamma V(s')]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx V^\pi$

A Small Gridworld





$r = -1$
on all transitions

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged

$$P_{5,6}^{right} = 1, \;\; P_{5,10}^{right} = 0 \;\text{ and }\; P_{7,7}^{right} = 1$$

- Reward is –1 until the terminal state is reached

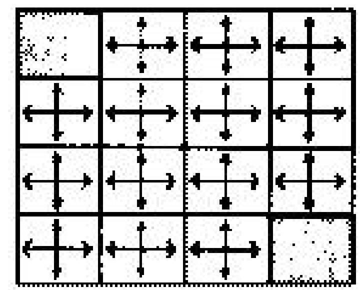
Iterative Policy Eval for the Small Gridworld

$\pi =$ random (uniform) action choices

V_k for the random policy | Greedy policy w.r.t. V_k

$k = 0$

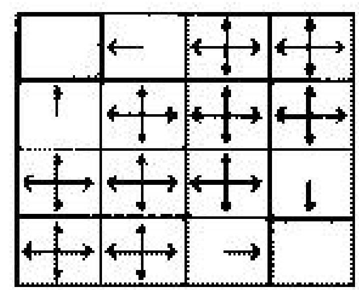
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0



random policy

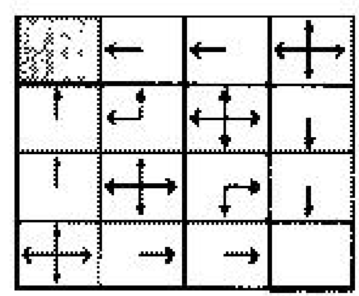
$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0



$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0



Iterative Policy Eval for the Small Gridworld

$k = 3$

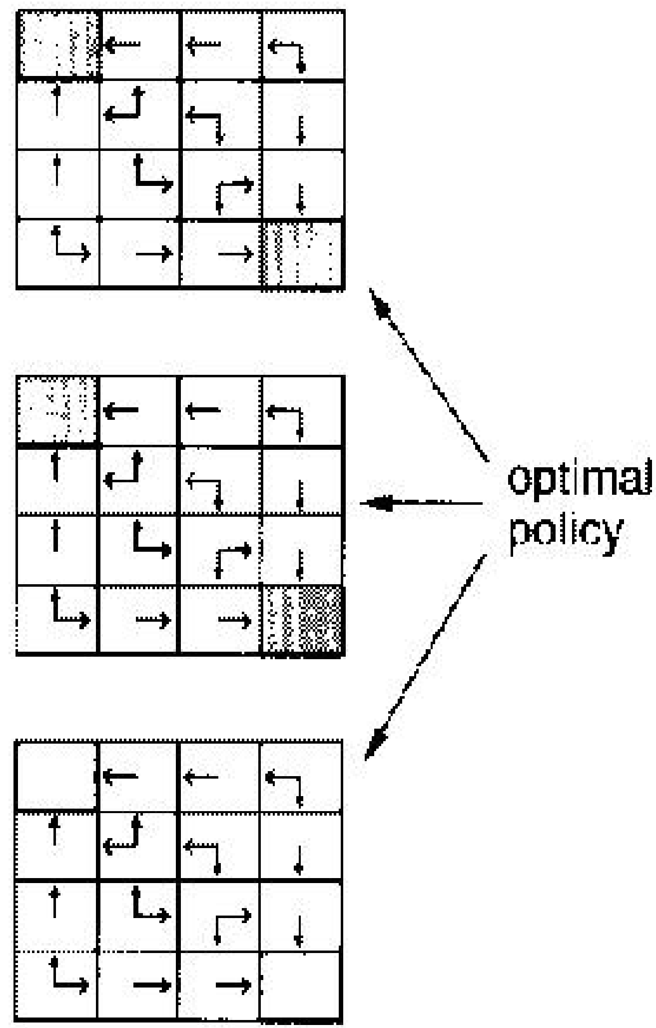
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



Policy Improvement

Suppose we have computed V^π for a deterministic policy π.

For a given state s,
would it be better to do an action $a \neq \pi(s)$?

The value of doing a in state s is:

$$Q^\pi(s,a) = E_\pi \left\{ r_{t+1} + \gamma V^\pi(s_{t+1}) \,\middle|\, s_t = s, a_t = a \right\}$$
$$= \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V^\pi(s') \right]$$

It is better to switch to action a for state s if and only if

$$Q^\pi(s,a) > V^\pi(s)$$

Policy improvement theorem

Let π and π' be any pair of deterministic policies such that, for all $s \in S$,

$$Q^{\pi}(s, \pi'(s)) \geq V^{\pi}(s)$$

$$\Downarrow$$

$$\boxed{\pi' \geq \pi \quad \Longleftrightarrow \quad V^{\pi'}(s) \geq V^{\pi}(s) \quad \forall s \in S}$$

This means, the policy π’ must be as good as, or better than, π. That is, it must obtain greater or equal expected return from all states s ∈ S:

Policy improvement theorem

$$
\begin{aligned}
V^{\pi}(s) &\leq Q^{\pi}(s, \pi'(s)) \\
&= E_{\pi'}\{r_{t+1} + \gamma V^{\pi}(s_{t+1}) \mid s_t = t\} \\
&\leq E_{\pi'}\{r_{t+1} + \gamma Q^{\pi}(s_{t+1}, \pi'(s_{t+1})) \mid s_t = t\} \\
&= E_{\pi'}\{r_{t+1} + \gamma E'_{\pi}\{r_{t+2} + \gamma V^{\pi}(s_{t+2})\} \mid s_t = t\} \\
&= E_{\pi'}\{r_{t+1} + \gamma r_{t+2} + \gamma^2 V^{\pi}(s_{t+2}) \mid s_t = t\} \\
&\leq E_{\pi'}\{r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \gamma^3 V^{\pi}(s_{t+3}) \mid s_t = t\} \\
&\vdots \\
&\leq E_{\pi'}\{r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \gamma^3 r_{t+4} + \cdots \mid s_t = t\} \\
&= V^{\pi'}(s) .
\end{aligned}
$$

Policy Improvement Cont.

Do this for all states to get a new policy π' that is **greedy** with respect to V^π :

$$\begin{aligned}\pi'(s) &= \arg\max_a Q^\pi(s,a) \\ &= \arg\max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V^\pi(s') \right]\end{aligned}$$

Then $V^{\pi'} \geq V^\pi$

Policy Improvement Cont.

What if $V^{\pi'} = V^{\pi}$?

i.e., for all $s \in S$, $V^{\pi'}(s) = \max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V^{\pi}(s') \right]$?

But this is the Bellman Optimality Equation.

So $V^{\pi'} = V^*$ and both π and π' are optimal policies.

Policy improvement thus must give us a strictly better policy except when the original policy is already optimal.

Theorem

$$V^{\pi'}(s) = V^{\pi}(s) \quad \forall s \in S \quad \Rightarrow \quad \pi' = \pi = \pi^*$$

Proof:

$$V^{\pi'}(s) = \max_{a \in A(s)} Q^{\pi}(s,a) = \max_{a \in A(s)} \sum_{s' \in S} P^a_{ss'} \cdot \left[R^a_{ss'} + \gamma \cdot V^{\pi}(s') \right] = \max_{a \in A(s)} \sum_{s' \in S} P^a_{ss'} \cdot \left[R^a_{ss'} + \gamma \cdot V^{\pi'}(s') \right]$$

$$V^{\pi'}(s) = \max_{a \in A(s)} \sum_{s' \in S} P^a_{ss'} \cdot \left[R^a_{ss'} + \gamma \cdot V^{\pi'}(s') \right]$$

Bellmann Optimality Equation

$$V^{\pi'}(s) = V^*(s) \qquad \pi' = \pi = \pi^*$$

optimal policy

Policy Improvement - nondeterministic policies

Let π and π' be any pair of arbitrary policies such that, for all $s \in S$

$$\sum_{a \in A(s)} \pi'(s,a) \cdot Q^{\pi}(s,a) \geq V^{\pi}(s)$$

$$\Downarrow$$

$$\boxed{\pi' \geq \pi \quad \Longleftrightarrow \quad V^{\pi'}(s) \geq V^{\pi}(s) \quad \forall s \in S}$$

Policy Iteration

$$\pi_0 \rightarrow V^{\pi_0} \rightarrow \pi_1 \rightarrow V^{\pi_1} \rightarrow \cdots \pi^* \rightarrow V^* \rightarrow \pi^*$$

policy evaluation

policy improvement
“greedification”

Policy Iteration

1. Initialization

   $V(s) \in \Re$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation

   Repeat

   $\quad \Delta \leftarrow 0$

   $\quad$ For each $s \in \mathcal{S}$:

   $\quad\quad v \leftarrow V(s)$

   $\quad\quad V(s) \leftarrow \sum_{s'} \mathcal{P}_{ss'}^{\pi(s)} \left[\mathcal{R}_{ss'}^{\pi(s)} + \gamma V(s') \right]$

   $\quad\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$

   until $\Delta < \theta$ (a small positive number)

3. Policy Improvement

   policy-stable $\leftarrow$ *true*

   For each $s \in \mathcal{S}$:

   $\quad b \leftarrow \pi(s)$

   $\quad \pi(s) \leftarrow \arg\max_a \sum_{s'} \mathcal{P}_{ss'}^{a} \left[\mathcal{R}_{ss'}^{a} + \gamma V(s') \right]$

   $\quad$ If $b \neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*

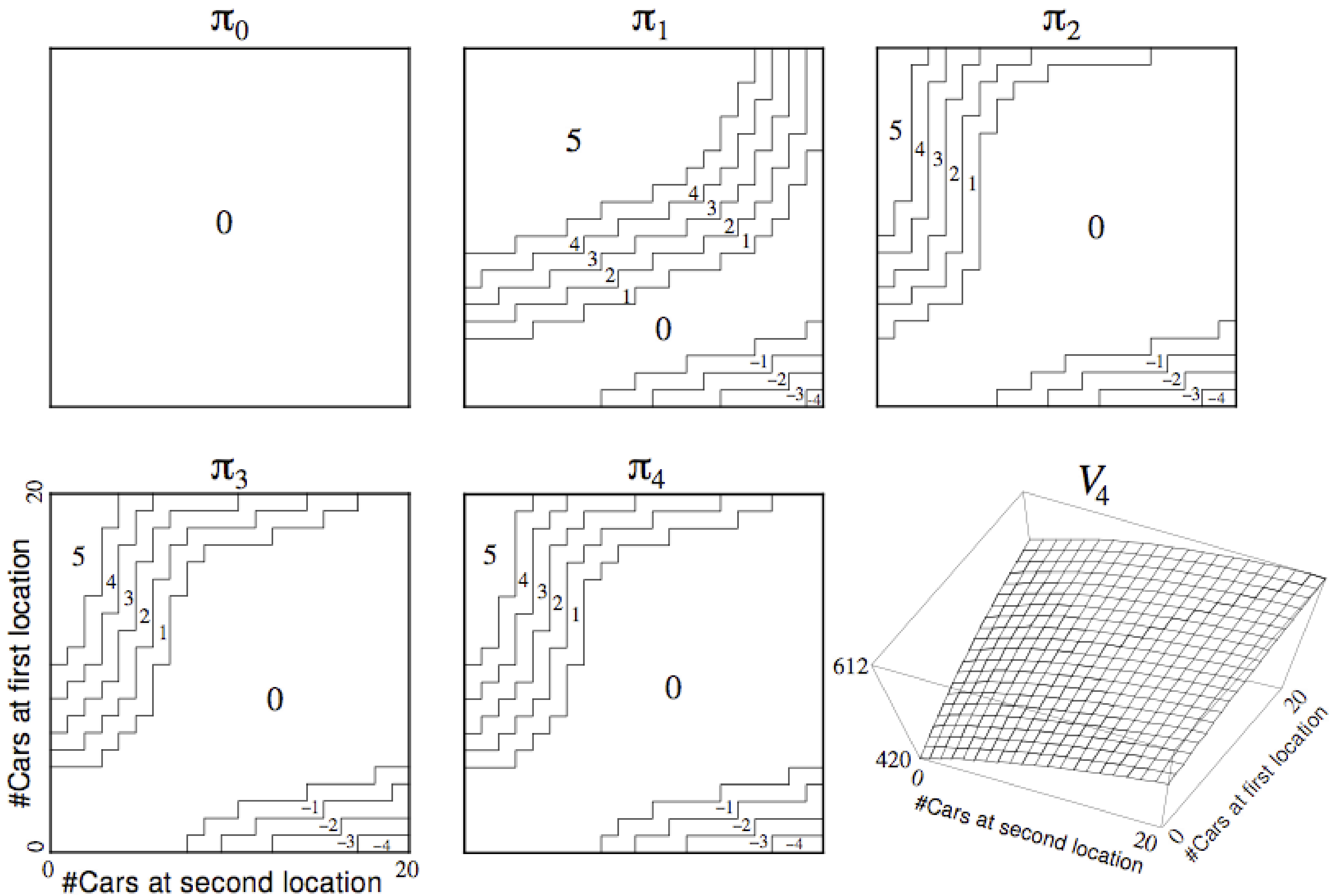
   If *policy-stable*, then stop; else go to 2

Jack's Car Rental

- $10 for each car rented (must be available when request rec'd)
- Two locations, maximum of 20 cars at each
- Cars returned and requested randomly
  - Poisson distribution, *n* returns/requests with prob $\frac{\lambda^n}{n!}e^{-\lambda}$
  - 1st location: average requests = 3, average returns = 3
  - 2nd location: average requests = 4, average returns = 2
- Can move up to 5 cars between locations overnight

- States, Actions, Rewards?
- Transition probabilities?

Jack's Car Rental



Jack’s CR Exercise

- Suppose the first car moved is free
  - From 1st to 2nd location
  - Because an employee travels that way anyway (by bus)
- Suppose only 10 cars can be parked for free at each location
  - More than 10 cost $4 for using an extra parking lot

- Such arbitrary nonlinearities are common in real problems

Value Iteration

Recall the **full policy-evaluation backup**:

$$V_{k+1}(s) \leftarrow \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V_k(s') \right]$$

Here is the **full value-iteration backup**:

$$\boxed{V_{k+1}(s) \leftarrow \max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V_k(s') \right]}$$

Value Iteration Cont.

Initialize V arbitrarily, e.g., $V(s) = 0$, for all $s \in \mathcal{S}^+$

Repeat
$\Delta \leftarrow 0$
For each $s \in \mathcal{S}$:
$v \leftarrow V(s)$
$V(s) \leftarrow \max_a \sum_{s'} \mathcal{P}^a_{ss'} [\mathcal{R}^a_{ss'} + \gamma V(s')]$
$\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

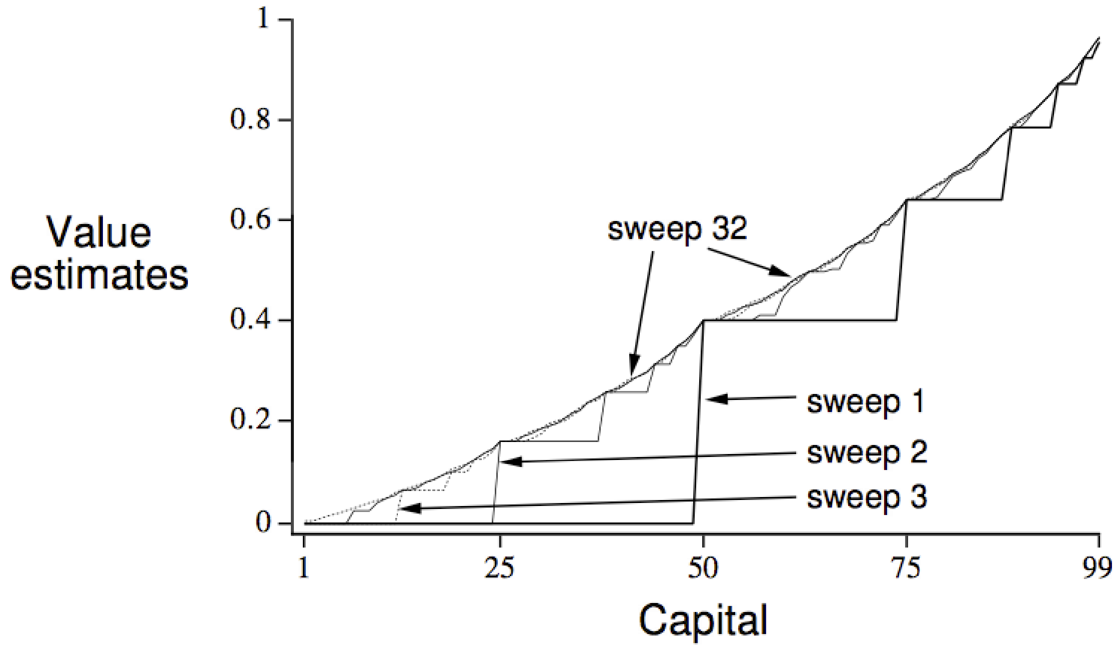
Output a deterministic policy, π, such that
$\pi(s) = \arg\max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V(s')\right]$

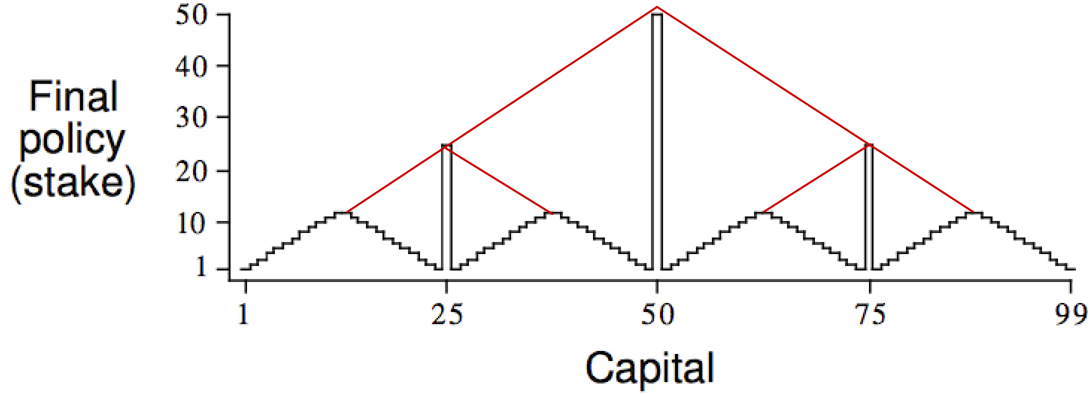
Gambler's Problem

- Gambler can repeatedly bet \$ on a coin flip
- Heads he wins his stake, tails he loses it
- Initial capital $\in$ {\$1, \$2, … \$99}
- Gambler wins if his capital becomes \$100 loses if it becomes \$0
- Coin is unfair
  - Heads (gambler wins) with probability p = .4

- States, Actions, Rewards?

Gambler's Problem Solution



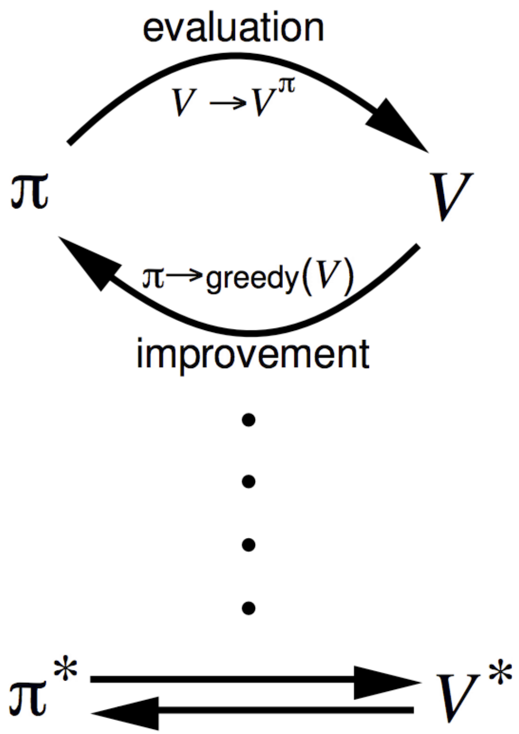


Asynchronous Dynamic Programming

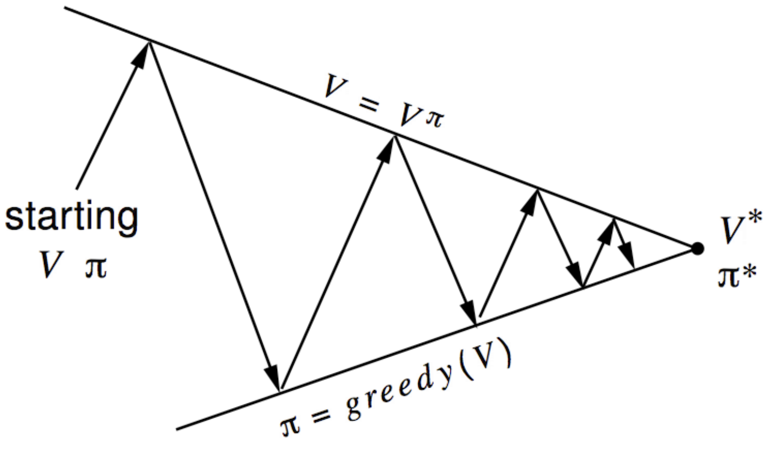
- All the DP methods described so far require exhaustive sweeps of the entire state set.
- Asynchronous DP does not use sweeps. Instead it works like this:
  - Repeat until convergence criterion is met:
    - Pick a state at random and apply the appropriate backup
- Still need lots of computation, but does not get locked into hopelessly long sweeps
- Can you select states to backup intelligently? YES: an agent’s experience can act as a guide.

Generalized Policy Iteration

Generalized Policy Iteration (GPI):
any interaction of policy evaluation and policy improvement, independent of their granularity.



A geometric metaphor for convergence of GPI:



Efficiency of Dynamic Programming

- To find an optimal policy is polynomial in the number of states…
- BUT, the number of states is often astronomical, e.g., often growing exponentially with the number of state variables (what Bellman called “the curse of dimensionality”).
- In practice, classical DP can be applied to problems with a few millions of states.
- Asynchronous DP can be applied to larger problems, and appropriate for parallel computation.
- It is surprisingly easy to come up with MDPs for which DP methods are not practical.

Summary

- Policy evaluation: backups without a max
- Policy improvement: form a greedy policy, if only locally
- Policy iteration: alternate the above two processes
- Value iteration: backups with a max
- Full backups (to be contrasted later with sample backups)
- Generalized Policy Iteration (GPI)
- Asynchronous DP: a way to avoid exhaustive sweeps
- **Bootstrapping**: updating estimates based on other estimates